ARKIV FÖR MATEMATIK Band 8 nr 10

1.68 14 1 Communicated 13 November 1968 by ÅKE PLEIJEL and LENNART CARLESON

On a conjecture of V. Bernstein

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1. Introduction

In this paper, we shall be concerned with the Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s} = f(s), \tag{1.1}$$

where the sequence $\{\lambda_n\}$ increases and tends to infinity with n. Let $N(r)$ denote the number of λ_n which are less than r; then the number

$$D = \lim_{\xi\to 1}\{\limsup_{r\to\infty}\,[N(r)-N(\xi r)]/[r-\xi r]\} \tag{1.2}$$

is called the *maximum density* of the sequence $\{\lambda_n\}$. Whenever a Dirichlet series is mentioned in this paper, it will always be assumed to have a sequence of exponents with finite maximum density. We shall be particularly interested in series of the form (1.1) which satisfy Ostrowski's gap condition; that is to say, series which are such that there exists an increasing sequence of integers $\{n_k\}$ and a positive constant $\mathcal{D}$, such that

$$\lambda_{n_k+1}-\lambda_{n_k} \geqslant \mathcal{D}\lambda_{n_k} \tag{1.3}$$

A Dirichlet series may converge at no finite point in the plane, it may converge at every finite point in the plane or else there may exist a finite number, σ_c such that the series converges in the half-plane $\operatorname{Re}(s)>\sigma_c$ but diverges at every point which has real part less than σ_c; no other case can occur. In the third-mentioned case, we may take $\sigma_c=0$ without loss of generality. Let us then write

$$S_n(s)=\sum_{m=1}^{n} a_m e^{-\lambda_m s}; \tag{1.4}$$

we know that the sequence $\{S_n(s)\}$ cannot converge at any point outside the closure of the region of convergence, but it is possible that a subsequence, $\{S_{n_k}(s)\}$ may converge in a region D, larger than the region of convergence of (1.1); when this occurs, we say that (1.1) overconverges in D.

For power series, the phenomena of gaps and overconvergence are connected by the following well-known theorems of Ostrowski (see, for example Dienes [3]).

Theorem 1 (Ostrowski). *If a power series $\sum a_n z^n$ satisfies Ostrowski's gap condition, the sequence $\{\sum_1^{n_k} a_n z^n\}$ converges in some neighbourhood of each regular point on the circle of convergence of the series.*

Theorem 2 (Ostrowski). *If $\{\sum_1^{n_k} a_n z^n\}$ converges in a neighbourhood of a point on the circle of convergence of the series $\Sigma\, a_n z^n$, then this series may be written as the sum of two power series, one satisfying Ostrowski's gap condition and the other having radius of convergence larger than $\sum a_n z^n$.*

We are interested in investigating the connection between gaps and overconvergence in the more general context of Dirichlet series. It is, of course, well known that Theorem 1 holds for Dirichlet series and indeed for more general series (see, for example, Leont'ev [5]). On the other hand, Theorem 2 certainly does not hold for all Dirichlet series, since examples of such series are known which show overconvergence of a character quite different from the gap-type overconvergence of power series. Following Bernstein [1], we shall say that $\{S_{n_k}(s)\}$ is a *closely overconvergent* sequence of partial sums in the region R if the following conditions are satisfied:

(i) $\lambda_{n_{k+1}-1}/\lambda_{n_k} \to 1$ as $k \to \infty$;

(ii) $(n_{k+1} - n_k)/n_k \to 0$ as $k \to \infty$;

(iii) the sequence of partial sums $\{S_{n_k}(s)\}$ converges whenever s belongs to R.

We shall say that the series (1.1) is closely overconvergent in R if it possesses a sequence of partial sums closely overconvergent in R.

V. Bernstein [1, Ch. 2] defined a number δ, called the *index of condensation* of the sequence $\{\lambda_n\}$, which measures the extent to which the members of $\{\lambda_n\}$ cluster together and he showed that close overconvergence cannot occur in any half-plane larger than $\mathrm{Re}\,(s) > \sigma_c - \delta$, but that with the correct choice of the sequence of coefficients $\{a_n\}$, this half-plane is always a region of close overconvergence. In [1], Bernstein conjectured that Theorem 2 might hold for some significant class of Dirichlet series. A natural candidate for such a class is the class for which $\delta = 0$, since, if this condition is satisfied, close overconvergence cannot occur. This problem remained unsolved until the appearance of a paper by M. E. Noble [6] in which it was proved that Theorem 2 holds for Dirichlet series satisfying the condition

$$\lambda_{n+1} - \lambda_n \geqslant q > 0. \tag{1.5}$$

Noble's can be extended without great difficulty to the class of Dirichlet series for which $\delta = 0$ and this result is best-possible, in the sense that if we are given a sequence $\{\lambda_n\}$ with positive index of condensation, we can choose a sequence of coefficients $\{a_n\}$ such that Theorem 2 does not hold for the series $\sum a_n e^{-\lambda_n s}$.

There remains the possibility of obtaining Theorem 2 in a modified form, valid for Dirichlet series whose sequence of exponents has positive index of condensation. A natural modification is suggested when we come to consider the half-plane of holomorphy of the series (1.1). Let σ_H denote the infimum of the numbers σ such that the function defined by (1.1) is regular in $\mathrm{Re}\,(s) > \sigma$; then σ_H is called the *abscissa of holomorphy* and the half-plane $\mathrm{Re}\,(s) > \sigma_H$ is called the *half-plane of holomorphy*. The half-plane of holomorphy is the natural region of close overconvergence, since it always is a region of close overconvergence (Bernstein [1, Ch. 6, Th. 11]) and a theorem of Bourion [2] shows that this cannot be true of any larger region. Furthermore, Theorem 1 can be obtained relative to the abscissa of holomorphy instead of the abscissa of convergence (Bernstein [1, Ch. 6, Th. 18]). These facts lead one to suspect that Theorem 2 also, might hold relative to the abscissa of holomorphy.

This again was conjectured by Bernstein in [1]. The main task of this paper is to show that this conjecture holds in the following form.

Theorem 3. *Suppose that the series* (1.1) *has finite abscissa of holomorphy,* σ_H, *and that the maximum density of the sequence* $\{\lambda_n\}$ *is also finite. Suppose further, that there exists a sequence* $\{S_{n_k}(s)\}$, *of partial sums of* (1.1) *which converges in some neighbourhood of a point on the line* $\operatorname{Re}(s)=\sigma_H$. *Then* (1.1) *may be written as the sum of two Dirichlet series,* S_1 *and* S_2, *where* S_1 *satisfies Ostrowski's gap condition and* S_2 *is closely overconvergent in a half-plane larger than* $\operatorname{Re}(s)>\sigma_H$.

We note that, in general, it is not possible to draw the conclusion that S_2 converges in a half-plane larger than the half-plane of convergence of (1.1). This is clearly demonstrated by the series

$$\sum_{n=1}^{\infty}\frac{1}{P_n}\{e^{-s}(1-e^{-s})\}^{4^n}+\sum_{n=1}^{\infty}\{e^{-(n+\frac{1}{2})s}-e^{-(n+\frac{1}{2}+e^{-n})s}\},$$

where P_n denotes the modulus of the largest coefficient in the expansion of $(1-z)^{4^n}$ in powers of z. However, in a certain special case, Theorem 3 does imply that the half-plane of convergence of S_2 is larger than that of the series (1.1). Let σ_c' and σ_H' denote respectively the abscissae of convergence and holomorphy of the series S_2 and let δ' denote the index of condensation of the sequence of exponents of S_2. A theorem of V. Bernstein [1, Ch. 2, Th. 1] states that for all Dirichlet series ($D<\infty$), $\sigma_c-\sigma_H\leqslant\delta$. Now suppose that, for the series (1.1), $\sigma_c-\sigma_H=\delta$. Then $\sigma_c'\leqslant\delta'+\sigma_H'<\sigma_H+\delta$, since Theorem 3 shows that $\sigma_H'<\sigma_H$ and obviously $\delta'\leqslant\delta$. Hence $\sigma_c'<\sigma_c$, which is to say that the half-plane of convergence of S_2 is larger than that of the series (1.1). Thus we have the following corollary to Theorem 3.

Corollary. *If, for the series* (1.1), $\sigma_c-\sigma_H=\delta$, *then the series* S_2 *has a half-plane of convergence larger than that of the series* (1.1).

We remark that we certainly have $\sigma_c-\sigma_H=\delta$ in the case when $\delta=0$ and so we obtain Noble's result in this case.

In our proof of Theorem 3, we shall find it necessary to consider the affect of certain transformations on closely overconvergent Dirichlet series. Some of the results obtained seem to be of sufficient interest in themselves to warrant special mention.

Theorem 4. *Suppose that for each integer* k, *there are given* p_k+1 *complex numbers* $\lambda_{m_k},\ldots,\lambda_{m_k+p_k}$ *such that* $p_k/|\lambda_{m_k}|\to 0$ *as* $k\to\infty$ *and for* $0\leqslant r\leqslant p_k$, $|\lambda_{m_k+r}-\lambda_{m_k}|=O(p_k)$. *Suppose that, for* s *belonging to some fixed domain* D,

$$\left|\sum_{n=m_k}^{m_k+p_k}a_ne^{-\lambda_ns}\right|\leqslant M_k(s). \tag{1.6}$$

Suppose also that we are given a sequence of function $\{C_k(z,s)\}$ *each being a regular function of* z *for* z *in some convex domain containing the points* $\lambda_{m_k},\ldots,\lambda_{m_k+p_k}$ *and for* s *in* D. *Then, for* s *belonging to any fixed compact subset of* D,

$$\left|\sum_{m_k}^{m_k+p_k}a_nC_k(\lambda_n,s)e^{-\lambda_ns}\right|\leqslant\left(\frac{A\,|\lambda_{m_k}|}{p_k}\right)^{p_k}.\,M_k(s)\sup_{q\leqslant p_k}|C_k^{(q)}(\xi_k,s)|, \tag{1.7}$$

where ξ_k *is some point belonging to the convex hull of the points* $\lambda_{m_k},\ldots,\lambda_{m_k+p_k}$ *and* A *is a positive constant.*

If $\{\lambda_n\}$ is an increasing sequence, we may apply Theorem 4 with $C_k(z, s) = \exp[-z(s-s_0)]$ in order to obtain the following result.

Theorem 5. *Suppose that, for s belonging to some neighbourhood of the point* $s_0 = \sigma_0 + it_0$,

$$\left|\sum_{m_k}^{m_k+p_k} a_n e^{-\lambda_n s}\right| \leqslant M_k.$$

Then, for every $\varepsilon > 0$, *and every bounded set B, there exists a* $k_0(\varepsilon, B)$ *such that for* $k \geqslant k_0(\varepsilon, B)$ *and s belonging to B,*

$$\left|\sum_{m_k}^{m_k+p_k} a_n e^{-\lambda_n s}\right| \leqslant M_k e^{-(\sigma-\sigma_0-\varepsilon)\lambda_{m_k}},$$

where $\sigma = \mathrm{Re}(s)$.

Finally, we can obtain the result of Bourion [2] mentioned above, as an immediate corollary of Theorem 5.

Theorem 6. *Suppose that* $\{S_{n_k}(s)\}$ *is a closely overconvergent sequence of partial sums of* (1.1) *in the region R and let* $\sigma_R = \inf_{s \in R}(\mathrm{Re}(s))$. *Then* $\{S_{n_k}(s)\}$ *is also a closely overconvergent sequence of partial sums in the region* $\mathrm{Re}(s) > \sigma_R$. *In particular, the maximum possible region of close overconvergence of the series* (1.1) *is its half-plane of holomorphy.*

The principle of the proof of Theorem 3 is as follows. We must divide the series (1.1) into two series, S_1 and S_2. The series S_2 will be composed of the terms of (1.1) for which λ_n lies in one of the gaps. We need to prove that S_2 is closely overconvergent in a half-plane larger than $\mathrm{Re}(s) > \sigma_H$ and hence we must define a suitable sequence of intervals, S, such that

$$\sum{}^* \{\sum_{\lambda_n \in I} a_n e^{-\lambda_n s}\} \tag{1.8}$$

converges in such a half-plane, where $\sum^*$ is taken over all intervals, I, of S which are contained in one of the gaps of S_1. In order to show that (1.8) converges in the required region, we seek a method of estimating $|\sum_{\lambda_n \in I} a_n e^{-\lambda_n s}|$.

Suppose $I \subset (\lambda_{n_k}, (1+\alpha)\lambda_{n_k})$ and let $\varkappa(t)$ be an integral function of exponential type which has zeros at the points of $\{\lambda_n\}$ contained in $(\lambda_{n_k}, (1+\beta)\lambda_{n_k}) \cap \complement(I)$; a Fourier transform technique then gives us an estimate for $|\sum_I \{a_n \varkappa(\lambda_n) e^{-\lambda_n s}\}|$. We would like to then apply Theorem 4 with $C_k(z, s) = 1/\varkappa(z)$, in order to obtain the required estimate for $|\sum_I a_n e^{-\lambda_n s}|$, but unfortunately the derivatives of $1/\varkappa(z)$ at points of I may well well be too large. It can be proved that these derivatives will be sufficiently small if the distances between I and the other intervals of S are all greater than $|I|^2/d(I, 0)$ (where $|I|$ is the length of I), but we cannot, in general, obtain a sequence S with these properties. However, we can define S so that, for any two of its intervals, I and J,

$$d(I, J) \geqslant \min\{|I|^2/d(I, 0), |J|^2/d(J, 0)\}.$$

The construction of S is given in section 2. An important property of S is that any interval of S intersecting a certain neighbourhood of a fixed interval, I, of S must have length not greater than $|I|/4$. This allows us to define a partial ordering on S. Then with a modified kernal function $\varkappa(t)$, defined in section 3, which has zeros sufficiently far from I, we can obtain an estimate for $\sum_I a_n \varkappa(\lambda_n) e^{-\lambda_n s}$ in terms of sums over intervals close to I, i.e., in terms of sums of the form $\sum_J a_n \varkappa(\lambda_n) e^{-\lambda_n s}$ where J is close to I. We can then use the partial ordering and obtain inductively an estimate for $\sum_I a_n \varkappa(\lambda_n) e^{-\lambda_n s}$ which is of the required form. Theorem 4 then gives us a suitable bound for $\sum_I a_n e^{-\lambda_n s}$ and the theorem follows.

The proof of Theorem 3 has been inspired by M. E. Noble's paper [6]; in particular, Lemmas 6 and 8 and other parts of section 4 have been adapted from his paper. It scarcely seems necessary to point out that I have made liberal use of the ideas of V. Bernstein, as this must already be apparent to the reader.

The author would like to express his thanks to the Royal Institute of Technology in Stockholm, where most of the work for this paper was performed, and in particular to the members of the mathematics department, for their most cordial hospitality.

The letter 'A' will frequently be used in this paper to denote a positive constant, not necessarily the same at each appearance; $k(\varepsilon)$ will occasionally be used in the same way to denote a number depending only on ε.

2. Results concerning intervals

Following V. Bernstein, we shall give the sequence $\{\lambda_n\}$ a structure by defining a sequence of intervals on the real line. We shall require our sequence of intervals to have certain properties not possessed by Bernstein's intervals. If I and J are two intervals on the real line, we shall write $d(I, J)$ for the quantity $\inf_{y \in J}^{x \in I} |x-y|$ and $d(I, 0)$ for $\inf_{x \in I} |x|$.

We first give Bernstein's construction. He proved ([1], Ch. 2 and Ch. 6, Th. 11) that for each $q < 1/10D$, there exists a disjoint set of intervals, $E(q, \{\lambda_n\})$ with the following properties:

(i) each interval of the set $E(q, \{\lambda_n\})$ contains at least one point λ_n;

(ii) if L denotes the length of some particular interval of the set and k denotes the number of points of $\{\lambda_n\}$ which it contains, then

$$(k+1)q \leqslant L \leqslant 2kq; \tag{2.1}$$

(iii) if $\lambda_m, \ldots, \lambda_{m+k-1}$ belong to the same interval of $E(q, \{\lambda_n\})$ and z does not belong to that interval, then

$$|(\lambda_m - z) \ldots (\lambda_{m+k-1} - z)| \geqslant k!\, q^k;$$

(iv) if we write the intervals of $E(q, \{\lambda_n\})$ in a sequence $\{I_n\}$ such that $d(I_n, 0)$ increases with n, then the series

$$\sum_{N=1}^{\infty} \Big\{ \sum_{\lambda_n \in I_N} a_n e^{-\lambda_n s} \Big\} \tag{2.2}$$

converges in the half-plane of holomorphy of the function $f(s)$, defined by (1.1).

We first wish to modify the set $E(q, \{\lambda_n\})$ so that the inequality (2.1) can be replaced by the equality $L=2kq$. In order to achieve this, we surround each interval of $E(q, \{\lambda_n\})$ by a symmetrically placed interval of length $2kq$; some of these new intervals may intersect each other and when this occurs, we regard two mutually intersecting intervals as making up a combined interval. The inequality (2.1) holds for such a combined interval and so we may surround each of these by an interval of the required length and thus continue the process. Let R be any fixed number. By using (2.1) and the fact that $q<1/10D$, where D is the maximum density of the sequence $\{\lambda_n\}$, we see that after a finite number of stages, the process does not change any interval intersecting $(0, R)$ and in particular, the process cannot produce an interval of infinite length. Thus we obtain a new set of intervals $E'(q)$ for which $L=2kq$. Obviously the properties (i) and (iv) hold for the intervals of $E'(q)$ and it is easy to show that (iii) holds and also the following modified version of (iii):

(iii, *a*) if z does not belong to any of the intervals $I_{n_1}, I_{n_2}, \ldots, I_{n_m}$ of $E'(q)$ and k denotes the number of points of $\{\lambda_n\}$ in $I_{n_1} \cup \ldots \cup I_{n_m}$,

$$\left|\prod_{r=1}^{m} \prod_{\lambda_n \in I_{n_r}} (\lambda_N - z)\right| \geqslant k!(q/4)^k.$$

Lastly, density considerations show that for the intervals of $E'(q)$, the overconvergence of the partial sums given in (iv) is close overconvergence.

Suppose that we are given any sequence of mutually disjoint intervals $\{J_j\}$, where $J_j=(a_j, a_j+l_j)$ and suppose that the numbers l_j, $j=1, 2, \ldots$ have a positive lower bound. We then define the intervals $J_j'=(a_j, a_j+l_j')$ as follows. Suppose that there exist integers M and N with $M<N$, such that

$$d(J_M, J_N) < \min(l_M^2/a_M, l_N^2/a_N) \tag{2.3}$$

and let M be the smallest such integer. With the choice of M now fixed, there can only be a finite number of possible choices of N; let us choose N as large as possible. We then define the sequence $\{l_j'\}$ by the relations

$$\begin{aligned} l_j' &= l_j, \quad \text{for} \quad j<M \quad \text{or} \quad j>N \\ l_M &= a_N + l_N - a_M \\ l_j' &= 0 \quad \text{for} \quad M<j\leqslant N. \end{aligned}$$

If (2.3) is not satisfied for any M and N, then we define $l_j'=l_j$ for all j. Let us now take $q<1/(16e^4D+1)$ and let us consider the set of intervals $E'(q)=\{I_j\}$. We write $I_j=(A_j, A_j+L_j)$ and we then form the sequence of intervals $\{I_j'\}$. From this sequence, we can discard any intervals of zero length and then form the sequence $\{I_j''\}$; clearly, we can continue this process without limit. Thus for each fixed j, we can define the sequence $L_j, L_j', \ldots, L_j^n, \ldots$. Provided that we allow the value $+\infty$, the limit

$$\bar{L}_j = \lim_{n\to\infty} L_j^n \tag{2.4}$$

always exists, since either $L_j^n=0$ for all but a finite number of n or else L_j^n always increases with n.

Lemma 1. *The number $\bar{L}_j$ is finite for each j and $\bar{L}_j/A_j \to 0$ as $j\to\infty$.*

It is this lemma which provides the basis for the construction of our required set of intervals. The principle of our proof of Lemma 1 is to show that, if the result were false, the intervals of $E'(q)$ would necessarily be packed closer together than is possible with $\{\lambda_n\}$ having maximum density D. We note that we may assume, without loss of generality, that each L_j is equal to $2q$, because we may replace any interval of length $2kq$ by k adjacent intervals of length $2q$ and this clearly does not affect the properties of $\bar{L}_j$, $j=1, 2, \ldots$. Having made this assumption, let us suppose that Lemma 1 is false.

We can then find a positive constant $\varrho<1$ and sequences $\{m_k\}$, $\{n_k\}$ and $\{N_k\}$, with $N_k \leqslant n_k$, such that

$$(1+\varrho)A_{n_k} \leqslant L_{N_k}^{m_k}+A_{N_k}. \tag{2.5}$$

Because of the method of construction of $\{L_{N_k}^{m_k}\}$, there exists a sequence $\{r_k\}$ such that $A_{n_k+r_k}+L_{n_k+r_k}=A_{N_k}+L_{N_k}^{m_k}$ and then the interval $\Lambda_k=(A_{n_k}, A_{N_k}+L_{N_k}^{m_k})$ contains just the intervals $I_{n_k}, \ldots, I_{n_k+r_k}$ of $E'(q)$. If we denote the interval between I_{n_k+r} and I_{n_k+r+1} by $Q_r(k)$, then clearly

$$|\Lambda_k| = \sum_{r=0}^{r_k} L_{n_k+r} + \sum_{r=0}^{r_k-1} |Q_r(k)|, \tag{2.6}$$

where we have denoted the length of an interval J by $|J|$. Each $Q_r(k)$ is contained in an interval of $\{I_j^{m_k}\}$ but not in any interval of $\{I_j\}$ and hence there exists a smallest integer P, such that $Q_r(k)$ is contained in an interval of $\{I_j^P\}$. This implies that $Q_r(k)$ lies between two intervals, I_M^{P-1} and I_N^{P-1}, of $\{I_j^{P-1}\}$ which satisfy (2.3). Suppose that the number of intervals of $E'(q)$ contained in I_M^{P-1} and I_N^{P-1} is respectively $n(M)$ and $n(N)$. We then define the order of $Q_r(k)$ to be the minimum of $n(M)$ and $n(N)$.

All the L_j's are equal to $2q$ and hence, using (2.3), we see that no first order $Q_r(k)$ can have length greater that $4q^2/A_{n_k}$. Therefore the sum of $\sum_0^{r_k} L_{n_k+r}$ and the total length of all the first order $Q_r(k)$, $0 \leqslant r \leqslant r_k-1$, is not greater than $(1+2q/A_{n_k})\sum_0^{r_k} L_{n_k+r}$. Similarly the addition of the total length of all the second order $Q_r(k)$'s will increase this number by a factor of at most $(1+2q/A_{n_k})$. In general, the sum of $\sum_0^{r_k} L_{n_k+r}$ and the total length of all the $Q_r(k)$ having order not greater than T is at most $(1+2q/A_{n_k})^T \sum L_{n_k+r}$. We may assume without loss of generality, that $L_{N_k}^{m_k} \leqslant 5A_{n_k}$ for k sufficiently large; for, if $L_{N_k}^{m_k} > 5A_{n_k}$, then we can find an $m_k' < m_k$ and an N_k with $N_k \leqslant N_k' < N_{k+1}$ such that

$$(1+\varrho)A_{n_k} < 2A_{n_k} \leqslant L_{N_k'}^{m_k'} \leqslant 5A_{n_k}.$$

With this assumption, no $Q_r(k)$ can have order greater than $2A_{n_k}/q$. Hence, by (2.6),

$$|\Lambda_k| \leqslant (1+2q/A_{n_k})^{2A_{n_k}/q} \sum_{R=0}^{r_k} L_{n_k+R} \leqslant e^4 \sum_{R=0}^{r_k} L_{n_k+R}. \tag{2.7}$$

Let us denote by $N(\Lambda_k)$, the number of points of $\{\lambda_n\}$ in the interval Λ_k. Then, because of the method of construction of $E'(q)$,

$$N(\Lambda_k) = (1/2q) \sum_{R=0}^{r_k} L_{n_k+R}$$

and hence, by (2.7),

$$N(\Lambda_k) \geqslant |\Lambda_k|/2e^4q \geqslant (8D + \tfrac{1}{2}e^4)|\Lambda_k|. \tag{2.8}$$

It follows from the definition of D([1] Note 1) that, given any positive ε, there exists a $k_0(\varepsilon)$ such that for $A_{n_k} = d(\Lambda_k, 0) \geqslant k_0$,

$$N(\Lambda_k) \leqslant D|\Lambda_k| + (d(\Lambda_k, 0) + |\Lambda_k|)\varepsilon = (D+\varepsilon)|\Lambda_k| + \varepsilon A_{n_k}.$$

Let us choose $\varepsilon = \varrho D$; then, since $|\Lambda_k| \geqslant \varrho A_{n_k}$,

$$N(\Lambda_k) \leqslant (D + \varepsilon + \varepsilon/\varrho)|\Lambda_k| \leqslant (2+\varrho) D |\Lambda_k|.$$

This contradicts (2.8) and thus Lemma 1 is proved.

Let S denote the sequence obtained by removing all intervals of zero length from the sequence $\{(A_n, A_n + L_n)\}$. S is our required sequence of intervals and we shall proceed to obtain some of its properties. It is clear that the method of construction of S ensures that if I_M and I_N are two intervals of S,

$$d(I_M, I_N) \geqslant \min\left\{\frac{|I_M|^2}{d(I_M, 0)}, \frac{|I_N|^2}{d(I_N, 0)}\right\}; \tag{2.9}$$

We are going to define a partial ordering on the set S. We shall suppose that for all n, $L_n/A_n \leqslant \frac{1}{3}$. This involves no loss of generality since we may always ensure that the inequality is satisfied by removing a finite number of λ_n. We shall write IpJ whenever I and J belong to S and

$$J \frown \left(I + \frac{|I|^2}{32\, d(I,0)} E\right) \neq \phi, \tag{2.10}$$

where $E = \{x;\ |x| < 1\}$. We shall write $J \leqslant I$ whenever there exist intervals $I = I_1, I_2, \ldots, I_M = J$ of S such that $I_r p I_{r+1}$, $1 \leqslant r \leqslant M-1$.

Lemma 2. *If I and J are intervals of S, the following statements hold:*

(*i*) *if IpJ and $I \neq J$, then $|J| \leqslant |I|/4$;*
(*ii*) *the relation $\leqslant$ is a partial ordering on S;*
(*iii*) *if $J \leqslant I$, then $J \subset I + |I| E$;*
(*iv*) *there are at most $|I|/q$ intervals J such that $J \leqslant I$, $J \neq I$.*

To see (i), let us first observe that $d(J, 0) \leqslant 2d(I, 0)$, for

$$\begin{aligned} d(J,0) &\leqslant d(I,0) + |I| + |I|^2/32d(I,0) \\ &\leqslant d(I,0)\cdot(1 + \tfrac{1}{3} + 1/288), \end{aligned}$$

since $|I| \leqslant d(I, 0)/3$ by assumption. But then, by (2.10),

$$\frac{|I|^2}{32\, d(I,0)} \geqslant d(I,J) \geqslant \frac{|J|^2}{d(J,0)} \geqslant \frac{|J|^2}{2\, d(I,0)}.$$

This proves (i), which immediately shows that we cannot have $J \leqslant I$ and $I \leqslant J$ without having $I = J$. Part (ii) is then obvious. Part (iii) is trivial if $J = I$. If $J \neq I$, let $I_1, \ldots, I_M$ be the intervals mentioned above. Then

$$\begin{aligned}\sup_{x \in J} \{d(x,I)\} &\leqslant |J| + d(J, I_{M-1}) + |I_{M-1}| + d(I_{M-1}, I_{M-2}) + \ldots + |I_2| + d(I_2, I)\\ &\leqslant |I|(1/4 + \ldots + (1/4)^M) + \frac{|I|^2}{16 d(I,0)}(1 + 1/4 + \ldots + (1/4)^{M-1})\\ &\leqslant 2|I|/3.\end{aligned}$$

This suffices to prove part (iii) and, since part (iv) is a trivial consequence of this, we have proved Lemma 2.

We next give a general lemma concerning partial orderings on finite sets. We shall later apply this lemma to certain subsets of S.

Lemma 3. *Suppose that A is a finite set on which is defined a partial ordering relation $\leqslant$. If a and b are members of A, we shall write $a < b$ if $a \leqslant b$ and $a \neq b$. Suppose also that we are given a positive real-valued function $\lambda : A \to R$ and a positive real-valued function $\beta : A \times A \to R$, such that, for all $a \in A$,*

$$\lambda(a) \leqslant \sum_{b<a} \beta(a,b)\,\lambda(b) + r, \tag{2.11}$$

where r is a fixed real number. For each $a \in A$, let $n(a)$ denote the number of $b \in A$ such that $b < a$. Then

$$\lambda(a) \leqslant r \cdot 2^{n(a)} \text{ supremum } \{\beta(a, b)\beta(b, c) \ldots \beta(y, z)\}, \tag{2.12}$$

where the supremum is taken over all subsets, $\{b, c, \ldots y, z\}$ of A such that

$$z < y < \ldots < c < b < a.$$

To each $b < a$, we may apply (2.11) and obtain

$$\lambda(b) \leqslant \sum_{c<b} \beta(b,c)\,\lambda(c) + r.$$

On substituting this expression in (2.11), we obtain

$$\lambda(a) \leqslant r\{1 + \sum_{b<a} \beta(a,b)\} + \sum_{b<a} \sum_{c<b} \beta(a,b)\beta(b,c)\,\lambda(c). \tag{2.13}$$

We can now use (2.11) to estimate the numbers $\lambda(c)$ which occur in (2.13). We may apply this technique successively until, after a finite number of steps, we obtain an expression of the form

$$\lambda(a) \leqslant r\{1 + \sum_{b<a} \beta(a,b) + \ldots + \sum_{b<a} \sum_{c<b} \ldots \sum_{z<y} \beta(a,b)\beta(b,c) \ldots \beta(y,z)\}. \tag{2.14}$$

The total number of terms in the bracket on the right-hand side of (2.13) is not more than

$$1 + n + n(n-1)/2 + \ldots + \binom{r}{n} + \ldots + 1 = 2^n, \tag{2.15}$$

where $n = n(a)$. The inequality (2.12) now follows from (2.14) and (2.15).

3. Results concerning transforms

We are going to isolate certain blocks of terms from the series (1.1) by applying a transform. It is, therefore, important for us to obtain conditions under which the modulus of the transformed block of terms is comparable with the modulus of the original block. The result we require is Theorem 4.

Proof of Theorem 4

We commence by introducing some notation. Let

$$P_k(s) = (1/p_k!) \prod_{p=0}^{p_k} (s - \lambda_{m_k+p}). \tag{3.1}$$

Let $b_k(z, W)$ denote the Borel transform with respect to s of $\{P_k(s) - P_k(z)\}/(s-z)$. Let

$$g_k(s) = \sum_{p=0}^{p_k} a_{m_k+p} e^{-\lambda_{m_k+p} s}, \tag{3.2}$$

and let us choose η to be any positive number and set

$$\phi_k(s,z) = \frac{1}{2\pi i} \int_{|\zeta|=\eta} g_k(s-\zeta) b_k(z,\zeta)\, d\zeta. \tag{3.3}$$

We show that

$$\phi_k(s, \lambda_{m_k+p}) = P_k'(\lambda_{m_k+p}) a_{m_k+q} e^{-\lambda_{m_k+q} s}. \tag{3.4}$$

On substituting (3.2) in (3.3), we obtain

$$\phi_k(s,z) = \sum_{p=0}^{p_k} \left\{ a_{m_k+p} e^{-\lambda_{m_k+p} s} \cdot \frac{1}{2\pi i} \int_{|\xi|=\eta} b_k(z,\xi) e^{\lambda_{m_k+p} \xi} d\xi \right\}. \tag{3.5}$$

Because $P_k(s)$ is a polynomial, $b_k(z, \zeta)$ is a regular function of ζ at every point except 0. Hence the standard result concerning the inversion of the Borel transform (see [1] Note 3) tells us that

$$\frac{1}{2\pi i} \int_{|\xi|=\eta} b_k(z,\xi) e^{\lambda_{m_k+p} \xi} d\xi = \{P_k(\lambda_{m_k+p}) - P_k(z)\}/(\lambda_{m_k+p} - z)$$

and (3.5) then becomes

$$\phi_k(s,z) = \sum_{p=0}^{p_k} a_{m_k+p} e^{-\lambda_{m_k+p} s} \{P_k(\lambda_{m_k+p}) - P_k(z)\}/(\lambda_{m_k+p} - z). \tag{3.6}$$

The equality (3.4) now follows on putting $z = \lambda_{m_k+p}$ in (3.6).

Next we require an estimate for the partial derivatives of $\phi_k(s, z)$, which is given by the following lemma.

Lemma 4. *Suppose that the points $\lambda_{m_k}, \ldots, \lambda_{m_k+p_k}$ satisfy the hypotheses of Theorem 4. Suppose that s belongs to some fixed compact subset C of D and that z belongs to the convex hull, A_k, of the points $\lambda_{m_k}, \ldots, \lambda_{m_k+p_k}$. Then, if $q \leqslant p_k$,*

$$\left|\left[\frac{\partial^q \phi_k(s,Z)}{\partial Z^q}\right]_{Z=z}\right| \leq M_k(s)\cdot(A\,|\lambda_{m_k}|/p_k)^{p_k}, \tag{3.7}$$

provided that k is larger than a certain number, $k_0(C)$.

For $\operatorname{Re}(\zeta)>0$,

$$b_k(z,\xi)=\int_0^\infty \frac{P_k(w)-P_k(z)}{w-z}\,e^{-w\xi}\,dw. \tag{3.8}$$

If we expand $P_k(w)$ as a power series in $(w-z)$ and substitute the resulting expression in (3.8), we obtain, for $\operatorname{Re}(\zeta)>0$,

$$\begin{aligned} b_k(z,\zeta) &= \sum_{q=0}^{p_k} P_k^{(q+1)}(z)\int_0^\infty \frac{(w-z)^q}{(q+1)!}\cdot e^{-w\zeta}\,dw \\ &= \sum_{q=0}^{p_k} \zeta^{-(q+1)}P_k^{(q+1)}(z)\,\{1-z\zeta+\ldots+(-z\zeta)^q/q!\}/(q+1) \end{aligned} \tag{3.9}$$

and by analytic continuation, the same formula holds for all $\zeta \neq 0$. If X belongs to the set $A_k+2p_k\bar{E}$, then by the hypotheses of Theorem 4, the diameter of A_k is of the order of p_k and so $|\lambda_{m_k+r}-X|\leq Ap_k$, for $0\leq r\leq p_k$; hence

$$|P_k(X)|\leq (Ap_k)^{p_k+1}/p_k!\leq A^{p_k}.$$

On applying Cauchy's inequality, we see that for $Z\in A_k+p_k\bar{E}$ and $0\leq q\leq p_k+1$,

$$|P_k^{(q)}(Z)|\leq q!\,A^{p_k}/p_k^q\leq A^{p_k}. \tag{3.10}$$

In (3.3), let us take $\eta=\frac{1}{2}d(C,\complement(D))$; then, if $|\zeta|=\eta$,

$$|-Z\zeta|^q/q!\leq (A\,|\lambda_{m_k}|)^q/q!\leq (A\,|\lambda_{m_k}|)^{p_k}/p_k!\leq (A\,|\lambda_{m_k}|/p_k)^{p_k}, \tag{3.11}$$

since we may assume that $p_k<A\,|\lambda_{m_k}|$. Hence, by (3.9), (3.10) and (3.11),

$$|b_k(Z,\zeta)|\leq (A\,|\lambda_{m_k}|/p_k)^{p_k}, \tag{3.12}$$

for $Z\in A_k+p_k\bar{E}$ and $|\zeta|=\eta$. Now suppose that $z\in A_k$, $s\in C$ and that $0\leq q\leq p_k$. Then, using (3.3), we see that if k is sufficiently large,

$$\begin{aligned} \left|\left[\frac{\partial^q \phi_k(s,Z)}{\partial Z^q}\right]_{Z=z}\right| &\leq 2\pi\eta M_k(s)\sup_{|\zeta|=\eta}\left|\left[\frac{\partial^q b_k(Z,\zeta)}{\partial Z^q}\right]_{Z=z}\right| \\ &\leq 2\pi\eta M_k(s)\cdot(q!/p_k^q)\ \operatorname*{supremum}_{Z\in A_k+p_k\bar{E}}\ \{\sup_{|\zeta|=\eta}|b_k(Z,\zeta)|\} \\ &\leq \quad M_k(s)\cdot(A\,|\lambda_{m_k}|/p_k)^{p_k} \end{aligned}$$

and this completes the proof of Lemma 4.

We now state a lemma which is due to Jensen [4].

Lemma 5 (Jensen). *Suppose that $x_1, \ldots, x_Q$ are points of the complex plane and that $F(z)$ is a function analytic in some convex domain containing these points. Let us define*

$$F_Q = \sum_{\nu=1}^{Q} F(x_\nu)/\prod{}'(x_\nu),$$

where $\Pi(x) = \Pi_{p=1}^{Q}(x - x_p)$. Then there exists a complex number $\mathcal{D}$, with modulus not greater than one and a point ξ_Q belonging to the convex hull of the points $x_1, x_2, \ldots, x_Q$ such that

$$F_Q = \frac{\mathcal{D}}{(Q-1)!}\left[\frac{d^{Q-1}}{dz^{Q-1}}\{F(z)\}\right]_{z=\xi_Q}.$$

Using the equality (3.4), we may write

$$\sum_{n=m_k}^{m_k+p_k} a_n C_k(\lambda_n, s)\, e^{-\lambda_n s} = \sum_{n=m_k}^{m_k+p_k} \phi_k(s, \lambda_n)\, C_k(\lambda_n, s)/P_k'(\lambda_n). \tag{3.13}$$

Applying Lemma 5 to the right-hand side of (3.13) yields

$$\sum_{m_k}^{m_k+p_k} a_n C_k(\lambda_n, s)\, e^{-\lambda_n s} = \mathcal{D}\left[\frac{\partial^{p_k}}{\partial Z^{p_k}}\{\phi_k(s, Z)\, C_k(Z, s)\}\right]_{Z=\xi_k}.$$

Theorem 4 now follows on application of Lemma 4 and Leibnitz formula.

We recall that, in section 1, we mentioned the possibility of using a function $\varkappa(t)$ which is an integral function of exponential type having zeros at the points of $\{\lambda_n\}$ contained in $(\lambda_{n_k}, (1+\beta)\lambda_{n_k}) \cap \complement(I)$. We shall however, use a slightly different function, namely one which has zeros at the points of $\{\lambda_n\}$ contained in $(\lambda_{n_k}$ $(1+\beta)\lambda_{n_k}) \cap \complement(U(I))$ but not at the points λ_n contained in $U(I)$; here we have used $U(I)$ to denote the union of all the intervals of S which are less than or equal to I in the sense of the partial ordering of section 2. This will enable us to obtain a formula to which we can profitably apply Lemma 3.

Let T_k denote the set of intervals, I of S, such that $I + |I|\,\bar{E}$ is contained in $(\lambda_{n_k}, (1+\mu)\lambda_{n_k})$ and let $V(I)$ denote the set of intervals $\{J, J \leqslant I\}$, whose union is equal to $U(I)$. For each $I \in T_k$, we define

$$J_k(I) = T_k \cap \complement(V(I)) \tag{3.14}$$

and

$$K(I, t) = \prod_{J \in J_k(I)} \prod_{\lambda_n \in J} \frac{\sin^2\{\delta(t - \lambda_n)/\lambda_{n_k}\}}{\delta(t - \lambda_n)/\lambda_{n_k}}, \tag{3.15}$$

where δ is a constant to be defined later in such a manner that $A/\mu < \delta < \pi/2\mu$.

Lemma 6. *Suppose that $I \in T_k$, $z \in I$ and that Q is less than the number of λ_n contained in I. Then, provided that k is sufficiently large,*

$$\left|\left[\frac{d^Q}{dt^Q}(1/K(I, t))\right]_{t=z}\right| \leqslant \exp\,(A\mu\lambda_{n_k}).$$

Suppose that u belongs to the interval $I+(|I|^2/32d(I,0))E$. Then, by definition of $V(I)$, u cannot belong to any interval of $J_k(I)$. Therefore, since property (iii, a) of section 2 holds for the intervals of S, we must have

$$\prod_{J\in J_k(I)}\prod_{\lambda_n\in J}|u-\lambda_n|\geqslant N!(q/4)^N, \tag{3.16}$$

where $N=N(I)$ is the number of λ_n contained in the intervals of $J_k(I)$. If λ_n belongs to some interval of $J_k(I)$, then $|\delta(u-\lambda_n)/\lambda_{n_k}|<\pi/2$ and hence

$$|1/K(I,u)|\leqslant\prod_{J\in J_k(I)}\prod_{\lambda_n\in J}\left|\frac{\pi^2\lambda_{n_k}}{4\delta(u-\lambda_n)}\right|\leqslant\left(\frac{\pi^2\lambda_{n_k}}{\delta q}\right)^N\cdot\frac{1}{N!}\leqslant\left(\frac{e\pi^2\lambda_{n_k}}{\delta qN}\right)^N, \tag{3.17}$$

by (3.16). Now, viewed as a function of X, $(P/X)^X$ has its maximum when $X=P/e$ and therefore

$$|1/K(I,u)|\leqslant\exp\{\pi^2\lambda_{n_k}/(\delta q)\}, \tag{3.18}$$

whenever u belongs to $I+(|I|^2/32d(I,0))E$. Lemma 6 now follows after the application of Cauchy's inequality.

Lemma 7. *Suppose that $I\in T_k$ and that $J<I$ (in the sense of section 2). Let $n(I)$ denote the number of λ_n in $U(I)$ and let $\bar{n}(I,J)=n(I)-n(J)$. Suppose also that $z\in J$ and that $q\leqslant n(J)$. Then, for k sufficiently large,*

$$\left[\frac{d^q}{dt^q}\left(\frac{K(I,t)}{K(J,t)}\right)\right]_{t=z}\leqslant\left(\frac{A\lambda_{n_k}}{\bar{n}(I,J)}\right)^{\bar{n}(I,J)}\cdot\left(\frac{\lambda_{n_k}}{Bn(J)}\right)^{n(J)}.$$

The proof of this lemma is very similar to the proof of Lemma 6 and will, therefore, not be given.

We may assume that the number $N(I)$ of the inequality (3.16) is not smaller than 2, for if $N(I)<2$ for more than a finite number of k, then Theorem 3 is trivially true. With this assumption, $\int_{-\infty}^{\infty}|K(I,t)|\,dt<\infty$ and hence we may set

$$k(I,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}K(I,t)\,e^{itx}dt. \tag{3.19}$$

Lemma 8. *For k sufficiently large,*

$$|k(I,x)|\leqslant A\lambda_{n_k}.$$

We have

$$\begin{aligned}|k(I,x)|&\leqslant\int_{-\infty}^{\infty}|K(I,u)|\,du\\&=\int_{|u|\leqslant 4\lambda_{n_k}/\delta}|K(I,u)|\,du+\int_{|u|>4\lambda_{n_k}/\delta}|K(I,u)|\,du\\&\leqslant 8\lambda_{n_k}/\delta+\int_{|u|>4\lambda_{n_k}/\delta}(2\lambda_{n_k}/\delta u)^{N(I)}du\end{aligned}$$

and the lemma follows immediately from here, since $N(I)\geqslant 2$.

4. Completion of the proof of Theorem 3

In order to simplify our notation, we shall now assume that σ_H, the abscissa of holomorphy of the function $f(s)$, is equal to zero and that the sequence of partial sums $\{S_{n_k}(s)\}$ converges in a neighbourhood of the origin. It is clear that these assumptions involve no loss of generality. The aforementioned neighbourhood of the origin must contain a square of the form $|\sigma| \leqslant \Delta$, $|t| \leqslant \Delta$. Let us choose $T < \Delta$ and let us define the number δ of (3.15) by

$$\delta = \frac{\min(T/2, \pi/2)}{2(D+1)\mu}, \tag{4.1}$$

where D is given by (1.2). Then, if $I \in T_k$, the function $K(I, t)$ is an integral function of order 1, with type not exceeding T. The theorem of Paley and Wiener ([7], p. 16) then assures us that $k(I, x)$ vanishes for $|x| \geqslant T$. We define

$$R_k(s) = f(s) - \sum_1^{n_k} a_n e^{-\lambda_n s} \tag{4.2}$$

and

$$f_{n_k}(s) = R_k(s) e^{\lambda_{n_k} s}. \tag{4.3}$$

For $I \in T_k$ and $\eta' > 0$, we set $\eta = \eta' + i\eta''$ and then we see that

$$\int_{-T}^{T} f_{n_k}(\eta + it) k(I,t) e^{-\lambda_{n_k} t} dt = \sum_{n_k+1}^{\infty} a_n e^{(\lambda_{n_k} - \lambda_n)\eta} \int_{-T}^{T} k(I,t) e^{i\lambda_n t} dt$$

$$= \sum_{n_k+1}^{\infty} a_n K(I, \lambda_n) e^{(\lambda_{n_k} - \lambda_n)\eta}.$$

But $K(I, \lambda_n) = 0$ whenever $\lambda_n \varepsilon J_k(I)$ and therefore, if I_{N_k} denotes the first interval of S such that $I_{N_k} + |I_{N_k}| \bar{E}$ intersects the interval $[(1+\mu)\lambda_{n_k}, \infty)$,

$$\int_{-T}^{T} f_{n_k}(\eta + it) k(I,t) e^{-i\lambda_{n_k} t} dt =$$

$$\int_{-T}^{T} k(I,t) e^{\lambda_{n_k}\eta} \left\{ \sum_{N=N_k}^{\infty} \sum_{\lambda_n \in I_N} a_n e^{-\lambda_n(\eta+it)} \right\} dt + \sum_{I_N \in V(I)} \sum_{\lambda_n \in I_N} a_n K(I, \lambda_n) e^{(\lambda_{n_k} - \lambda_n)\eta}. \tag{4.4}$$

Lemma 9. *There exists a positive number $A_0 = A_0(\eta, T, \Delta)$ and a positive integer $k_0 = k_0(\eta, T, \Delta, \mu)$ such that for $k \geqslant k_0$,*

$$\left| \sum_{I_N \in V(I)} \sum_{\lambda_n \in I_N} a_n K(I, \lambda_n) e^{(\lambda_{n_k} - \lambda_n)\eta} \right| \leqslant A \lambda_{n_k}^2 \{ e^{-A_0 \lambda_{n_k}} + e^{-\frac{1}{2}\eta' \mu \lambda_{n_k}} \}.$$

We are going to use the equality (4.4). We begin by obtaining an estimate for $|R_k(s)|$ on the segment $|t| \leqslant T' = (T + \Delta)/2$ of the imaginary axis. We write the intervals of S in a sequence $\{I_p\}$ such that $d(I_p, 0)$ increases with p and we denote by m_ν the first integer such that $\lambda_{m_\nu} \in I_\nu$. We recall that $|I_\nu| = o(\lambda_{m_\nu})$. Let M_k be the smallest member of the sequence $\{m_\nu\}$ such that $\lambda_{M_k} \geqslant \lambda_{n_k}$. Then, for $\mathrm{Re}(s) > 0$, we may write

$$R_k(s) = \sum_{n_k+1}^{M_k-1} a_n e^{-\lambda_n s} + \sum_{m_\nu \geqslant M_k} \sum_{m_\nu}^{m_{\nu+1}-1} a_n e^{-\lambda_n s}. \tag{4.5}$$

Let C denote the disc $|s-\frac{1}{8}\Delta|<\Delta/16$; then C is contained in the square $|\sigma|\leqslant\Delta$, $|t|\leqslant\Delta$ and hence if $s\in C$ there is a $k_1(\Delta, s)$ such that for $k\geqslant k_1(\Delta, s)$,

$$\left|\sum_{n_k+1}^{\infty} a_n e^{-\lambda_n s}\right| \leqslant \tfrac{1}{2}.$$

But C is contained in the half-plane of holomorphy of the function $f(s)$. Therefore for each $s\in C$ there is a $k_2(\Delta, s)$ such that for $k\geqslant k_2(\Delta, s)$,

$$\left|\sum_{M_k}^{\infty} a_n e^{-\lambda_n s}\right| \leqslant \tfrac{1}{2}.$$

Writing $k'(\Delta,s)=\max\,(k_1,k_2)$, we have

$$\left|\sum_{n_k+1}^{M_k-1} a_n e^{-\lambda_n s}\right| \leqslant 1 \tag{4.5a}$$

for $s\in C$ and $k\geqslant k'$. Let $Q(K)$ denote the set of points of C for which (4.5a) holds for all $k\geqslant K$. $Q(K)$ is clearly closed in C and $C=\bigcup_{k=1}^{\infty}Q(K)$. The Baire category theorem then tells us that one of the sets $Q(K)$ contains an open disc U; i.e. there is a $k''(\Delta)$ such that (4.5a) holds for any s in U and any $k\geqslant k''(\Delta)$. Similarly, we can show that there is an integer $K'(\Delta)$ and an open disc U' contained in U such that

$$\left|\sum_{m_\nu}^{m_{\nu+1}-1} a_n e^{-\lambda_n s}\right| \leqslant 1$$

for $m_\nu\geqslant M_k$, $k\geqslant K'$ and $s\in U'$.

Now suppose that $\mathrm{Re}(s)=\Delta$; then using Theorem 5 with $\varepsilon=\frac{1}{2}\Delta$, we see that

$$|R_k(s)| \leqslant e^{-\lambda_{n_{k+1}}\Delta/2} + \sum_{m_\nu\geqslant M_k} e^{-\lambda_{m_\nu}\Delta/2}$$

and since $\liminf\,(\lambda_{m_{\nu+1}}-\lambda_{m_\nu})\geqslant q>0$, this implies that

$$|R_k(s)| \leqslant \exp(-q\Delta\lambda_{n_k}/4). \tag{4.6}$$

Let γ_1 denote the segment $|t|\leqslant\Delta$ of the line $\sigma=\Delta$ and let γ_2 denote the remaining part of the boundary of the square $|\sigma|\leqslant\Delta$, $|t|\leqslant\Delta$. The inequality (4.6) gives us an estimate for $|R_k(s)|$ on γ_1 and, by hypothesis, $|R_k(s)|\leqslant 1$ on γ_2, if k is sufficiently large. The Two Constants Theorem then tells us that there exists $w=w(\Delta, T)$ with $0<w<1$, such that for $k\geqslant k_0(\Delta)$,

$$\sup_{|t|\leqslant T'} |R_k(it)| \leqslant \exp\,(-q\Delta w\lambda_{n_k}/4). \tag{4.7}$$

Next, we use (4.7) to obtain an upper bound for $|f_{n_k}(\eta+it)|$ as t varies in $|t|\leqslant T$. This will, in turn, give us an estimate for the integral on the left of the equality (4.4). Let us suppose first, that $\mathrm{Re}(s)\geqslant 2\varepsilon$. We multiply the equality (4.5) by $e^{\lambda_{n_k}s}$ and then apply Theorem 5. This yields

$$|f_{n_k}(s)|\leqslant e^{\lambda_{n_k}\sigma}\left\{e^{-\lambda_{n_{k+1}}(\sigma-\varepsilon)}+\sum_{m_\nu\geqslant M_k}e^{-\lambda_{m_\nu}(\sigma-\varepsilon)}\right\}$$

$$\leqslant e^{\lambda_{n_k}\varepsilon}\left\{1+\sum_{p=0}^{\infty}e^{-pq(\sigma-\varepsilon)}\right\}$$

and hence, for $\mathrm{Re}(s)\geqslant 2\varepsilon$ and $k\geqslant k_0(\varepsilon)$,

$$|f_{n_k}(s)|\leqslant e^{2\varepsilon\lambda_{n_k}}. \tag{4.8}$$

Now suppose that $0\leqslant \mathrm{Re}\,(s)\leqslant 2\varepsilon$ and that $|t|\leqslant T'$. By an application of Theorem 5,

$$\left|\sum_1^{n_k}a_n\right|=0(e^{\varepsilon\lambda_{n_k}})$$

and hence

$$|f_{n_k}(s)|\leqslant e^{2\varepsilon\lambda_{n_k}}\left\{|f(s)|+\left|\sum_1^{n_k}a_n\right|\right\}\leqslant e^{3\varepsilon\lambda_{n_k}}, \tag{4.9}$$

for $k\geqslant k_0(\varepsilon)$. On combining (4.8) and (4.9), we see that (4.9) holds whenever s belongs to the rectangle $0\leqslant\mathrm{Re}\,(s)<2\eta$, $|t|<T'$ and $k\geqslant k_0(\varepsilon)$. But, by (4.7),

$$\sup_{|t|\leqslant T'}|f_{n_k}(it)|=\sup_{|t|\leqslant T'}|R_k(it)|\leqslant \exp\,(-q\Delta w\lambda_{n_k}/4). \tag{4.10}$$

Therefore, by the Two Constants Theorem, there exists a positive $A_0=A_0(\eta,T,\Delta)$ such that, for k sufficiently large and $\eta''\leqslant(T'-T)/2$

$$\sup_{|t|\leqslant T}|f_{n_k}(\eta+it)|\leqslant \exp\,(-A_0\lambda_{n_k}). \tag{4.11}$$

By combining (4.11) with Lemma 8, we can now obtain an estimate for the integral on the left-hand side of (4.4). We have

$$\left|\int_{-T}^{T}f_{n_k}(\eta+it)\,k(I,t)\,e^{-i\lambda_{n_k}t}dt\right|\leqslant A\lambda_{n_k}^2e^{-A_0\lambda_{n_k}}, \tag{4.12}$$

for k sufficiently large.

We now consider the first term on the right-hand side of (4.4). Applying Theorem 5 as above, we obtain

$$\begin{aligned}\sum_{N\geqslant N_k}\left|\sum_{\lambda_n\in I_N}a_ne^{(\lambda_{n_k}-\lambda_n)(\eta+it)}\right|&\leqslant e^{\lambda_{n_k}\eta'}\sum_{\nu=N_k}^{\infty}e^{-(\eta'-\varepsilon)\lambda_{m_\nu}}\\&\leqslant A\,\exp\,\{\lambda_{n_k}\eta'-(\eta'-\varepsilon)\lambda_{Q_k}\},\ (Q_k=m_{N_k})\\&\leqslant A\,\exp\,\{\varepsilon\lambda_{n_k}+(\eta'-\varepsilon)(\lambda_{n_k}-\lambda_{Q_k})\},\end{aligned} \tag{4.13}$$

for $k \geqslant k_0(\varepsilon, \eta)$. But, for large $k, \lambda_{Q_k} - \lambda_{n_k} \geqslant \mu\lambda_{n_k} - \varepsilon\lambda_{n_k}$ and therefore

$$\sum_{N \geqslant N_k} \left| \sum_{\lambda_n \in I_N} a_n e^{(\lambda_{n_k} - \lambda_n)(\eta + it)} \right| \leqslant A \exp \{\lambda_{n_k}[\varepsilon - (\eta' - \varepsilon)(\mu - \varepsilon)]\}$$

$$\leqslant A \exp(-\eta'\mu\lambda_{n_k}/2), \qquad (4.14)$$

provided that we choose $\varepsilon \leqslant \eta'\mu/\{2(\eta' + \mu + 1)\}$ and k larger than a certain $k_0(\eta, \mu)$. On combining (4.14) with Lemma 8, we have, for $k \geqslant k_0(\eta, \mu)$,

$$\left| \int_{-T}^{T} \sum_{N=N_k}^{\infty} \sum_{\lambda_n \in I_N} a_n e^{(\lambda_{n_k} - \lambda_n)(\eta + it)} k(I, t) e^{-i\lambda_{n_k} t} dt \right| \leqslant A\lambda_{n_k}^2 \exp(-\eta'\mu\lambda_{n_k}/2). \qquad (4.15)$$

Lemma 9 follows from (4.12) and (4.15).

When we use the ordering relation defined in section 2, Lemma 9 gives

$$\left| \sum_{\lambda_n \in I} a_n K(I, \lambda_n) e^{(\lambda_{n_k} - \lambda_n)\eta} \right| \leqslant \sum_{J < I} \left| \sum_{\lambda_n \in J} a_n K(J, \lambda_n) e^{(\lambda_{n_k} - \lambda_n)\eta} \frac{K(I, \lambda_n)}{K(J, \lambda_n)} \right|$$

$$+ A\lambda_{n_k}^2 \{e^{-A_0\lambda_{n_k}} + e^{-\eta'\mu\lambda_{n_k}/2}\}, \qquad (4.16)$$

for any $I \in T_k$. By theorem 4,

$$\left| \sum_{\lambda_n \in J} a_k K(J, \lambda_n) e^{(\lambda_{n_k} - \lambda_n)\eta} \cdot \frac{K(I, \lambda_n)}{K(J, \lambda_n)} \right| \leqslant \left| \sum_{\lambda_n \in J} a_n K(J, \lambda_n) e^{(\lambda_{n_k} - \lambda_n)\eta} \right|$$

$$\times \left(\frac{A\lambda_{n_k}}{n(J)} \right)^{n(J)} \sup_{z \in J} \left\{ \max_{p \leqslant n(J)} \left| \left[\frac{d^p}{dt^p} \left(\frac{K(I, t)}{K(J, t)} \right) \right]_{t=z} \right| \right\}.$$

If we substitute this expression in (4.16) and use Lemma 7, we obtain

$$\left| \sum_{\lambda_n \in I} a_n K(I, \lambda_n) e^{(\lambda_{n_k} - \lambda_n)\eta} \right| \leqslant A\lambda_{n_k}^2 \{e^{-A_0\lambda_{n_k}} + e^{-\eta'\mu\lambda_{n_k}/2}\}$$

$$+ \sum_{J < I} \left(\frac{A\lambda_{n_k}}{\bar{n}(I, J)} \right)^{\bar{n}(I, J)} \left(\frac{\lambda_{n_k}}{Bn(J)} \right)^{n(J)} \left| \sum_{\lambda_n \in J} a_n K(J, \lambda_n) e^{(\lambda_{n_k} - \lambda_n)\eta} \right|. \qquad (4.17)$$

Since (4.17) holds for any choice of the interval $I \in T_k$, we may apply Lemma 3, which yields

$$\left| \sum_{\lambda_n \in I} a_n K(I, \lambda_n) e^{(\lambda_{n_k} - \lambda_n)\eta} \right| \leqslant A\lambda_{n_k}^2 \{e^{-A_0\lambda_{n_k}} + e^{-\eta'\mu\lambda_{n_k}/2}\} \cdot 2^{r(I)}$$

$$\times \operatorname*{supremum}_{W < V < \dots < J < I} \left[\left(\frac{A\lambda_{n_k}}{\bar{n}(I, J)} \right)^{\bar{n}(I, J)} \cdots \left(\frac{A\lambda_{n_k}}{\bar{n}(V, W)} \right)^{\bar{n}(V, W)} \cdot \left(\frac{\lambda_{n_k}}{n(J)_B} \right)^{n(J)} \cdots \left(\frac{\lambda_{n_k}}{n(W)_B} \right)^{n(W)} \right] \qquad (4.18)$$

where $r(I)$ denotes the number of intervals of S which are less than I. We shall obtain an upper bound for the supremum appearing in (4.18). Suppose that $W<V<\dots<J<I$ and let us write

$$\bar{m}(I)=\bar{n}(I,J)+\dots+\bar{n}(V,W)$$

and

$$m(I)=n(J)+\dots+n(V)+n(W). \tag{4.20}$$

The quantities $\bar{m}(I)$ and $m(I)$ depend on the choice of the intervals $J, \dots, V, W$ but there is a number C, depending only on the sequence $\{\lambda_n\}$, such that $\bar{m}(I)\leqslant C|I|$ and $m(I)\leqslant C|I|$. Next, we note that

$$\left(\frac{A\lambda_{n_k}}{\bar{n}(I,J)}\right)^{\bar{n}(I,J)}\dots\left(\frac{A\lambda_{n_k}}{\bar{n}(V,W)}\right)^{\bar{n}(V,W)}=\left(\frac{A\lambda_{n_k}}{\bar{m}(I)}\right)^{\bar{m}(I)}\frac{\bar{m}(I)!}{\bar{n}(I,J)!\dots\bar{n}(V,W)!}\{1+o(\lambda_{n_k})\} \tag{4.21}$$

and similarly

$$\left(\frac{\lambda_{n_k}}{Bn(J)}\right)^{n(J)}\dots\left(\frac{\lambda_{n_k}}{Bn(W)}\right)^{n(W)}=\left(\frac{\lambda_{n_k}}{Bm(I)}\right)^{m(I)}\frac{m(I)!}{n(J)!\dots n(W)!}\{1+o(\lambda_{n_k})\}. \tag{4.22}$$

Now Lemma 2 ensures that, if $K<J<I$, then $n(K)\leqslant n(J)/4\leqslant n(I)/16$ and hence, as is easily deduced, $\bar{n}(J,K)\leqslant\bar{n}(I,J)/2$. We are, therefore, entitled to use the following lemma, which is easily proved by induction.

Lemma 10. *If we are given n numbers, $a_1, \dots, a_n$ such that, for $1\leqslant r\leqslant n-1$, $0<a_r\leqslant a_{r+1}/2$, then*

$$\frac{(a_1+\dots+a_n)!}{a_1!\dots a_n!}\leqslant 4^{(a_1+\dots+a_n)}.$$

On applying Lemma 10, we obtain from (4.18), (4.21) and (4.22)

$$\left|\sum_{\lambda_n\in I}a_n K(I,\lambda_n)\,e^{(\lambda_{n_k}-\lambda_n)\eta}\right|\leqslant e^{o(\lambda_{n_k})}\{e^{-A_0\lambda_{n_k}}+e^{-\eta'\mu\lambda_{n_k}/2}\}. \tag{4.23}$$

Since (4.23) holds in a neighbourhood, we may apply Theorem 4 with $C_k(z,s)=e^{-zs}\,e^{(z-\lambda_{n_k})\eta}/K(I,z)$. We then use Lemma 6 and Leibnitz formula and we see that,

$$\left|\sum_{\lambda_n\in I}a_n e^{-\lambda_n s}\right|\leqslant e^{A_1\mu\lambda_{n_k}}\{e^{-A_0\lambda_{n_k}}+e^{-\eta'\mu\lambda_{n_k}/2}\}\,e^{(\xi_k-\lambda_{n_k})(\eta'-\sigma)}\,e^{-\lambda_{n_k}\sigma},\quad(\xi_k\in I). \tag{4.24}$$

We choose μ so small that $A_1\mu\leqslant A_0/3$ and then, keeping μ fixed, we choose η' larger than $6A_1$. Then

$$\left|\sum_{\lambda_n\in I}a_n e^{-\lambda_n s}\right|\leqslant\{e^{-2A_0\lambda_{n_k}/3}+e^{-\eta'\mu\lambda_{n_k}/3}\}\,e^{(\xi_k-\lambda_{n_k})(\eta'-\sigma)}\,e^{-\lambda_{n_k}\sigma}$$

and hence

$$\left|\sum_{\lambda_n \in I} a_n e^{-\lambda_n s}\right| \leqslant \{e^{-A_0 \lambda_{n_k}/3} + e^{-\eta' \mu \lambda_{n_k}/12}\} e^{-\lambda_{n_k} \sigma}, \tag{4.25}$$

provided that $\sigma > -\eta'$ and

$$\sup_{\xi_k \in I} (\xi_k - \lambda_{n_k}) \leqslant \lambda_{n_k} \min(\mu/12, A_0/6\eta'). \tag{4.26}$$

We denote by U_k, the set of $I \in T_k$ for which (4.26) holds. Then, by (4.25), the double series

$$\sum_{k=1}^{\infty} \sum_{I \in U_k} \left[\sum_{\lambda_n \in I} \{a_n e^{-\lambda_n s}\}\right]$$

converges in some half-plane of the form $\operatorname{Re}(s) > -d$, $(d > 0)$; here we have used Theorem 5. Since (4.26) gives us gaps of the required type, this suffices to prove Theorem 3.

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Tryckt den 17 december 1969

Uppsala 1969. Almqvist & Wiksells Boktryckeri AB