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On the Hellinger integrals and interpolation of q-variate stationary stochastic processes

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Introduction

Let $(X_t)_{-\infty}^{\infty}$ be a q-variate continuous parameter, mean continuous, weakly stationary stochastic process (SP) with the spectral distribution measure F defined on $\mathcal{B}$ the Borel family of subsets of the real line; cf. [1]. It is known [10] that for matrix-valued measures M and N the Hellinger integral $(M, N) = \int_{-\infty}^{\infty} (dMdN^*/dF)$ ($^* =$ conjugate) may be defined in such a way that $H_{2,F}$ the space of all matrix-valued measures M for which $(M, M)_F = \int_{-\infty}^{\infty} (dMdM^*/dF)$ exist becomes a Hilbert space under the inner product $\tau(M, N)_F$ ($\tau =$ trace). The significance of these integrals when M and N are complex-valued measures and F is a non-negative real-valued measure has been pointed out by H. Cramér [2, p. 487] and U. Grenander [3, p. 207; 4, p. 195] in relation to unvariate SP's. The importance of Hellinger integrals with regard to the theory of interpolation of a q-variate weakly stationary SP with discrete time has been discussed by the author in [11]. In this paper we propose to use the Hellinger integrals and obtain similar results concerning the interpolability of a q-variate continuous parameter, mean continous, weakly stationary SP. The question of interpolability of a univariate SP with continuous time has been looked at by K. Karhunen [6]. Our results extend his work in a natural way.

Let K be any bounded measurable subset of the real line. K' will denote the complement of K in the set of the real numbers. $\mathfrak{M}_K$ and $\mathfrak{M}_{K'}$ will denote the (closed) subspaces spanned by $X_t, t \in K$ and $X_t, t \in K'$ respectively, i.e., $\mathfrak{M}_K = \mathfrak{S}\{X_t, t \in K\}$ and $\mathfrak{M}_{K'} = \mathfrak{S}\{X_t, t \in K'\}$. $\mathfrak{M}_\infty$ will denote $\mathfrak{S}\{X_t, t \text{ real}\}$ and finally $\mathfrak{N}_K$ will denote $\mathfrak{M}_\infty \cap \mathfrak{M}_{K'}^\perp$, where $\mathfrak{M}_{K'}^\perp$ denotes the orthogonal complement of $\mathfrak{M}_{K'}$ in a fixed Hilbert space $\mathcal{H}^q$ containing the SP $(X_t)_{-\infty}^{\infty}$.

Definition 1. We say that (a) K is interpolable with respect to (w.r.t.) $(X_t)_{-\infty}^{\infty}$ if $\mathfrak{N}_K = \{0\}$.

(b) $(X_t)_{-\infty}^{\infty}$ is interpolable if each bounded measurable subset K of the real line is interpolable w.r.t. $(X_t)_{-\infty}^{\infty}$.

For each $X \in \mathfrak{M}_\infty$, (X, X_t) is a continuous function on $(-\infty, \infty)$. Moreover, $(X, X_t) = 0$ iff $t \in K'$. Thus the following definition makes sense.

Definition 2. For each $X \in \mathcal{N}_K$, we let

$$P_X(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda t}(X, X_t)\,dt$$
$$= \int_K e^{-i\lambda t}(X, X_t)\,dt.$$

The properties of P_X are given in the next lemma.

Lemma 1. (a) *The entries of the matrix-valued function P_X are integrable* w.r.t. *Lebesgue measure. Hence for each $B \in \mathcal{B}$, the integral $\int_B P_X(\lambda)\,d\lambda$ exists.*

(b) *If for each $B \in \mathcal{B}$ we define*

$$M_{P_X}(B) = \int_B P_X(\lambda)\,d\lambda,$$

then $M_{P_X} \in H_{2,F}$.

Proof. (a) Let $X \in \mathcal{N}_K$ and Ψ be in $L_{2,F}$ such that $V\Psi = X$, where V is the isomorphism on $L_{2,F}$ onto $\mathcal{M}_\infty$ [9, pp. 279–98]. Then

$$(X, X_t) = (V\Psi, Ve^{-i\lambda t}) = \frac{1}{2\pi}(\Psi, e^{-it\lambda})_F$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi(\lambda)\,dF(\lambda)\,e^{it\lambda} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\lambda}\Psi(\lambda)\,dF(\lambda). \tag{1}$$

Also by definition of P_X

$$P_X(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda t}(X, X_t)\,dt. \tag{2}$$

By (1) and (2) it follows that for each $B \in \mathcal{B}$

$$\int_B P_X(\lambda)\,d\lambda = \int_B \Psi(\lambda)\,dF(\lambda). \tag{3}$$

Thus (a) follows from (3).

(b) Since by (a) for each $B \in \mathcal{B}$, $\int_B P_X(\lambda)\,d\lambda$ exists, therefore M_{P_X} is a matrix-valued measure on $\mathcal{B}$. By the definition of M_{P_X}, (3) and [10, Theorem 2] (b) follows. (Q.E.D.)

Thus the following definition makes sense.

Definition 3. We define the operator T_K on $\mathcal{N}_K$ into $H_{2,F}$ as follows: for each $X \in \mathcal{N}_K$

$$T_K X = \frac{1}{\sqrt{2\pi}} M_{P_X}.$$

The important properties of T_k are given in the following theorem.

Theorem 1. (a) *Let* $X \in \mathcal{N}_K$ *and* $\Psi \in L_{2,F}$ *such that* $V\Psi = X$, *where* V *is the isomorphism on* $L_{2,F}$ *onto* $\mathcal{M}_\infty$ *[9, pp. 297–98]. Then for each* $B \in \mathcal{B}$, $M_{P_X}(B) = \int_B \Psi dF$.

(b) T_K *is an isometry on* $\mathcal{N}_K$ *into* $H_{2,F}$. *In fact for all* X *and* Y *in* $\mathcal{N}_K$

$$(X, Y) = (T_K X, T_K Y)_F.$$

(c) *The range of* T_K *is a closed subspace of the Hilbert space* $H_{2,F}$.

Proof. (a) follows from the proof of Lemma 1.

(b) Let X and Y be in $\mathcal{N}_K$, and let Φ and Ψ be in $L_{2,F}$ such that $V\Phi = X$ and $V\Psi = Y$. Then by (a) and [10, Theorem 1]

$$2\pi(T_X, T_Y)_F = (\Phi, \Psi)_F. \tag{1}$$

Also by [9, p. 297]

$$2\pi(X, Y) = (\Phi, \Psi)_F. \tag{2}$$

From (1) and (2), (b) follows.

(c) Since $\mathcal{N}_K$ is a (closed) subspace and since by (b) T_K is an isometry on $\mathcal{N}_K$ into $H_{2,F}$, therefore range of T_K is a closed subspace of $H_{2,F}$. (Q.E.D.)

It is convenient at this point to introduce the following definition.

Definition 4. (a) A $q \times q$ matrix-valued function P on $(-\infty, \infty)$ is called time-limited if

(i) The entries of P are integrable w.r.t. Lebesgue measure.

(ii) $$P(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda t} G(t)\, dt,$$

where G is a $q \times q$ matrix-valued function whose entries have bounded supports and are square-integrable w.r.t. Lebesgue measure.

(b) $\mathcal{L}$ will denote the class of all time-limited $q \times q$ matrix-valued functions on $(-\infty, \infty)$.

(c) for each $P \in \mathcal{L}$ the matrix-valued measure M_P is defined on $\mathcal{B}$ as follows; for each $B \in \mathcal{B}$

$$M_P(B) = \int_B P(\lambda)\, d\lambda.$$

We note that if $X \in \mathcal{N}_K$ and $P_X(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda t}(X, X_t)\, dt$, then by Lemma 1 (a), $P_X \in \mathcal{L}$.

Lemma 2. *Let* $X \in \mathcal{N}_K \cap \mathcal{N}_L$. *Then*

$$T_K X = T_K Y.$$

Proof. It is clear that $\mathcal{N}_K \cap \mathcal{N}_L = \mathcal{N}_{K\cap L}$. Hence $T_K X = T_{K\cap L} X = T_L X$. (Q.E.E.)
Making use of this lemma, T_K's may be put together to introduce a well-defined operator with a bigger domain. This is done in the following theorem.

Theorem 2. *Let $\mathcal{N} = \bigcup \mathcal{N}_K$, where K is a bounded measurable subset of $(-\infty, \infty)$. Define the operator T on $\mathcal{N}$ by*

$$TX = T_K X, \text{ if } X \in \mathcal{N}_K.$$

Then

(a) *$\mathcal{N}$ is a linear manifold in $\mathcal{M}_\infty$, i.e., $X, Y \in \mathcal{N}$ and A, B matrices $\Rightarrow AX + BY \in \mathcal{N}$.*

(b) *T is a single-valued linear operator on $\mathcal{N}$, i.e., if $X, Y \in \mathcal{N}$ and A, B are matrices, then*

$$A(AX + BY) = ATX + BTY.$$

(c) *T is an isometry on $\mathcal{N}$ into $H_{2,F}$. In fact for $X, Y \in \mathcal{N}$*

$$(X, Y) = (TX, TY)_F.$$

(d) *The range of T consists of all matrix-valued measures M_P for which the Hellinger integrals $\int_{-\infty}^{\infty} (dM_P dM_P^* / df)$ exist where $P \in \mathcal{L}$, $\mathcal{L}$ is as in definition 4*(b) *and M_P is related to P as in definition 4* (c).

Proof. (a) follows from the fact that $\mathcal{N}_K \cup \mathcal{N}_L \subseteq \mathcal{N}_{K\cup L}$.
(b) and (c) are consequences of Lemma 1 and Theorem 1.
(d) Let $X \in \mathcal{N}$. Then $X \in \mathcal{N}_K$ for some K. It then follows from the definition of T that

$$TX = T_K X = M_{P_X}, \tag{1}$$

where for each $B \in \mathcal{B}$, $M_{P_X}(B) = \int_B P_X(\lambda)\, d\lambda$ and $P_X(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda t}(X, X_t)\, dt$. Since by Theorem 1 (a) the entries of P_X are integrable w.r.t. Lebesgue measure, hence $P_X \in \mathcal{L}$. From (1) and (c) it follows that $(X, X) = (M_{P_X}, M_{P_X})_F$ and hence (M_{P_X}, M_{P_X}) is Hellinger integrable w.r.t. F.
Conversely let M_P be a matrix-valued measure such that for each $B \in \mathcal{B}$

$$M_P(B) = \int_B P(\lambda)\, d\lambda,$$

where $P \in \mathcal{L}$ and $\int_{-\infty}^{\infty} (dM_P dM_P^* / df)$ exists. Then by [10, Theorem 1 (c)], $\Phi = (dM_P/d\mu)(dF/d\mu)^- \in L_{2,F}$, where μ is any σ-finite non-negative real-valued measure w.r.t. which M_P and F are a.c. $\{(dF/d\mu)^-$ denotes the generalized inverse of $dF/d\mu$; cf. [8]$\}$. If $X \in \mathcal{M}_\infty$ such that $V\Phi = X$, where V is as in Theorem 1, then by [9, p. 297] and [10, Theorem 2]

$$\begin{aligned}(X, X_t) &= \frac{1}{2\pi}(\Phi, e^{-i\lambda t})_F \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(dM_P/d\mu)\,(dF/d\mu)^{-}(dF/d\mu)\,e^{i\lambda t}d\mu \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\lambda t}(dM_P/d\mu)\,d\mu \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\lambda t}dM_P = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\lambda t}P(\lambda)\,d\lambda. \qquad (1)\end{aligned}$$

Since $P \in \mathcal{L}$ then

$$P(\lambda) = \int_{-\infty}^{\infty} G(t)\,e^{-it\lambda}dt,$$

where the entries of G have bounded supports and are square-integrable w.r.t. Lebesgue measure. It then follows that

$$G(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda t}P(\lambda)\,d\lambda. \qquad (2)$$

By (1) and (2) we conclude that

$$(X, X_t) = G(t) \text{ a.e.}$$

Therefore the entries of (X, X_t) have bounded supports and hence their supports are contained in $[-\varepsilon, \varepsilon]$ for some $\varepsilon > 0$. Since X is in $\mathcal{M}_\infty$ it follows that $X \in \mathcal{N}_{[-\varepsilon,\varepsilon]}$ and therefore $X \in \mathcal{N} = \cup_K \mathcal{N}_K$. It is clear that $M_{P_X} = M_P$ and the result follows. (Q.E.D.)

We are now ready to give a characterization for the interpolability of a SP.

Theorem 3. *$(X_t)_{-\infty}^{\infty}$ is interpolable iff for any time-limited matrix-valued function P for which M_P is not a null-point in $H_{2,F}$, (M_P, M_P) is not Hellinger w.r.t. F.*

Proof. ($\Leftarrow$) If K is any bounded measurable subset of $(-\infty, \infty)$, it is a consequence of Lemma 1 (b) and Theorem 2 (d) that $\mathcal{N}_K = \{0\}$. Hence by definition 1 (a) K is interpolable w.r.t. $(X_t)_{-\infty}^{\infty}$. Since K is arbitrary it follows that $\mathcal{N} = \cup_K = \{0\}$ so that by definition 1 (b), $(X_t)_{-\infty}^{\infty}$ is interpolable.

($\Rightarrow$) It follows that $\mathcal{N} = \{0\}$. Hence by Theorem 2 (d) range of $T = \{0\}$. The result follows from Theorem 2 (c). (Q.E.D.)

Remark 1. Since $\cup_\varepsilon \mathcal{N}_{[-\varepsilon,\varepsilon]} = \cup_K \mathcal{N}_K = \mathcal{N}$ and since by [7, Theorem 10] P is a time-limited matrix-valued function in the form $P(\lambda) = \int_{-\varepsilon}^{\varepsilon} e^{-it\lambda}G(t)\,dt$ if the entries of $P(\lambda)$ are integrable as well as square-integrable w.r.t. Lebesgue measure and $P(z) = o(^{\varepsilon|z|})$, where $P(z)$ is the unique analytic extension of $P(\lambda)$, we immediately obtain the following theorem, which generalizes the corresponding result for the unvariate case due to Karhunen [6].

Theorem 4. *$(X_t)_{-\infty}^{\infty}$ is interpolable iff for any analytic matrix-valued function $P(z)$ of the form $P(z)=o(e^{\varepsilon|z|})$ such that the entries of $P(\lambda)$ are integrable as well as square-integrable w.r.t. Lebesgue measure if M_P is not a null-point in $H_{2,F}$, then (M_P, M_P) is not Hellinger integrable w.r.t. F.*

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