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On the Laplace transform of functionals on classes of infinitely differentiable functions

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The purpose of this note is to study functionals on quasi-analytic and non-quasi-analytic classes of infinitely differentiable functions, equipped with suitable topologies, and in particular to prove theorems of the Payley–Wiener type connecting properties of functionals with the behaviour of their Laplace transforms. This has been done in the non-quasi-analytic case by Roumieu [10], who has studied so called ultra-distributions. For a related (and partially equivalent) definition of generalised distributions, see e.g. Björck [2].

In this note the interest lies in the quasi-analytic case, although the theorems do not exclude non-quasi-analytic classes. After some elementary definitions and properties of the spaces and functionals to be considered we state two "Payley–Wiener theorems" in section 1. These theorems are proved in section 2 essentially with methods taken from Hörmander [4]. In section 3 we prove some approximation theorems, which are used to guarantee that a functional is uniquely determined by its Laplace transform.

1. Functionals on c_L and C_L

Let Ω be an open set in $\mathbf{R}^d$. Then $C^\infty(\Omega)$ denotes the space of complex-valued functions with continuous derivatives of every order in Ω. If $\alpha=(\alpha_1, \ldots, \alpha_d)$ is a multi-index ($\alpha_j=0, 1, \ldots$), we write $D^\alpha=D_1^{\alpha_1}\ldots D_d^{\alpha_d}$ where $D_j=\partial/\partial x_j$. Similarly $\zeta^\alpha=\zeta_1^{\alpha_1}\ldots\zeta_d^{\alpha_d}$ if $\zeta=(\zeta_1, \ldots, \zeta_d)\in\mathbf{C}^d$. We shall also write $|\alpha|=\alpha_1+\ldots+\alpha_d$ and $\alpha!=\alpha_1!\ldots\alpha_d!$

Let $L=(L_\alpha)_\alpha$ be a family of positive real numbers defined for all multi-indices $\alpha=(\alpha_1, \ldots, \alpha_d)$. Then $C_L(\Omega)$ denotes the set of $f\in C^\infty(\Omega)$, such that for every compact set K in Ω there are constants $a>0$ and C such that

$$\forall\alpha:\quad \sup_K |D^\alpha f|\leqslant Ca^{|\alpha|}L_\alpha. \tag{1.1}$$

$c_L(\Omega)$ denotes the set of $f\in C^\infty(\Omega)$, such that for every compact set K in Ω and every $a>0$ there is a constant C such that (1.1) is valid.

It is clear that $C_L(\Omega)$ and $c_L(\Omega)$ are complex linear spaces.

A natural topology on $c_L(\Omega)$ is defined by the set of all semi-norms

$$f \to \sup_{\alpha} \sup_{K} |D^{\alpha} f| \frac{h^{|\alpha|}}{L_{\alpha}}, \tag{1.2}$$

where K is a compact set in Ω and where $h > 0$. We shall use the equivalent set of semi-norms

$$f \to \|f\|_{L,K,h} = \sum_{\alpha} \sup_{K} |D^{\alpha} f| \frac{h^{|\alpha|}}{L_{\alpha}}. \tag{1.3}$$

It is easy to see that $c_L(\Omega)$ is a Fréchet space with this topology.

On $C_L(\Omega)$ we use the topology which is defined by all semi-norms p on $C_L(\Omega)$, such that there is a compact set K in Ω and for every $h > 0$ a constant C such that

$$\forall f \in C_L(\Omega): \quad p(f) \leqslant C \|f\|_{L,K,h}. \tag{1.4}$$

(Of course we can here use the semi-norms in (1.2) instead of $\|f\|_{L,K,h}$.)

Given two families $L = (L_{\alpha})_{\alpha}$ and $M = (M_{\alpha})_{\alpha}$ we write $L \prec M$, if there are constants $a > 0$ and C such that

$$\forall \alpha: \quad L_{\alpha} \leqslant C a^{|\alpha|} M_{\alpha}. \tag{1.5}$$

We write $L \prec\prec M$ if for every $a > 0$ there is a constant C such that (1.5) is valid.

It is clear that $L \prec M$ implies that $c_L(\Omega) \subset c_M(\Omega)$ and $C_L(\Omega) \subset C_M(\Omega)$ and that $L \prec\prec M$ implies that $C_L(\Omega) \subset c_M(\Omega)$. We also see that the corresponding inclusion maps are continous. The converse implications are true, when the family L is logarithmically convex, i.e. when $\log L_{\alpha}$ is a convex function of α, which is equivalent to

$$\forall \alpha: \quad L_{\alpha} = \sup_{r} \inf_{\beta} \frac{r^{\alpha}}{r^{\beta}} L_{\beta}, \tag{A}$$

where r runs over all $r = (r_1, \ldots, r_d)$ with $r_j > 0$ (cf. Bang [1], § 3).

If L satisfies

$$\forall \alpha: \quad L_{\alpha+\beta} \leqslant b^{|\alpha|+1} L_{\alpha} \quad \text{if} \quad |\beta| = 1 \tag{B}$$

with some $b > 0$, then $c_L(\Omega)$ and $C_L(\Omega)$ are closed under differentiation. (B) is also a necessary condition when L satisfies (A) (cf. Bang [1], § 4).

$c_L(\Omega)$ and $C_L(\Omega)$ are closed under multiplication, if L satisfies

$$\forall \alpha, \forall \beta: \quad L_{\alpha} L_{\beta} \leqslant C c^{|\alpha|+|\beta|} L_{\alpha+\beta} \tag{C}$$

with some constants C and $c > 0$. For then it follows by means of Leibniz' formula for differentiation that

$$\|fg\|_{L,K,h} \leqslant C \|f\|_{L,K,2ch} \|g\|_{L,K,2ch}. \tag{1.6}$$

In the particular case when $L_\alpha = l_{|\alpha|}$ for all α, where $(l_n)_0^\infty$ is a sequence of positive real numbers, the condition (C) is a consequence of (A). For then $(l_n)_0^\infty$ is logarithmically convex, and this implies $l_m l_n \leqslant l_0 l_{m+n}$. However, in general (C) does not follow from (A) (a counter-example can be found in Roumieu [10], p. 159).

We recall the theorem of Denjoy–Carleman in the following general form proved by Lelong [7]. See also Roumieu [10], Th. 1.

$C_L(\Omega)$ does not contain any function with compact support contained in Ω (except the zero function), if and only if

$$\sum_{n=1}^{\infty} \bar{L}_n / \bar{L}_{n+1} = +\infty, \tag{D}$$

where the sequence $\bar{L} = (\bar{L}_n)_0^\infty$ is the largest logarithmically convex minorant sequence of $(\inf_{|\alpha|=n} L_\alpha)_{n=0}^\infty$, i.e. $\bar{L}$ is given by

$$\bar{L}_n = \sup_{t>0} \inf_{\alpha} t^{n-|\alpha|} L_\alpha.$$

The statement is true also when $C_L(\Omega)$ is replaced by $c_L(\Omega)$.

$C_L(\Omega)$ and $c_L(\Omega)$ are called quasi-analytic when L satisfies (D).

A linear form u on $c_L(\Omega)$ is continuous if and only if there are a compact set K in Ω and constants $h > 0$ and C such that

$$|u(f)| \leqslant C \|f\|_{L,K,h} \tag{1.7}$$

for all $f \in c_L(\Omega)$. A linear form u on $C_L(\Omega)$ is continuous if and only if there are a compact set K in Ω and for every $h > 0$ a constant C such that (1.7) is valid for all $f \in C_L(\Omega)$.

We denote by $c'_L(\Omega)$ and $C'_L(\Omega)$ the topological dual spaces of $c_L(\Omega)$ and $C_L(\Omega)$ resp.

It is clear that $c_L(\Omega) \subset C_L(\Omega) \subset C^\infty(\Omega)$ with continuous inclusion maps, if we give $C^\infty(\Omega)$ the usual topology defined by all semi-norms $f \to \sum_{|\alpha| \leqslant m} \sup_K |D^\alpha f|$ where K is a compact subset of Ω and m a non-negative integer. Therefore the restriction to $c_L(\Omega)$ or $C_L(\Omega)$ of a distribution with compact support in Ω is a continuous linear form on $c_L(\Omega)$ and $C_L(\Omega)$ resp. However, the formula

$$u(f) = \sum_\alpha D^\alpha \mu_\alpha(f) = \sum_\alpha (-1)^{|\alpha|} \mu_\alpha(D^\alpha f) \tag{1.8}$$

defines a continuous linear form u on $c_L(\Omega)$, whenever all μ_α are measures with support in some compact set K in Ω and with total mass $\|\mu_\alpha\| \leqslant C h^{|\alpha|} / L_\alpha$ (K, C and h independent of α). Therefore there are functionals on $c_L(\Omega)$ and on $C_L(\Omega)$, which can not be extended to distributions.

Using the Hahn–Banach theorem one can see that every $u \in c'_L(\Omega)$ has the form (1.8).

We say that a compact set K_0 in Ω is a carrier of or carries a functional $u \in c'_L(\Omega)$, if for every compact neighbourhood $K \subset \Omega$ of K_0 there are constants $h > 0$ and C such that (1.7) is valid for all $f \in c_L(\Omega)$. Similarly K_0 carries $u \in$

$C'_L(\Omega)$, if for every compact neighbourhood $K \subset \Omega$ of K_0 and every $h>0$ there is a constant C such that (1.7) is valid for all $f \in C_L(\Omega)$.

In the non-quasi-analytic case there is also the concept of support of a functional u on $c_L(\Omega)$ or $C_L(\Omega)$: At least if L also satisfies (C) we can define supp u as the smallest compact subset K of Ω, such that $u(f)=0$ when $f=0$ in some neighbourhood of K. It is clear that supp u is contained in every carrier of u. Conversely, supp u is a carrier of u, because for every compact neighbourhood K of supp u one can find $\varphi \in c_L(\Omega)$ with supp $\varphi \subset K$ and $\varphi = 1$ in a neighbourhood of supp u. Then

$$|u(f)| = |u(\varphi f)| \leqslant C' \|\varphi\|_{L,K,2ch} \|f\|_{L,K,2ch}$$

with some constant C' follows from (1.7) anp (1.6).

We define the Laplace transform $\tilde{u}$ of a functional u on $c_L(\mathbf{R}^d)$ or $C_L(\mathbf{R}^d)$ by

$$\forall \zeta \in \mathbf{C}^d: \quad \tilde{u}(\zeta) = u(x \to e^{\langle x, \zeta \rangle}), \tag{1.9}$$

where $\langle x, \zeta \rangle = x_1\zeta_1 + \ldots + x_d\zeta_d$. When $u \in C'_L(\mathbf{R}^d)$ we must require that $\inf_\alpha a^{|\alpha|} L_\alpha > 0$ for some $a>0$, so that $f(x) = e^{\langle x, \zeta \rangle}$ belongs to $C_L(\mathbf{R}^d)$ for all $\zeta \in \mathbf{C}^d$. If $u \in c'_L(\mathbf{R}^d)$ we must require that $\inf_\alpha a^{|\alpha|} L_\alpha > 0$ for all $a>0$. Then it is clear that $\tilde{u}$ is an entire function in $\mathbf{C}^d$, because the Taylor series of $e^{\langle x, \zeta \rangle}$ is convergent in $C_L(\mathbf{R}^d)$ and in $c_L(\mathbf{R}^d)$ resp.

The inequality (1.7) implies (with $\zeta = \xi + i\eta$)

$$|\tilde{u}(\zeta)| \leqslant C \sum_\alpha \sup_K |D^\alpha e^{\langle x, \zeta \rangle}| \frac{h^{|\alpha|}}{L_\alpha} = C \sum_\alpha \frac{h^{|\alpha|}|\zeta^\alpha|}{L_\alpha} \sup_K e^{\langle x, \xi \rangle} = C q_L(h\zeta)\, e^{H_K(\xi)}, \tag{1.10}$$

where
$$q_L(\zeta) = \sum_\alpha \frac{|\zeta^\alpha|}{L_\alpha} \quad \text{and} \quad H_K(\xi) = \sup_{x \in K} \langle x, \xi \rangle. \tag{1.11}$$

H_K is the supporting function of (the closed convex hull of) K and is continuous, convex and positively homogeneous of degree 1.

If $u \in c'_L(\mathbf{R}^d)$ is carried by K, we know that for every $\varepsilon>0$ there are constants $h>0$ and C such that (1.7) is valid for all $f \in c_L(\mathbf{R}^d)$ with K replaced by $K_\varepsilon = \{x \in \mathbf{R}^d\colon d(x, K) \leqslant \varepsilon\}$. If $u \in C'_L(\mathbf{R}^d)$ is carried by K, there is a constant C for every $\varepsilon>0$ and $h>0$ such that (1.7) is valid for all $f \in C_L(\mathbf{R}^d)$ with K replaced by K_ε. Therefore, if we replace K by K_ε in (1.10) and use the equality $H_{K_\varepsilon}(\xi) = H_K(\xi) + \varepsilon|\xi|$ we have proved the first parts of the following two theorems.

Theorem 1. *Suppose that* $\inf_\alpha a^{|\alpha|} L_\alpha > 0$ *for all* $a>0$. *If* $u \in c'_L(\mathbf{R}^d)$ *is carried by a compact set* K *in* $\mathbf{R}^d$, *then for every* $\varepsilon>0$ *there are constants* $h>0$ *and* C *such that* $U = \tilde{u}$ *satisfies*

$$\forall \zeta \in \mathbf{C}^d: \quad |U(\zeta)| \leqslant C q_L(h\zeta)\, e^{H_K(\xi) + \varepsilon|\xi|}. \tag{1.12}$$

Conversely, if L *also satisfies* (A) *and* (B) *and if* U *is an entire function in* $\mathbf{C}^d$ *such that* (1.12) *is fulfilled for all* $\varepsilon>0$ *with a convex compact set* K *in* $\mathbf{R}^d$

(C and $h>0$ depending on ε), then there is a unique functional $u \in c'_{L'}(\mathbf{R}^d)$ such that $\tilde{u}=U$ and u is carried by K. Here $L'=(L'_\alpha)_\alpha$ is defined by $L'_{\alpha'}=L_{\alpha+(1,\dots,1)}$.

Theorem 2. *Suppose that $\inf_\alpha a^{|\alpha|}L_\alpha>0$ for all $a>0$. If $u \in C'_L(\mathbf{R}^d)$ is carried by a compact set K in $\mathbf{R}^d$, then for every $\varepsilon>0$ and $h>0$ there is a constant C such that $U=\tilde{u}$ satisfies (1.12).*

Conversely, if L also satisfies (A) and (B) and if U is an entire function in $\mathbf{C}^d$ such that (1.12) is fulfilled for all $\varepsilon>0$ and $h>0$ with a convex compact set K in $\mathbf{R}^d$ (C depending on ε and h), then there is a unique functional $u \in C'_{L'}(\mathbf{R}^d)$ such that $\tilde{u}=U$. u is carried by K at least if L also satisfies (C) and $(\alpha!)_\alpha \prec L$.

Remark 1. From (B) follows that $c_{L'}(\mathbf{R}^d) \subset c_L(\mathbf{R}^d)$ and $C_{L'}(\mathbf{R}^d) \subset C_L(\mathbf{R}^d)$ with continuous inclusion maps. Equality holds if L also satisfies $L \prec L'$ and then the topologies coincide too. This means that we can replace L' by L everywhere in Theorems 1 and 2, if we add the condition $L \prec L'$. When $d=1$ or L_α depends only on $|\alpha|$, $L \prec L'$ follows from (A).

If L depends only on $|\alpha|$, say $L_\alpha=l_n$ when $|\alpha|=n$, then we can replace $q_L(h\zeta)$ in (1.12) by $q_l(h|\zeta|)$, where $q_l(t)=\sum_{n=0}^\infty t^n/l_n$, because

$$q_l(|\zeta|/\sqrt{d}) \leqslant q_L(\zeta) \leqslant C q_l(|\zeta|)$$

with C depending only on d.

Remark 2. When L satisfies only (A) and (B), we can see that u in Theorem 2 is carried by K if K is a closed rectangle in $\mathbf{R}^d$ with sides parallel to the coordinate planes. See the proof of Theorem 2. The stronger conditions on L for arbitrary convex compact sets K are used when we approximate by means of Theorem 4 but they should not be the best possible.

However, our notion of carrier does not seem to be very interesting for functionals on $c_L(\Omega)$ or $C_L(\Omega)$, when these spaces are contained in $c_{(\alpha!)}(\Omega)$. It is well-known that $C_{(\alpha!)}(\Omega)$ is the space of real analytic functions in Ω, and when Ω is connected, $c_{(\alpha!)}(\Omega)$ is the space of restrictions to Ω of entire functions in $\mathbf{C}^d$. The restriction mapping is an isomorphism of the space $A(\mathbf{C}^d)$ of entire functions in $\mathbf{C}^d$ onto $c_{(\alpha!)}(\Omega)$. It is also an homeomorphism, if $A(\mathbf{C}^d)$ has the usual topology, which is defined by all norms $f \to \|f\|_K=\sup_K|f|$, where K is a compact set in $\mathbf{C}^d$. In fact, it follows from Cauchy's inequalities and Taylor's formula that

$$\|f\|_{(\alpha!),K,h} \leqslant C\|f\|_{K'} \quad \text{and} \quad \|f\|_K \leqslant C\|f\|_{(\alpha!),x,h} \tag{1.13}$$

for all $f \in A(\mathbf{C}^d)$. In the first inequality C and the compact set K' in $\mathbf{C}^d$ depend on $h>0$ and the compact set K in Ω, and in the second inequality C and $h>0$ depend on $x \in \Omega$ and the compact set K in $\mathbf{C}^d$. (1.13) also shows that every $u \in c'_{(\alpha!)}(\Omega)$ is carried by every non-empty compact set in Ω, if Ω is connected. The same statement is true for $C'_L(\Omega)$, when $L \prec\prec (\alpha!)_\alpha$, and for $c'_L(\Omega)$, when $L \prec (\alpha!)_\alpha$, at least if L also satisfies $L_{\alpha+\beta} \leqslant a^{|\alpha|+|\beta|+1}\alpha! L_\beta$ with some $a>0$.

These properties of a functional u on $c_L(\Omega)$, when $L \prec (\alpha!)_\alpha$, or on $C_L(\Omega)$, when $L \prec\prec (\alpha!)_\alpha$, are also reflected in the estimate (1.12). For if $L_\alpha \leqslant C a^{|\alpha|}\alpha!$ then

$q_L(\zeta) \geq C^{-1} \exp(\sum_{j=1}^{d} |\zeta_j|/a)$, so that (1.12) can not tell anything precise about the carrier of u.

On the other hand, no estimate $\|f\|_{L,K,h} \leq C\|f\|_{L,K',k}$ with $K \not\subset K'$ can hold in $c_L(\Omega)$ when $(\alpha!)_\alpha \prec\prec L$ or in $C_L(\Omega)$ when $(\alpha!)_\alpha \prec L$. To see this we can choose $a \in K \setminus K'$ and define $f(x) = (|x-a|^2 + i\delta)^{-1}$, where $\delta > 0$ can be chosen arbitrarily.

For functionals on $c_{(\alpha!)}(\Omega)$ one should instead use carriers defined by means of the norms on $A(\mathbf{C}^d)$, i.e. carriers of analytic functionals. Such carriers have been studied e.g. by Martineau [8] and Kiselman [5] and [6].

2. Proofs of Theorems 1 and 2

The main step is the following lemma.

Lemma 1. *Suppose that L satisfies* (B), *that K is a convex compact set in R^d and that U is an entire function in $\mathbf{C}^d$, such that*

$$\forall \zeta \in \mathbf{C}^d: \quad |U(\zeta)| \leq C q_L(h\zeta)\, e^{H_K(\xi)} \tag{2.1}$$

for some $h > 0$ and C. Then there is an entire function W in $\mathbf{C}^{2d} = \mathbf{C}^d \times \mathbf{C}^d$ such that

$$\forall \zeta \in \mathbf{C}^d: \quad W(\zeta, \zeta) = U(\zeta) \tag{2.2}$$

and

$$\forall (\zeta, \zeta') \in \mathbf{C}^{2d}: \quad |W(\zeta, \zeta')| \leq \frac{C'}{|\zeta_1 \cdots \zeta_d|} q_L(ah\zeta)\, e^{H_K(\xi')}(1 + |\zeta'|)^{3d}, \tag{2.3}$$

where C' depends on C, h, L, K and d and a on L and d.

Let us first see how we can use Lemma 1.

The function W which we get in the lemma can be developped in a Taylor series

$$W(\zeta, \zeta') = \sum_\alpha \zeta^\alpha U_\alpha(\zeta'),$$

where all U_α are entire functions in $\mathbf{C}^d$. By Cauchy's inequalities and (2.3) we get

$$\begin{aligned} |U_\alpha(\zeta')| &\leq \sup_{|\zeta_j| \leq r_j} |W(\zeta, \zeta')| \frac{1}{r^\alpha} \\ &\leq C'(1 + |\zeta'|)^{3d} e^{H_K(\xi')} q_L(ahr) \frac{1}{r^\alpha r_1 \cdots r_d} \\ &\leq 2^d C'(1 + |\zeta'|)^{3d} e^{H_K(\xi')} \sup_\beta \frac{(2ahr)^\beta}{L_\beta} \frac{1}{r^\alpha r_1 \cdots r_d}, \end{aligned} \tag{2.4}$$

where $r = (r_1, \ldots, r_d)$ with $r_j > 0$. Now if L satisfies (A) we get from (2.4)

$$\forall \zeta \in \mathbf{C}^d: \quad |U_\alpha(\zeta)| \leqslant 2^d C' \frac{(2ah)^{|\alpha|+d}}{L_{\alpha+(1,\ldots,1)}} e^{H_K(\xi)}(1+|\zeta|)^{3d}. \tag{2.5}$$

We shall now use the Payley-Wiener theorem for distributions (see e.g. Hörmander [3], Th. 1.7.7). Thereby we get for every α a distribution u_α with compact support contained in K and with $\tilde{u}_\alpha = U_\alpha$. (Observe that $\hat{u}_\alpha(\zeta) = \tilde{u}_\alpha(-i\zeta)$.) If we choose $\varphi \in C^\infty(\mathbf{R}^d)$ with compact support contained in K_ε and with $\varphi = 1$ in a neighbourhood of K, then we get from (2.5)

$$\begin{aligned} |u_\alpha(f)| = |u_\alpha(\varphi f)| &= (2\pi)^{-d} \left| \int \hat{u}_\alpha(-\xi) \widehat{\varphi f}(\xi) \, d\xi \right| \\ &\leqslant \pi^{-d} C' \frac{(2ah)^{|\alpha|+d}}{L'_\alpha} \int (1+|\xi|)^{3d} |\widehat{\varphi f}(\xi)| \, d\xi \\ &\leqslant C_1 \frac{(2ah)^{|\alpha|}}{L'_\alpha} \sum_{|\beta| \leqslant N} \sup_{K_\varepsilon} |D^\beta f|, \end{aligned} \tag{2.6}$$

where C_1 depends only on C', h, a and d and where $N = 4d + 1$.

If L satisfies (B), so does L' with some $b \geqslant 1$. Then we can define

$$u(f) = \sum_\alpha u_\alpha(D^\alpha f) \tag{2.7}$$

with absolute convergence for all $f \in C^\infty(\mathbf{R}^d)$ such that

$$\| f \|_{L', K_\varepsilon, h'} < +\infty, \tag{2.8}$$

where $h' = 2b^N ah$. For by (2.6) we get

$$\begin{aligned} \sum_\alpha |u_\alpha(D^\alpha f)| &\leqslant C_1 \sum_{|\beta| \leqslant N} \sum_\alpha \frac{(2ah)^{|\alpha|}}{L'_\alpha} \sup_{K_\varepsilon} |D^{\alpha+\beta} f| \\ &\leqslant C_1 \sum_{|\beta| \leqslant N} (2ah)^{-|\beta|} \sum_\alpha \sup_{K_\varepsilon} |D^\alpha f| \frac{(2b^N ah)^{|\alpha|}}{L'_\alpha} = C_2 \| f \|_{L', K_\varepsilon, h'} \end{aligned} \tag{2.9}$$

where C_2 depends on C_1, h and d.

It is clear that

$$\tilde{u}(\zeta) = u(x \to e^{\langle x, \zeta \rangle}) = \sum_\alpha u_\alpha(x \to \zeta^\alpha e^{\langle x, \zeta \rangle}) = \sum_\alpha \zeta^\alpha \tilde{u}_\alpha(\zeta) = \sum_\alpha \zeta^\alpha U_\alpha(\zeta) = W(\zeta, \zeta) = U(\zeta) \tag{2.10}$$

by means of (2.2).

Proof of Theorem 1. Suppose that U satisfies the hypothesis in the second part of Theorem 1. Then we can use Lemma 1 and the procedure described after it with K replaced by K_ε (we remember that $H_{K_\varepsilon}(\xi) = H_K(\xi) + \varepsilon|\xi|$). If we change C_2 to C and h' to h in (2.9), we get for every $\varepsilon>0$ a functional u_ε, defined and satisfying

$$|u_\varepsilon(f)| \leqslant C \| f \|_{L', K_{2\varepsilon}, h} \tag{2.11}$$

for all $f \in c_{L'}(\mathbf{R}^d)$; C and h depend on ε. Furthermore $\tilde{u}_\varepsilon = U$ for all ε. However we do not know from the construction described above that the functionals u_ε are all identical. We need to know that a functional $u \in c'_{L'}(\mathbf{R}^d)$ is uniquely determined by its Laplace transform $\tilde{u}$. Now we can see that $u(f)$ is uniquely determined by $\tilde{u}$, when f is a polynomial, because the Taylor series of $e^{\langle x, \zeta \rangle}$ is convergent in $c_{L'}(\mathbf{R}^d)$. By the Hahn-Banach Theorem it is then necessary and sufficient to know that the polynomials are dense in $c_{L'}(\mathbf{R}^d)$. Therefore when we have proved Corollary 3a in the following section, we can conclude that all u_ε are identical. Denoting the common value by u we get a unique $u \in c'_{L'}(\mathbf{R}^d)$ such that $\tilde{u} = U$. From (2.11) also follows that u is carried by K.

Proof of Theorem 2. Suppose that U satisfies the hypothesis in the second part of Theorem 2. Then using Theorem 1 we get a unique functional $u \in c'_{L'}(\mathbf{R}^d)$ such that $\tilde{u} = U$. The proof also shows that

$$|u(f)| \leqslant C_{\varepsilon, h} \| f \|_{L', K_{2\varepsilon}, h} \tag{2.12}$$

for all $f \in c_{L'}(\mathbf{R}^d)$ and all $\varepsilon>0$ and $h>0$. Using (2.12) and approximation by means of Theorem 3 in section 3, we can then extend u uniquely to a continuous linear form on $C_{L'}(\mathbf{R}^d)$, which we denote by u too. More precisely $|u(f)| \leqslant C_{\varepsilon, h} \| f \|_{L', I, h}$ if $f \in C_{L'}(\mathbf{R}^d)$ and I is a rectangle (with sides parallel to the coordinate planes) such that $K_{2\varepsilon} \subset I$. Hence u is carried by the smallest such rectangle containing K.

If L also satisfies (C) and $(\alpha!)_\alpha \prec L$ then so does L'. Therefore $c_{(\alpha!)}(\mathbf{R}^d) \subset c_{L'}(\mathbf{R}^d)$ and we can use Theorem 4 in section 3 with L replaced by L'. Let K' be a compact neighbourhood of K such that Theorem 4 is applicable (with K' instead of K). Then $u_{K'}(f) = \lim_{s \to +\infty} u(T_{K', s} f)$ exists and satisfies

$$|u_{K'}(f)| \leqslant C_{\varepsilon, k} \| f \|_{L', K_{2\varepsilon}, k} \leqslant C_{\varepsilon, k} \| f \|_{L', K', h} \tag{2.13}$$

for all $f \in C_{L'}(\mathbf{R}^d)$, if $K_{3\varepsilon} \subset K'$ and $k \leqslant h$ is sufficiently small (depending on h, L' and ε). Therefore $u_{K'} \in C'_{L'}(\mathbf{R}^d)$ and so $u_{K'} = u$, because it is clear that $u_{K'} = u$ on $c_{L'}(\mathbf{R}^d)$ and the extension of u to a continuous linear form on $C_{L'}(\mathbf{R}^d)$ is unique. Hence (2.13) shows that u is carried by K. The proof of Theorem 2 is concluded.

Remark. If $L \prec L'$, i.e. if $L_\alpha \leqslant Ct^{|\alpha|} L'_\alpha$ for some constants C and $t>0$, then we can replace L'_α by L_α in (2.6), if we also replace C_1 by $C_1 C$ and h by th. After that the proofs of Theorems 1 and 2 work with L instead of L'.

Proof of Lemma 1. The idea is taken from Hörmander [4], 4.5 and the proof in based on the following lemma, which is Theorem 4.4.3 in Hörmander's book.

Lemma 2. *Let S and S' be complementary complex linear subspaces of $\mathbf{C}^n$ and let φ be a pluri-subharmonic function in $\mathbf{C}^n$ such that*

$$|\varphi(z+z')-\varphi(z)|\leqslant B \quad \text{if} \quad z\in\mathbf{C}^n,\ z'\in S' \quad \text{and} \quad |z'|\leqslant 1 \tag{2.14}$$

for some constant B. Then if V is an analytic function in S such that

$$\int_S |V|^2 e^{-\varphi}d\sigma < +\infty, \tag{2.15}$$

where σ is the Lebesgue measure in S, there is an entire function W in $\mathbf{C}^n$ such that $W=V$ in S and

$$\int |W|^2 e^{-\varphi}(1+|z|^2)^{-3k}dm \leqslant A\int_S |V|^2 e^{-\varphi}d\sigma, \tag{2.16}$$

where m is the Lebesgue measure in $\mathbf{C}^n$, k is the complex dimension of S' and A depends on B, S and S'.

For the definition and properties of of pluri-subharmonic functions we refer to Hörmander [4], 2.6 (and 1.6). The condition (2.14) on φ is weaker than the condition (4.4.9) in Hörmander's book, but an examination of the proof there shows that our condition is sufficient. However, our constant A is not $(6\pi e^B)^k$ when S and S' are not orthogonal.

In Lemma 1 we suppose that L satisfies (B) for some $b\geqslant 1$. Therefore

$$|\zeta^\beta|\,q_L(\zeta)=\sum_\alpha \frac{|\zeta^{\alpha+\beta}|}{L_\alpha}\leqslant \sum_\alpha \frac{|(b^{|\beta|}\zeta)^{\alpha+\beta}|}{L_{\alpha+\beta}}\leqslant q_L(b^{|\beta|}\zeta)$$

and from this follows

$$(1+|\zeta|)^n q_L(\zeta)\leqslant C_1 q_L(b^n\zeta), \tag{2.17}$$

where C_1 depends only on n and d.

Now suppose that U satisfies the hypothesis in Lemma 1. Let S be the subspace $\{(\zeta,\zeta): \zeta\in\mathbf{C}^d\}$ of $\mathbf{C}^{2d}$ and define an analytic function V in S by $V(\zeta,\zeta)=U(\zeta)$ for all $\zeta\in\mathbf{C}^d$.

Using (2.1) and (2.17) (with $n=d+1$) we obtain

$$\begin{aligned}\int_S |V(\zeta,\zeta)|^2 q_L(b^{d+1}h\zeta)^{-2}e^{-2H_K(\xi)}d\sigma &\leqslant C_1^2\int_S |V(\zeta,\zeta)|^2 q_L(h\zeta)^{-2}e^{-2H_K(\xi)}(1+h|\zeta|)^{-2d-2}d\sigma\\ &\leqslant C_1^2C^2\int_S (1+h|\zeta|)^{-2d-2}d\sigma = C_2 < +\infty,\end{aligned} \tag{2.18}$$

where σ is the Lebesgue measure in S.

We shall now use Lemma 2 with $n=2d$, $S'=\{(0,\zeta)\colon \zeta\in\mathbf{C}^d\}$ and φ defined in $\mathbf{C}^{2d}$ by

$$\varphi(\zeta,\zeta')=2\log q_L(b^{d+1}h\zeta)+2H_K(\xi').$$

By means of the rules in Hörmander [4], 1.6 and 2.6, we can see that φ is pluri-subharmonic in $\mathbf{C}^{2d}$, because

$$\log q_L(b^{d+1}h\zeta)=\sup_N \log \sum_{|\alpha|\leqslant N}\frac{|(b^{d+1}h\zeta)^\alpha|}{L_\alpha},$$

where each $\log|(b^{d+1}h\zeta)^\alpha|/L_\alpha$ is pluri-subharmonic, and H_K is convex. Furthermore φ satisfies (2.14) for some B since $q_L(b^{d+1}h\zeta)$ does not depond on ζ' and H_K is uniformly continuous.

(2.18) means that V satisfies (2.15) and so Lemma 2 gives an entire function W in $\mathbf{C}^{2d}$ such that $W(\zeta,\zeta)=V(\zeta,\zeta)=U(\zeta)$ when $\zeta\in\mathbf{C}^d$ and

$$\int|W(\zeta,\zeta')|^2q_L(b^{d+1}h\zeta)^{-2}e^{-2H_K(\xi')}(1+|\zeta|^2+|\zeta'|^2)^{-3d}dm$$

$$\leqslant A\int_S|V(\zeta,\zeta)|^2q_L(b^{d+1}h\zeta)^{-2}e^{-2H_K(\xi)}d\sigma\leqslant AC_2 \tag{2.19}$$

in view of (2.16) and (2.18). Here m is the Lebesgue measure in $\mathbf{C}^{2d}$.

Repeated use of the inequality

$$|u(0)|\leqslant(\pi r^2)^{-1}\int_{|z|\leqslant r}|u(z)|\,dx\,dy,$$

which is valid when u is analytic for $|z|\leqslant r$ in $\mathbf{C}$, then gives

$$|W(\zeta,\zeta')|^2\leqslant\pi^{-2d}|\zeta_1\dots\zeta_d|^{-2}\int_{\substack{|z_j-\zeta_j|\leqslant|\zeta_j|\\|z_j'-\zeta_j'|\leqslant 1}}|W(z,z')|^2dm$$

$$\leqslant AC_2\pi^{-2d}|\zeta_1\dots\zeta_d|^{-2}(1+2|\zeta|)^{6d}q_L(2b^{d+1}h\zeta)^2\sup_{|z_j'-\zeta_j'|\leqslant 1}e^{2H_K(x')}(1+|z'|)^{6d} \tag{2.20}$$

in view of (2.19) and the inequality $1+\lambda^2+\mu^2\leqslant(1+\lambda)^2(1+\mu)^2$ for $\lambda\geqslant 0$ and $\mu\geqslant 0$. Using (2.17) (with $n=3d$) and the uniform continuity of H_K and $\log(1+|z'|)$ we can from (2.20) conclude that

$$|W(\zeta,\zeta')|\leqslant C'|\zeta_1\dots\zeta_d|^{-1}q_L(2b^{4d+1}h\zeta)e^{H_K(\xi')}(1+|\zeta'|)^{3d},$$

where C' depends on A, C_2, L, d, h and K. So if we put $a=2b^{4d+1}$, we have proved (2.3) in Lemma 1.

3. Approximation theorems

If f is defined in the closed intervall $[a, b]$, the Bernstein polynomials $P_n f$ $(n=1,2,\ldots)$, are defined by

$$P_n f(x) = \sum_{k=0}^{n} \binom{n}{k} f\left(a + \frac{k(b-a)}{n}\right) \left(\frac{x-a}{b-a}\right)^k \left(\frac{b-x}{b-a}\right)^{n-k}. \tag{3.1}$$

It is well-known that $P_n f \to f$ uniformly in $[a, b]$ when $n \to \infty$, if f is continuous in $[a, b]$ (see e.g. Meinardus [9], 2.2.). A simple calculation shows that

$$D^j P_n f(x) = \frac{n!}{(b-a)^j (n-j)!} \sum_{k=0}^{n-j} \binom{n-j}{k} \Delta^j f\left(a + \frac{k(b-a)}{n}\right) \left(\frac{x-a}{b-a}\right)^k \left(\frac{b-x}{b-a}\right)^{n-j-k} \tag{3.2}$$

where $\Delta^j f$ is defined recursively by $\Delta^0 f = f$ and

$$\Delta^j f(x) = \Delta^{j-1} f\left(x + \frac{b-a}{n}\right) - \Delta^{j-1} f(x).$$

When $f \in C^\infty([a, b])$ we have

$$\Delta^j f(x) = \left(\frac{b-a}{n}\right)^j \int_0^1 \ldots \int_0^1 D^j f\left(x + \frac{b-a}{n} \sum_1^j t_i\right) dt_1, \ldots, dt_j. \tag{3.3}$$

When f is defined in a rectangle $I = [a_1, b_1] \times \ldots \times [a_d, b_d] \subset \mathbf{R}^d$ and $\nu = (\nu_1, \ldots, \nu_d)$ $(\nu_j = 1, 2, \ldots$ for $j = 1, \ldots, d)$, we define

$$P_\nu f(x) = \sum_{0 \leqslant \beta \leqslant \nu} \binom{\nu}{\beta} f\left(a + \frac{\beta(b-a)}{\nu}\right) \frac{(x-a)^\beta (b-x)^{\nu-\beta}}{(b-a)^\nu}, \tag{3.4}$$

where $a = (a_1, \ldots, a_d)$, $b = (b_1, \ldots, b_d)$,

$$\frac{\beta(b-a)}{\nu} = (\beta_1(b_1 - a_1)/\nu_1, \ldots, \beta_d(b_d - a_d)/\nu_d),$$

$$\binom{\nu}{\beta} = \binom{\nu_1}{\beta_1} \cdots \binom{\nu_d}{\beta_d}$$

and $0 \leqslant \beta \leqslant \nu$ means that $0 \leqslant \beta_j \leqslant \nu_j$ for $j = 1, \ldots, d$. $P_\nu f$ is constructed by successive applications of formula (3.1) with respect to the variables $x_1, \ldots, x_d$ and with $n = \nu_1 \ldots, \nu_d$, $a = a_1, \ldots, a_d$ and $b = b_1 \ldots, b_d$ resp.

If f is continuous in I, it follows that $P_\nu f \to f$ uniformly in I when $\nu \to \infty$ (in the sense that $\min(\nu_1, \ldots, \nu_d) \to \infty$). This is proved by the same methods as in the one-dimensional case and should be well-known.

When $f \in C^\infty(I)$, we get by combining (3.1)–(3.4)

$$D^\alpha P_\nu f(x)=\frac{\nu!}{\nu^\alpha(\nu-\alpha)!}\sum_{0\leqslant\beta\leqslant\nu-\alpha}\binom{\nu-\alpha}{\beta}f_\alpha\left(a+\frac{\beta(b-a)}{\nu}\right)\frac{(x-a)^\beta(b-x)^{\nu-\alpha-\beta}}{(b-a)^{\nu-\alpha}},\tag{3.5}$$

where $$f_\alpha(x)=\int_0^1\cdots\int_0^1 D^\alpha f\left(x+\frac{\tau(b-a)}{\nu}\right)dt_{11}\ldots dt_{1,\alpha_1}\ldots dt_{d1}\ldots dt_{d,\alpha_d}\tag{3.6}$$

with $$\frac{\tau(b-a)}{\nu}=\left(\frac{b_1-a_1}{\nu_1}\sum_1^{\alpha_1}t_{1j},\ldots,\frac{b_d-a_d}{\nu_d}\sum_1^{\alpha_d}t_{dj}\right).$$

It follows from (3.6) that

$$\left|f_\alpha\left(a+\frac{\beta(b-a)}{\nu}\right)\right|\leqslant\sup_I|D^\alpha f|\tag{3.7}$$

and $$\left|f_\alpha\left(a+\frac{\beta(b-a)}{\nu}\right)-D^\alpha f\left(a+\frac{\beta(b-a)}{\nu-\alpha}\right)\right|$$

$$\leqslant\sup_{\substack{|y_j-z_j|\leqslant\alpha_j(b_j-a_j)/\nu_j\\ y,z\in I}}|D^\alpha f(y)-D^\alpha f(z)|=c_{\alpha,\nu},\tag{3.8}$$

when $0\leqslant\beta\leqslant\nu-\alpha$. From (3.5), (3.6) and (3.7) follows

$$|D^\alpha P_\nu f(x)|\leqslant\frac{\nu!}{\nu^\alpha(\nu-\alpha)!}\sup_I|D^\alpha f|\sum_{0\leqslant\beta\leqslant\nu-\alpha}\binom{\nu-\alpha}{\beta}\frac{(x-a)^\beta(b-x)^{\nu-\alpha-\beta}}{(b-a)^{\nu-\alpha}}$$

$$=\frac{\nu!}{\nu^\alpha(\nu-\alpha)!}\sup_I|D^\alpha f|\leqslant\sup_I|D^\alpha f|\quad\text{if}\quad x\in I\tag{3.9}$$

and from (3.5), (3.6) and (3.8) follows

$$|D^\alpha P_\nu f(x)-D^\alpha f(x)|$$

$$\leqslant\frac{\nu!}{\nu^\alpha(\nu-\alpha)!}c_{\alpha,\nu}\sum_{0\leqslant\beta\leqslant\nu-\alpha}\binom{\nu-\alpha}{\beta}\frac{(x-a)^\beta(b-x)^{\nu-\alpha-\beta}}{(b-a)^{\nu-\alpha}}$$

$$+\frac{\nu!}{\nu^\alpha(\nu-\alpha)!}|P_{\nu-\alpha}D^\alpha f(x)-D^\alpha f(x)|+\left(1-\frac{\nu!}{\nu^\alpha(\nu-\alpha)!}\right)|D^\alpha f(x)|\quad\text{if}\quad x\in I$$

and from this follows

$$\sup_I|D^\alpha P_\nu f-D^\alpha f|\leqslant c_{\alpha,\nu}+\sup_I|P_{\nu-\alpha}D^\alpha f-D^\alpha f|+\left(1-\frac{\nu!}{\nu^\alpha(\nu-\alpha)!}\right)\sup_I|D^\alpha f|.\tag{3.10}$$

Here $c_{\alpha,\nu}\to0$ when $\nu\to\infty$ with α fixed because of (3.8) and the uniform continuity of $D^\alpha f$ In I. The continuity of $D^\alpha f$ in I also implies that $P_{\nu-\alpha}D^\alpha f\to D^\alpha f$

uniformly in I when $\nu \to \infty$ with α fixed, as we have already observed. Finally $\nu!/\nu^\alpha(\nu-\alpha)! \to 1$ when $\nu \to \infty$ with α fixed. Therefore we can conclude from (3.10) that for fixed α

$$\sup_I |D^\alpha P_\nu f - D^\alpha f| \to 0 \quad \text{when} \quad \nu \to \infty. \tag{3.11}$$

In section 2 we use the following approximation theorem.

Theorem 3. *If $L=(L_\alpha)_\alpha$ and $h>0$ are given and if $f \in C^\infty(I)$ satisfies*

$$\|f\|_{L,I,h} < +\infty \tag{3.12}$$

then

$$\|P_\nu f - f\|_{L,I,h} \to 0 \quad \textit{when} \quad \nu \to \infty \tag{3.13}$$

(i.e. when $\min(\nu_1, \ldots, \nu_d) \to \infty$*). Here I is a rectangle in $\mathbf{R}^d$ and $P_\nu f$ is defined by* (3.4).

Proof. From (3.9) follows that

$$\|P_\nu f - f\|_{L,I,h} \leqslant \sum_{|\alpha|\leqslant N} \sup_I |D^\alpha P_\nu f - D^\alpha f| \frac{h^{|\alpha|}}{L_\alpha} + 2 \sum_{|\alpha|>N} \sup_I |D^\alpha f| \frac{h^{|\alpha|}}{L_\alpha}. \tag{3.14}$$

The second sum is independent of ν and tends to 0 when $N \to \infty$ because of (3.12). The first sum tends to 0 when $\nu \to \infty$ and N is fixed because of (3.11). Hence (3.13) follows from (3.14).

Corollary 3a. *The polynomials form a dense subspace of $c_L(\mathbf{R}^d)$.*

Corollary 3b. *The polynomials form a dense subspace of $C_L(\mathbf{R}^d)$.*

In section 2 we also use an approximation theorem, which works for more general compact sets than rectangles in $\mathbf{R}^d$. Therefore let K be a compact set in $\mathbf{R}^d$ and let

$$\varphi(x) = (2\pi)^{-d/2} e^{-|x|^2/2}.$$

We have defined φ so that

$$\int \varphi \, dx = 1. \tag{3.15}$$

Then if f is a continuous function in K, we can define

$$T_s f(x) = T_{K,s} f(x) = \int_K f(y) \varphi(s(x-y)) s^d dy \tag{3.16}$$

for all $s>0$. $T_s f$ is the restriction to $\mathbf{R}^d$ of an entire function in $\mathbf{C}^d$ because φ is such a function and we integrate over a compact subset of $\mathbf{R}^d$.

From (3.15) and (3.16) follows immediately

$$\sup_{\mathbf{R}^d} |T_s f| \leqslant \sup_K |f|. \tag{3.17}$$

Now suppose that K_0 is a compact subset of the interior of K and that $\delta = d(K_0, \complement K)$. Then we get from (3.15) and (3.16)

$$\begin{aligned} |T_s f(x) - f(x)| &= \left| \int_{s(x-K)} f(x-u/s)\varphi(u)\,du - \int f(x)\varphi(u)\,du \right| \\ &\leqslant \sup_{|u|\leqslant r} |f(x-u/s) - f(x)| \int_{|u|\leqslant r} \varphi(u)\,du + 2\sup_K |f| \int_{|u|\geqslant r} \varphi(u)\,du \\ &\leqslant \sup_{|y-x|\leqslant r/s} |f(y) - f(x)| + 2\sup_K |f| \int_{|u|\geqslant r} \varphi(u)\,du \end{aligned} \tag{3.18}$$

if $x \in K_0$ and $r/s \leqslant \delta$. If we choose $r = \sqrt{s}$, it follows from (3.18) that

$$\sup_{K_0} |T_s f - f| \to 0 \quad \text{when} \quad s \to +\infty \tag{3.19}$$

for every compact subset K_0 of the interior of K, because f is uniformly continuous in K and $\int_{|u|\geqslant r} \varphi(u)\,du \to 0$ when $r \to +\infty$.

Now we suppose that $f \in C^\infty$ in a neighbourhood of K and that K is so regular that we can use Stokes' formula for K and its boundary ∂K (oriented with the normal pointing outwards). For our purposes it is sufficient that K is the union of a finite number of a rectangles. From (3.16) we then obtain by Stokes' formula

$$\begin{aligned} D_j T_s f(x) &= \int_K f(y) D_j \varphi(s(x-y)) s^{d+1} dy \\ &= \int_K D_j f(y) \varphi(s(x-y)) s^d dy - \int_K \frac{\partial}{\partial y_j} (f(y)\varphi(s(x-y)))\, s^d dy \\ &= T_s D_j f(x) + (-1)^j s^d \int_{\partial K} f(y)\varphi(s(x-y))\, d\hat{y}_j, \end{aligned} \tag{3.20}$$

where $d\hat{y}_j = dy_1 \wedge \ldots \wedge dy_{j-1} \wedge dy_{j+1} \wedge \ldots \wedge dy_d$. The interpretation of (3.20) when the dimension is 1 is obvious. We also get

$$\frac{\partial}{\partial x_k} \int_{\partial K} f(y)\varphi(s(x-y))\, d\hat{y}_j = s \int_{\partial K} f(y) D_k \varphi(s(x-y))\, d\hat{y}_j. \tag{3.21}$$

Using (3.20) and (3.21) we see by induction that

$$D^{\alpha}T_s f(x) = T_s D^{\alpha} f(x)$$

$$+\sum_{j=1}^{d}\sum_{k=0}^{\alpha_j-1}(-1)^j s^{k+|\alpha_j''|+d}\int_{\partial K} D^{\alpha_j'}D_j^{\alpha_j-k-1}f(y)\,D^{\alpha_j''}D_j^k\varphi(s(x-y))\,d\hat{y}_j, \tag{3.22}$$

where $\alpha_j' = (\alpha_1 \ldots, \alpha_{j-1}, 0, \ldots, 0)$ and $\alpha_j'' = (0, \ldots, 0, \alpha_{j+1}, \ldots, \alpha_d)$. If $\alpha_j = 0$ then the corresponding sum over over k in (3.22) shall be 0.

Now let K_0 be a compact subset of the interior of K and put $\delta = d(K_0, \complement K)$. Then it follows from (3.22) that

$$|D^{\alpha}T_s f(x) - D^{\alpha}f(x)| \leqslant |T_s D^{\alpha}f(x) - D^{\alpha}f(x)|$$

$$+\sum_{j=1}^{d}\sum_{k=0}^{\alpha_j-1} s^{k+|\alpha_j''|+d}\sup_K |D^{\alpha_j'}D_j^{\alpha_j-k-1}f| \sup_{|u|\geqslant s\delta}|D^{\alpha_j''}D_j^k\varphi(u)|\,A(\partial K)$$

$$\text{if} \quad x \in K_0, \tag{3.23}$$

where $A(\partial K)$ is the $(d-1)$-dimensional measure of ∂K.

From (3.19) and (3.23) follows that

$$\sup_{K_0} |D^{\alpha}T_s f - D^{\alpha} f| \to 0 \quad \text{when} \quad s \to +\infty \tag{3.24}$$

for every fixed α, because

$$r^n \sup_{|u|\geqslant r} |D^{\beta}\varphi(u)| \to 0 \quad \text{when} \quad r \to \infty$$

if $n \geqslant 0$ and β is a multi-index.

Theorem 4. *Suppose that $f \in C^{\infty}$ in a neighbourhood of a compact subset K of $\mathbf{R}^d$, which is so regular that Stokes' formula is applicable, and suppose that*

$$\|f\|_{L,K,h} < +\infty, \tag{3.25}$$

where $h > 0$ and $L = (L_{\alpha})_{\alpha}$ satisfies (C) *and $(\alpha!)_{\alpha} \prec L$, which implies that there are constants C and $a > 0$ such that $|\alpha|! \leqslant Ca^{|\alpha|}L_{\alpha}$ for all α. Then*

$$\|T_s f - f\|_{L,K_0,h/c} \to 0 \quad \text{when} \quad s \to +\infty \tag{3.26}$$

if $T_s f$ is defined by (3.16), *K_0 is a compact subset of the interior of K and $h < (5a)^{-1}d(K_0, \complement K)$. c is the constant in* (C); *here we suppose that $c \geqslant 1$.*

In the proof we need the following estimate of the derivatives of φ.

Lemma 3. *If $\varphi(u) = (2\pi)^{-d/2}e^{-|u|^2/2}$ $(u \in \mathbf{R}^d)$, then for every m there is a constant C such that*

$$\forall \alpha: \quad r^{|\alpha|+m} \sup_{|u|\geqslant r} |D^{\alpha}\varphi(u)| \leqslant C5^{|\alpha|}|\alpha|! \tag{3.27}$$

Proof. φ is also defined by

$$\varphi(u)=(2\pi)^{-d}\int e^{i\langle u,\xi\rangle}e^{-|\xi|^2/2}d\xi.$$

It follows that

$$r^{|\alpha|+m}|D^\alpha\varphi(u)|=r^{|\alpha|+m}(2\pi)^{-d}\left|\int(i\xi)^\alpha e^{i\langle u,\xi\rangle}e^{-|\xi|^2/2}d\xi\right|$$

$$=(2\pi)^{-d}r^{|\alpha|+m}\left|\int(i\xi-\eta)^\alpha e^{i\langle u,\xi\rangle-\langle u,\eta\rangle}e^{-|\xi|^2/2+|\eta|^2/2-i\langle\xi,\eta\rangle}d\xi\right.$$

$$\leqslant(2\pi)^{-d}r^{|\alpha|+m}e^{-\langle u,\eta\rangle+|\eta|^2/2}\int(|\xi|+|\eta|)^{|\alpha|}e^{-|\xi|^2/2}d\xi$$

$$\leqslant Cr^{|\alpha|+m}e^{-r^2/2}\int_0^{+\infty}(t+r)^{|\alpha|}e^{-t^2/2}t^{d-1}dt$$

$$\leqslant 2^{|\alpha|}C\left(r^{2|\alpha|+m+d-1}e^{-r^2/2}\int_0^r e^{-t^2/2}dt+e^{-r^2/2}\int_r^{+\infty}t^{2|\alpha|+m+d-1}e^{-t^2/2}dt\right)$$

$$\text{if}\quad |u|\geqslant r. \tag{3.28}$$

Here we have moved the integration to the hyperplane $\mathbf{R}^d+i\eta$ in $\mathbf{C}^d$, where η is the vector in $\mathbf{R}^d$ which has the same direction as u and length r. It is obvious that this is possible by Cauchy's integral theorem.

Using the inequality

$$r^{2k+n}e^{-r^2/2}\leqslant\sqrt{(2k+n)!}\leqslant 2^k k!\sqrt{(2k+1)\dots(2k+n)}$$

and the equality

$$\int_0^{+\infty}t^{2k+n}e^{-t^2/2}dt=2^{k+(n-1)/2}\int_0^{+\infty}t^{k+(n-1)/2}e^{-t}dt=2^{k+(d-1)/2}\Gamma(k+(d+1)/2)$$

we obtain (3.27) from (3.28) with a new constant C.

Proof of Theorem 4. If $\delta=d(K_0,\complement K)$, we get from (3.23), (3.17) and Lemma 3

$$\sup_{K_0}|D^\alpha T_s f-D^\alpha f|\leqslant 2\sup_K|D^\alpha f|+C_1\sum_{j=1}^{d}\sum_{k=0}^{\alpha_j-1}\sup_K|D^{\alpha_j'}D_j^{\alpha_j-k-1}f|\left(\frac{5}{\delta}\right)^{k+|\alpha_j''|}(k+|\alpha_j''|)! \tag{3.29}$$

with a new constant C_1 not depending on α or s.

Now suppose that f satisfies (3.25). Then using the condition (C) with $c\geqslant 1$ and $|\alpha|!\leqslant Ca^{|\alpha|}L_\alpha$ we see that (3.29) implies that

$$\sum_{|\alpha|>N} \sup_{K_0} |D^\alpha T_s f - D^\alpha f| \frac{(h/c)^{|\alpha|}}{L_\alpha} \leqslant 2 \sum_{|\alpha|>N} \sup_{K} |D^\alpha f| \frac{(h/c)^{|\alpha|}}{L_\alpha}$$

$$+ C_1 C^2 \sum_{j=1}^{d} \sum_{|\alpha|>N} \sum_{k=0}^{\alpha_j - 1} \sup_K |D^{\alpha_j'} D_j^{\alpha_j - k - 1} f| \frac{h^{|\alpha|}(5/\delta)^{k+|\alpha_j''|} a^{k+1+|\alpha_j''|}}{L_{(\alpha_1, \ldots, \alpha_{j-1}, \alpha_j - k - 1, 0, \ldots, 0)}}$$

$$\leqslant 2 \sum_{|\alpha|>N} \sup_K |D^\alpha f| \frac{h^{|\alpha|}}{L_\alpha} + C_2 \sum_\alpha \left(\frac{5ah}{\delta}\right)^{|\alpha|} \sum_{|\beta|>N-|\alpha|-1} \sup_K |D^\beta f| \frac{h^{|\beta|}}{L_\beta}, \tag{3.30}$$

where the new constant C_2 is independent of N and s. In the last sum we have changed $(\alpha_1, \ldots, \alpha_{j-1}, \alpha_j - k - 1, 0, \ldots, 0)$ to β. Suppose that $h < \delta/5a$. Then (3.30) implies that

$$\| T_s f - f \|_{L, K_0, h/c} \leqslant \sum_{|\alpha| \leqslant N} \sup_{K_0} |D^\alpha T_s f - D^\alpha f| \frac{(h/c)^{|\alpha|}}{L_\alpha}$$

$$+ \left(2 + C_2 \sum_{|\alpha|<m} \left(\frac{5ah}{\delta}\right)^{|\alpha|}\right) \sum_{|\alpha| \geqslant N-m} \sup_K |D^\alpha f| \frac{h^{|\alpha|}}{L_\alpha} + C_2 \sum_{|\alpha| \geqslant m} \left(\frac{5ah}{\delta}\right)^{|\alpha|} \| f \|_{L, K, h}. \tag{3.31}$$

Here the middle and the last term tend to 0 when $m \to +\infty$ and $N - m \to +\infty$. They are both independent of s. The first term tends to 0 when $s \to +\infty$ for fixed N. Therefore (3.26) follows from (3.31) and Theorem 4 is proved.

Corollary 4a. *If L satisfies* (C) *and $(\alpha!)_\alpha \prec\prec L$, then the entire functions in $\mathbf{R}^d$ are dense in $c_L(\Omega)$, if Ω is an open set in $\mathbf{R}^d$.*

Corollary 4b. *If L satisfies* (C) *and $(\alpha!)_\alpha \prec L$, then the entire functions in $\mathbf{R}^d$ are dense in $C_L(\Omega)$, if Ω is an open set in $\mathbf{R}^d$.*

Corollary 4a shows that the image of $c_L(\mathbf{R}^d)$ (under the restriction mapping $c_L(\mathbf{R}^d) \to c_L(\Omega)$) is dense in $c_L(\Omega)$, if L satisfies (C) and $(\alpha!)_\alpha \prec\prec L$. By Corollary 4b the same statement is true for C_L, if L satisfies (C) and $(\alpha!)_\alpha \prec L$. On the other hand it is not true for $c_L(\Omega)$ if $L \prec (\alpha!)_\alpha$ or for $C_L(\Omega)$ if $L \prec\prec (\alpha!)_\alpha$ and Ω is not connected.

If $u \in c_L'(\mathbf{R}^d)$ is carried by a compact subset of $\Omega \subset \mathbf{R}^d$, then $u(f) = u(g)$ when f and $g \in c_L(\mathbf{R}^d)$ and $f = g$ in Ω. Hence we can identify the space of all such u with the space of continuous linear forms on the image of $c_L(\mathbf{R}^d)$ in $c_L(\Omega)$, and all these linear forms can be uniquely extended to $c_L(\Omega)$ if and only if the image of $c_L(\mathbf{R}^d)$ is dense in $c_L(\Omega)$. Therefore we can identify $c_L'(\Omega)$ with the space of all $u \in c_L'(\mathbf{R}^d)$ which are carried by compact subsets of Ω, at least if L satisfies (C) and $(\alpha!)_\alpha \prec\prec L$. This statement is not true when $L \prec (\alpha!)_\alpha$ and Ω is not connected.

Similarly $C_L'(\Omega)$ can be identified with the space of all $u \in C_L'(\mathbf{R}^d)$ which are carried by compact subsets of Ω, at least if L satisfies (C) and $(\alpha!)_\alpha \prec L$. It is not true when $L \prec\prec (\alpha!)_\alpha$ and Ω is not connected.

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