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The optimal number of faces in cubical complexes

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1. Introduction

Let a *cubical complex* C be a set of faces of an n-dimensional cube, such that if a face of dimension r, $1 \leqslant r \leqslant n$, belongs to C, then all lower dimensional faces of this r-face belong to C. J. B. Kruskal suggested in [7] the problem to optimize the number of s-dimensional faces for complexes which contain a fixed number of faces of dimension r. What is required is to determine the maximum possible number of s-faces if $r < s$ and the minimum possible number of s-faces if $r > s$ and the minimum possible number of s-faces if $r > s$ that C can have if the number of r-faces in C is given. In the special case $r=0$, $s=1$ this optimization problem has been solved by L. H. Harper and A. J. Bernstein in [4] and [1], and also by J. H. Lindsey II in [8].

For simplicial complexes a similar optimization problem was solved in full generality by J. B. Kruskal in [6]. G. Katona has also solved this problem in [5] not aware of Kruskal's solution. Another different solution can be obtained by the method used by G. F. Clements and B. Lindström in [2]. By a similar method I will be able to solve Kruskal's problem for cubical complexes for any r and s.

We shall also consider the problem to maximize a non-decreasing function of the dimensions of faces in a cubical complex and apply the result to a determinant defined by means of the Möbius function of the complex.

For the convenience of the reader we shall now give an outline of the method to be used in this paper.

A major step towards the solution of the problem is to find a suitable total ordering of all faces in the n-cube. Then we define the replacement operator R. If S_r is any set of cubical r-faces let RS_r be the $|S_r|$ first r-faces in the total ordering of faces ($|X|$ is the cardinality of the set X). We define the boundary operator ∂ such that ∂S_r is the set of all $(r-1)$-faces of elements in S_r.

The following inclusion is now crucial

$$\partial RS_r \subseteq R\partial S_r.$$

We shall prove this inclusion by induction over n, the dimension of the cube which contains the set S_r. To be able to use the induction hypothesis we have to introduce restricted replacement operators R_v, which operate in $n-1$ dimensions keeping the vth coordinate fixed. We first apply R_1 to S_r, then R_2 to $R_1 S_r$ etc. After applying R_n we apply R_1 and so on. It will turn out that the sets "converge" and we obtain a set T_r such that $R_v T_r = T_r$ for $v=1, \ldots, n$. In general is T_r distinct from RS_r. Therefore we have to an adjustment by replacing the first element in $RS_r - T_r$ by the last element in T_r and repeat this if necessary. Finally we determine the number of s-faces in the related complex $r(RS_r)$ as a function of the number of r-faces. The result can be expressed in terms of two functions introduced by J. B. Kruskal in [6] and [7].

2. Total ordering of the cubical faces

The vertices $v_i = (x_1, ..., x_n)$, $x_v = 0$ or 1 for $v = 1, ..., n$, are labelled such that

$$i = \sum_{v=1}^{n} x_v 2^{n-v},$$

or in binary notation $i = x_1 x_2 ... x_n$.

If $v_i = (x_1, ..., x_n)$ and $v_j = (y_1, ... y_n)$ are two vertices, then we shall write $v_i \prec v_j$ if $x_v \leqslant y_v$ for $v = 1, ..., n$ and $v_i \neq v_j$. This defines a partial ordering of all vertices of the n-cube.

Any face w_{ij} of the n-cube is determined by two vertices v_i and v_j such that $v_i \prec v_j$, and we shall also write $w_{ij} = (v_i, v_j)$. The vertices of the n-cube are considered as 0-faces and we write $v_i = w_{ii} = (v_i, v_i)$.

The faces of the n-cube are now totally ordered by $<$ such that

$$w_{i_1 j_1} < w_{i_2 j_2} \text{ if } j_1 < j_2 \text{ or } j_1 = j_2 \text{ and } i_2 < i_1,$$

where $j_1 < j_2$ and $i_2 < i_1$ are inequalities between integers.

3. Definition of the operators

The set of all $(r-1)$-faces of an r-face w_{ij} is called the *boundary* of w_{ij} and will be denoted by ∂w_{ij}. If S_r is any set of r-faces ∂S_r will be the set of all $(r-1)$-faces, which belong to at least one r-cube of S_r. $\partial^{-1} S_r$ will be the set of all $(r+1)$-faces all r-faces of which are in S_r. If the operator ∂ is applied p times to S_r, we shall write the result $\partial^p S_r$. $\partial^{-p} S_r$ is defined similarly. The boundary of a vertex is the zero set ϕ.

The operator ∂ applied to an arbitrary set of faces C yields ∂C. A cubical complex is a set C of faces such that $\partial C \subseteq C$. By the hull $h(C)$ of a cubical complex C we shall understand the set of faces w_{ij} for which $\partial w_{ij} \subseteq C$. $h(C)$ is then a cubical complex.

The number of elements in a set X is denoted by $|X|$. If we replace S_r by the $|S_r|$ first r-faces in the total ordering of faces, the resulting set of r-faces is denoted by RS_r.

If $v_i \prec v_j$ there are three possibilities concerning (x_v, y_v) for each $v = 1, ..., n: (x_v, y_v) = (0, 0), (1, 1)$ or $(0, 1)$. According to these we divide any set of r-faces S_r into three disjoint subsets $S_{r,v}(x_v, y_v)$. If we replace $S_{r,v}(x_v, y_v)$ for some fixed v by the $|S_{r,v}(x_v, y_v)|$ first r-faces in the total ordering, with the same combination (x_v, y_v) of coordinates, and take the union when $(x_v, y_v) = (0, 0), (1, 1), (0, 1)$, then the result will be denoted by $R_v S_r$.

4. The crucial inclusion*

The main result in this section is the following theorem:

Theorem 1. *Let S_r be a set or r-faces in the n-cube, $1 \leqslant r \leqslant n$. Let ∂ and R be the operators defined above. Then we have*

$$\partial R S_r \subseteq R \partial S_r.$$

* The proofs of Theorem 1 and Lemma 1, as presented in this section, have been essentially rewritten in August 1970.

From this theorem we obtain easily the following corollary, which shows how to find the optimal number of s-faces for cubical complexes for a given number of r-faces.

Corollary 1. $|\partial^p RS_r| \leqslant |\partial^p S_r|, \quad |\partial^{-p} S_r| \leqslant |\partial^{-p} RS_r|.$

Explicit formulas for the optimal number of faces will be derived in the last section of this paper.

Proof of Theorem 1. The proof will be by induction from $n-1$ to n. For $n=1$ and $n=2$ there is only a small number of cases to consider. We leave the proof of these cases for the reader.

The first few faces in the total ordering of faces are

$$w_{00}, w_{11}, w_{01}, w_{22}, w_{02}, w_{33}, w_{23}, w_{13}, w_{03}, \ldots$$

We may associate an ordinal number with each face. If S is any set of faces let $n(S)$ be the sum of all numbers associated with the faces in S.

Let S_r be an arbitrary set of r-faces in the n-cube. We apply the operators R_1, R_2, ..., R_n, R_1, ... one after the other and obtain

$$S_r^0 = S_r,\ S_r^1 = R_1 S_r,\ S_r^2 = R_2 S_r^1,\ \ldots,\ S_r^n = R_n S_r^{n-1},\ S_r^{n+1},\ = R_1 S_r^n,\ \ldots$$

If $S_r^{v+1} \neq S_r^v$ it follows that $n(S_r^{v+1}) < n(S_r^v)$. Since the sequence of integers $n(S_r^0)$, $n(S_r^1)$, ... etc. cannot decrease indefinitely there exists q such that

$$S_r^q = S_r^{q+1} = \ldots = S_r^{q+n}. \tag{4.1}$$

Assuming that the theorem is true for $(n-1)$-cubes, we shall now prove that

$$|\partial S_r^{v+1}| \leqslant |\partial S_r^v|, \quad v = 0, 1, 2, \ldots \tag{4.2}$$

There is no serious loss of generality if we assume that $v < n$.

If $(x_v, y_v) = (0, 0)$ or $(1, 1)$ then all r-cubes of $S_{r,v}^{v-1}(x_v, y_v)$ lie in an $(n-1)$-dimensional face, and we may apply the assumption on induction to obtain

$$\partial R_v S_{r,v}^{v-1}(x_v, y_v) \subseteq R_v \partial S_{r,v}^{v-1}(x_v, y_v) \tag{4.3}$$

Next we shall prove (4.3) also for $(x_v, y_v) = (0, 1)$. Each r-cube of $S_{r,v}^{v-1}(0, 1)$ has one $(r-1)$-face in each $(n-1)$-face for which $x_v = 0$ or $x_v = 1$. We choose the one with $x_v = 0$ to represent the r-cube. The remaining $(r-1)$-faces of the r-cube are represented by their $(r-2)$-faces in the $(n-1)$-face $x_v = 0$. The number of $(r-1)$-faces with $x_v = 0$ or $x_v = 1$ is not changed if we apply the operator R_v to the set of r-cubes. The number of $(r-2)$-faces of these $(r-1)$-cubes is not increased after applying R_v by the assumption on induction. Since they represent the above remaining $(r-1)$-faces, it follows that the number of these $(r-1)$-faces is not increased, and (4.3) follows for $(x_v, y_v) = (0, 1)$. We have proved (4.3), and (4.2) (for $v-1$) then follows by the definition of S_r^v and S_r^{v+1}.

From (4.1) and (4.2) we find that

$$R_v S_r^q = S_r^q, \quad v = 1, 2, \ldots, n \tag{4.4}$$

and

$$|\partial S_r^q| \leqslant |\partial S_r|. \tag{4.5}$$

In general is $RS_r^q \neq S_r^q$ so we must show how to get $RS_r^q = RS_r$ from S_r^q without increasing the number of $(r-1)$-faces of the boundary of the set.

Let $(a, a') = (a_1 \dots a_n, a'_1 \dots a'_n)$ be the last r-cube of S_r^q in the total ordering of faces. Let $(b, b') = (b_1 \dots b_n, b'_1 \dots b'_n)$ be the first r-cube not in S_r^q. Hence we have

$$(a, a') \in S_r^q, \ (b, b') \notin S_r^q. \tag{4.6}$$

Since $RS_r^q \neq S_r^q$ by the assumption, it follows that

$$(b, b') < (a, a') \tag{4.7}$$

First we assume that $a' = b'$. By (4.4), (4.6) and (4.7) it follows that $a_v \neq b_v (v = 1, \dots, n)$ and then, since $a_v \leqslant a'_v$ and $b_v \leqslant b'_v$, we get $a' = b' = 11 \dots 1$. We apply Lemma 4 in [2] to the complements of a and b (i.e. $a' - a$ and $b' - b$) and obtain $a_n = 0$, $b_n = 1$. Since $b_n = b'_n = 1$, it follows that every $(r-1)$-cube in $\partial(b, b')$ belongs to the boundary of an r-cube preceding (b, b') in the total ordering. It follows that $\partial(b, b')$ is contained in the boundary of $S_r^q - \{(a, a')\}$, since (b, b') is the first r-cube that is not in S_r^q. The number of $(r-1)$-cubes in the boundary is therefore not increased when (b, b') is adjoined to S_r^q and (a, a') is deleted.

Next assume that

$$a' > b'. \tag{4.8}$$

If $a'_j > b'_j$ and $a'_k > b'_k$ for two indices $j < k$ and if $a_k = a'_k$ then we get the following inequalities between r-faces (note that $b_k = b'_k = 0$)

$$(b, b') < (b_1 \dots b_{k-1} a_k b_{k+1} \dots b_n, b'_1 \dots b'_{k-1} a'_k b'_{k+1} \dots b'_n) < (a, a'). \tag{4.9}$$

These inequalities are in contradiction to (4.6) by (4.4) (for $v = 1$, k or n). A contradiction is also obtained when $(a_k, a'_k) = (0, 1)$ if we replace one pair $(b_i, b'_i) = (0, 1)$ by the pair $(1, 1)$ in order to keep the dimension r of the intermediary face in (4.9). Therefore it follows that

$$a'_\gamma > b'_\gamma \quad \text{for only one index } \gamma. \tag{4.10}$$

Let α be the largest index for which $a_\alpha = a'_\alpha$. We shall prove that $2(n-\alpha)$ $(r-1)$-cubes will disappear from the boundary of $S_r^q \cup \{(b, b')\}$ when (a, a') is deleted.

Let $(a_v, a'_v) = (0, 1)$ for an index $v < \alpha$. We obtain two $(r-1)$-faces of $\partial(a, a')$ by increasing a_v to 1 or decreasing a'_v to 0. But these two $(r-1)$-faces are even in the boundary of $S_r^q - \{(a, a')\}$. For if we replace (a_α, a'_α) by $(0, 1)$ and increase a_v to 1 or decrease a'_v to 0, then we find two r-faces which precede (a, a') in the total ordering of faces with at least one component (a_i, a'_i), $i \neq \alpha, v$, in common with (a, a'). These two r-faces belong to $S_r^q - \{(a, a')\}$ by (4.4) $(v = i)$ and the union of their boundaries contains the two $(r-1)$-faces in question.

If $i > \alpha$ then $(a_i, a'_i) = (0, 1)$. We find two $(r-1)$-faces of $\partial(a, a')$ by increasing a_i to 1 or decreasing a'_i to 0. If one or these two $(r-1)$-cubes is in the boundary of an r-cube $(c, c') \neq (a, a')$, then $c_k < c'_k$ and $a_k = a'_k$ for an index $k < \alpha$ and $(c_j, c'_j) = (a_j, a'_j)$ for $j \neq i, k$. Since $(c, c') > (a, a') > (b, b')$ and (a, a') is largest in S_r^q, it follows that the two $(r-1)$-faces are not in the boundary of $S_r^q \cup \{(b, b')\}$ even. We conclude that $2(n-\alpha)$ $(r-1)$-cubes disappear from the boundary of $S_r^q \cup \{(b, b')\}$ when (a, a') is deleted from this set. If $a_i \neq a'_i$ for all indices i, then the same conclusion holds as before if we put $\alpha = 0$.

Let β be the largest index for which $b_\beta = b'_\beta$. By the assumption that (b, b') is the first r-cube not in S_r^q, it follows that $(c, c') < (b, b')$ implies $(c, c') \in S_r^q$ when (c, c') is an r-cube. As in the above proof for (a, a') we find that indices $i < \beta$ do not contribute with new $(r-1)$-faces when (b, b') is adjoined to $S_r^q - \{(a, a')\}$. It follows that the number of $(r-1)$-faces is increased by $2(n-\beta)$ at most when (b, b') is adjoined to $S_r^q - \{(a, a')\}$.

The difference between the number of $(r-1)$-cubes in the boundary of $S_r^q \cup \{(b, b')\} - \{(a, a')\}$ and the number of $(r-1)$-cubes in ∂S_r^q is $2(\alpha - \beta)$ at most. In order to prove that the number of $(r-1)$-faces is not increased by adjoining (b, b') and deleting (a, a') simultaneously, it is sufficient to prove that

$$\alpha \leqslant \beta \tag{4.11}$$

We shall prove that the inequality $\alpha > \beta$ is contradictory. By the definition of β we find that

$$(b_\alpha, b'_\alpha) = (0, 1), \quad \text{if} \quad \alpha > \beta. \tag{4.12}$$

By (4.10) is $b_\gamma = b'_\gamma = 0$, whence $\beta \geqslant \gamma$. Assume that $\beta > \gamma$ and define the r-face (d, d') by (an index below a numeral indicates its position)

$$\begin{gathered}(d, d') = (b_1 \,..\, \underset{\gamma}{0} \,..\, \underset{\beta}{0} \,..\, \underset{\alpha}{\mathrm{A}} \,..\,, b'_1 \,..\, \underset{\gamma}{0} \,..\, \underset{\beta}{1} \,..\, \underset{\alpha}{\mathrm{A}} \,..\,),\\ (d_i, d'_i) = (b_i, b'_i) \text{ for } i \neq \alpha, \beta\\ d_\alpha = d'_\alpha = a_\alpha = a'_\alpha = A.\end{gathered} \tag{4.13}$$

We find then by (4.10) and since $b_\beta = b'_\beta$ that

$$(a, a') > (d, d') > (b, b'),$$

which is in contradiction to (4.6) by (4.4) ($v = \alpha, \gamma$). Hence we conclude that

$$\beta = \gamma \tag{4.14}$$

By (4.10) is $a'_i \leqslant b'_i$ for $i \neq \gamma$. If $b'_i = 0$ for an index $i \neq \gamma$, then it follows that $(a_i, a'_i) = (b_i, b'_i) = (0, 0)$, which is impossible by (4.4), (4.6) and (4.7). Hence

$$b'_i = 1 \quad \text{for} \quad i \neq \gamma, \quad b'_\gamma = 0. \tag{4.15}$$

By (4.8) and (4.10) it follows that

$$a'_i = b'_i \quad \text{for} \quad i = 1, \ldots, \gamma - 1.$$

We combine this fact with (4.15) and (4.10) and get

$$a'_i = 1 \quad \text{for} \quad i = 1, \ldots, \gamma. \tag{4.16}$$

By (4.4), (4.6), (4.7) and (4.14) it follows that

$$b_i < b'_i, \; a_i = a'_i \quad \text{for} \quad i = \gamma + 1, \ldots, n. \tag{4.17}$$

We shall prove these relations also for $i = 1, \ldots, \gamma - 1$. Assume that one could find an index k such that

$$b_k = b'_k, \; k < \gamma.$$

By (4.15) is then $b_k = b'_k = 1$, and it follows by (4.4), (4.6) and (4.16) that

$$(a_k, a'_k) = (0, 1).$$

We define an r-cube (f, f') by

$$(f, f') = (b_1 \,..\, \underset{k}{0} \,..\, \underset{\gamma}{0} \,..\, \underset{\alpha}{1} \,....,1 \,..\, \underset{k}{1} \,..\, 1\underset{\gamma}{0}1 \,..\, \underset{\alpha}{1} \,..\, 1),$$

$$(f_i, f'_i) = (b_i, b'_i) \quad \text{for} \quad i \neq k, \alpha.$$

Then we find by (4.15), (4.16) and (4.17) that

$$(a, a') > (f, f') > (b, b').$$

These inequalities are in contradiction to (4.6) by (4.4) (for $v = k, \gamma$) since $(a_k, a'_k) = (0, 1)$. Hence, the relation $b_k = b'_k$ is false, and (4.17) holds for every $i \neq \gamma$.

It follows that $r = \dim (a, a') \leq 1$ and $r = \dim (b, b') \geq n-1$, which is a contradiction since $n \geq 3$. This final contradiction shows that the relation $\alpha > \beta$ is false and (4.11) is proved.

Let $T = S_r^q \cup \{(b, b')\} - \{(a, a')\}$. We have proved above that $|\partial T| \leq |\partial S_r^q|$. It is easy to see that $R_v T = T$ for $v = 1, 2 \ldots, n$, by the definition of T. Hence, we can iterate the adjunction and deletion with T in place of S_r^q if $RT \neq T$ and repeat until we finally obtain the set RS_r. We obtain (cf. (4.5))

$$|\partial RS_r| \leq |\partial S_r|.$$

Theorem 1 follows from the last inequality if $R\partial RS_r = \partial RS_r$. This is true by the following lemma.

Lemma 1. If $RS_r = S_r$, then $R\partial S_r = \partial S_r$.

Proof of Lemma 1. Let (e, e') and (f, f') be two $(r-1)$-cubes such that $(e, e') < (f, f')$ and $(f, f') \in \partial(g, g')$, where $(g, g') \in S_r$. We shall prove that $(e, e') \in \partial S_r$. Consider two cases.

Case 1. Assume that $e' = g'$, whence $e' = f' = g'$. There is now an index α such that $f_\alpha = g_\alpha + 1$ and $f_v = g_v$ for $v \neq \alpha$. We define β such that $e_v = f_v$ for $v < \beta$ and $e_\beta > f_\beta$.

If $\alpha < \beta$ then we find that $e_1 \ldots (e_\alpha - 1) \ldots e_n > g$, hence $(e_1 \ldots (e_\alpha - 1) \ldots e_n, e') \in S_r$ and $(e, e') \in \partial S_r$.

If $\alpha > \beta$ and $e_\gamma = 1$ for an index $\gamma > \beta$, then it follows that

$$e_1 \ldots (e_\gamma - 1) \ldots e_n > f_1 \ldots (f_\alpha - 1) \ldots f_n = g, \text{ and } (e, e') \in \partial S_r.$$

If $\alpha > \beta$ and $e_v = 0$ for all $v > \beta$, then we find that $f_v = 0$ for $v > \beta$ except when $v = \alpha$ (e and f have the same numer of 1's). It follows that $g = e_1 \ldots e_{\beta-1} 0 \ldots 0$, and $(e, e') \in \partial(g, g')$ since $e_\beta = 1$.

Case 2. We shall next assume that $e' < g'$. If there is an index v such that $e_v = e'_v = 1$, then $(e, e') \in \partial S_r$ follows since $(e, e') \in \partial(e_1 \ldots \underset{v}{0} \ldots e_n, e'_1 \ldots \underset{v}{1} \ldots e'_n)$ and since

$$(e_1 \ldots \underset{v}{0} \ldots e_n, e'_1 \ldots \underset{v}{1} \ldots e'_n) < (g, g').$$

We now make the assumption that $(e_v, e'_v) = (0, 0)$ or $(0, 1)$ for $v = 1, \ldots, n$. We define α such that $e'_v = g'_v$ for $v < \alpha$ and $e'_\alpha < g'_\alpha$. If $(e_v, e'_v) = (0, 0)$ for some $v > \alpha$, then we obtain $(e, e') \in \partial S_r$ as before.

Let $(e_v, e'_v) = (0, 1)$ for every $v > \alpha$. Then we have

$$(e, e') = (0 \ldots 0, g'_1 \ldots g'_{\alpha-1} 01 \,..\, 1).$$

The dimension of (e, e') is $r-1$ and the dimension of (g, g') is r. Since $g'_\alpha = 1$, it follows then that

$$(g, g') = (0 \dots 0, g'_1 \dots g'_{\alpha-1} 1 \dots 1).$$

Hence $(e, e') \in \partial(g, g')$, and the lemma is proved.

This completes the proof of Theorem 1.

Proof of Corollary 1. By repeated application of the operator ∂ it follows from the theorem that

$$\partial^p RS_r \subseteq R\partial^p S_r, \quad p = 1, \dots, r. \tag{4.18}$$

In particular is $|\partial^p RS_r| \leqslant |\partial^p S_r|$ for $p = 1, \dots r$.

In order to prove the second inequality we first observe that

$$\partial^{-1} \partial S_r = S_r, \; \partial\partial^{-1} S_r \subseteq S_r. \tag{4.19}$$

If we replace S_r by $\partial^{-1} S_r$ in the theorem, we find that

$$\partial R \partial^{-1} S_r \subseteq R\partial\partial^{-1} S_r \subseteq RS_r. \tag{4.20}$$

If we apply ∂^{-1} in both members of (4.20), we find by (4.19) that

$$R\partial^{-1} S_r \subseteq \partial^{-1} RS_r. \tag{4.21}$$

Iterated application of (4.21) then yields the second inequality in Corrollary 1.

We shall apply Theorem 1 to prove an extremal property for cubical complexes. A similar result has been proved for simplicial complexes (see Theorem 1 in [10] and Corollary 3 in [2]).

Let $f(n_1, n_2, \dots, n_m)$ be a fucntion which is *symmetric* in its arguments, i. e. invariant by any permutation of its arguments. If $n_1, n_2, \dots, n_m$ are the dimensions of faces in a cubical complex C, we shall write briefly $f(C) = f(n_1, n_2, \dots, n_m)$. $f(C)$ is *non-decreasing* if it is non-decreasing in each of its arguments. The complex of the $|C|$ first cubical faces is denoted by FC.

Theorem 2. *Let $f(C)$ be a symmetric and non-decreasing function defined for cubical complexes of a fixed number of faces. Then we have*

$$f(C) \leqslant f(FC).$$

Proof. Let C_r be the subset of all faces of dimension r in C. Let

$$RC = \bigcup_{r=0}^{n} RC_r, \tag{4.22}$$

where RC_r denotes the $|C_r|$ first cubes of dimension r. RC is a cubical complex for we have by Theorem 1

$$\partial \bigcup_{r=0}^{n} RC_r = \bigcup_{r=0}^{n} \partial RC_r \subseteq \bigcup_{r=0}^{n} R\partial C_r \subseteq \bigcup_{r=0}^{n} RC_{r-1} \subseteq RC.$$

Let (a, a') be the last element in RC and let (b, b') be the first cube which is not in RC. If $RC \neq FC$ we find that $(b, b') < (a, a')$ and

$$\dim (a, a') \neq \dim (b, b'). \tag{4.23}$$

We want to prove that $\dim (a, a') < \dim (b, b')$ by a contradiction.

Assume that $\dim (a, a') > \dim (b, b')$, i. e.

$$\sum_{v=1}^{n} (a_v' - a_v) > \sum_{v=1}^{n} (b_v' - b_v). \tag{4.24}$$

If $a' > b'$ we can find k such that $a_k' > b_k'$ and $a_v' = b_v'$ for $v = 1, \dots, k-1$.

We define a function $g(v)$ such that $g(v) = 0$ if $(a_v, a_v') \geqslant (b_v, b_v')$ and $g(v) = 1$ in the other case.

If $g(v) = 0$ for $v = 1, \dots, n$ then (b, b') would be a face of (a, a') and $(b, b') \in RC$, since $(a, a') \in RC$ and RC is a complex. But $(b, b') \notin RC$, so there exists v such that $g(v) = 1$. By (4.24) it follows that

$$\sum_{v:g(v)=0} [(a_v' - a_v) - (b_v' - b_v)] > \sum_{v:g(v)=1} [(b_v' - b_v) - (a_v' - a_v)] \tag{4.25}$$

Because of (4.25) we can find $(c, c') = (c_1 \dots c_n, c_1' \dots c_n')$ such that

$$(b_v, b_v') \leqslant (c_v, c_v') \leqslant (a_v, a_v') \quad \text{if} \quad g(v) = 0,$$

$$c_k' = a_k' > b_k', \; c_v' \geqslant b_v' \quad \text{for} \quad v < k$$

$$(c_v, c_v') = (a_v, a_v') \quad \text{if} \quad g(v) = 1,$$

$$\sum_{v:g(v)=0} [(c_v' - c_v) - (b_v' - b_v)] = \sum_{v:g(v)=1} [(b_v' - b_v) - (a_v' - a_v)].$$

It follows that $\dim (b, b') = \dim (c, c')$ and $(c, c') > (b, b')$. Since $(c_v, c_v') \leqslant (a_v, a_v')$ for $v = 1, \dots, n$ and $(a, a') \in RC$, it follows that $(c, c') \in RC$ and then $(b, b') \in RC$ by (4.22). Since $(b, b') \notin RC$ we have the desired contradiction and the assumption $\dim (a, a') > \dim (b, b')$ is wrong. Since we cannot have $\dim (a, a') = \dim (b, b')$ by (4.23), we find that

$$\dim (a, a') < \dim (b, b'). \tag{4.26}$$

If $a' = b'$ then the proof of (4.26) parallels the proof of Corollary 3 on p. 233 in [2] and can be omitted here.

If (a, a') is deleted from RC and (b, b') is adjoined, we get a cubical complex K with $f(RC) \leqslant f(K)$. If $K \neq FC$ we can repeat the procedure until we obtain FC. It follows that

$$f(C) = f(RC) \leqslant f(FC),$$

which was to be proved.

As an application of Theorem 2, we shall optimize certain (± 1)-determinants defined with the aid of cubical complexes.

Example

A cubical complex C with the empty set ϕ adjoined is a semilattice with intersection of faces as product operation. It is easy to see by our representation of faces as pairs

(a, a') that C is a subsemilattice of the interval lattice of the (semi-)lattice of all subsets of some finite set. The Möbius function of the last mentioned lattice takes only the values 1 and -1 (see Corollary to Proposition 5 in [11]). From Theorem 6 in [3] it follows then that the Möbius function of our semilattice of cubical faces (and ϕ) assumes the values 1 and -1 and no other values. With the aid of the Corollary in [9], we obtain a determinant of the order $m=|C|+1$, with all entries 1 or -1 and the value

$$\prod_{v=1}^{m-1}(3^{r_v}+1) \tag{4.27}$$

where $r_1, r_2, \ldots, r_{m-1}$ are the dimensions of all faces in C (3^{r_v} is the number of subfaces of the vth face). The function (4.27) is symmetric and increasing function of its arguments. Theorem 2 shows how to find the cubical complex with $m-1$ faces, which maximizes (4.27).

5. A formula for the optimal number of faces

In [7] J. B. Kruskal conjectured a formula for the optimal number of faces (see Introduction). We shall here derive such a formula with the aid of our Theorem 1. It will be apparent that our result agrees with Kruskal's conjecture in some instances, but disagrees with it in other instances. It will be possible to formulate our results in terms of functions, which have been introduced by J. B. Kruskal.

For any positive integer m we define the *r-canonical representation* (see [6])

$$m=\binom{m_1}{r}+\binom{m_2}{r-1}+\ldots+\binom{m_i}{r-i+1}, \tag{5.1}$$

where we first choose m_1 as large as possible when $\binom{m_1}{r}\leqslant m$, and then we choose m_2 as large as possible such that $\binom{m_1}{r}+\binom{m_2}{r-1}\leqslant m$ and so on until we finally obtain equality. Then we have (a consequense of Pascal's triangle)

$$m_1>m_2>\ldots>m_i\geqslant r-i+1\geqslant 1. \tag{5.2}$$

It is easy to prove that the representation (5.1) is unique, when (5.2) holds (see Lemma 1 in [5]).

For *any* sequence of positive integers $m_1, m_2, \ldots, m_i$, we put

$$[m_1, m_2, \ldots, m_i]_r=\binom{m_1}{r}+\binom{m_2}{r-1}+\ldots+\binom{m_i}{r-i+1}. \tag{5.3}$$

Kruskal defines the *fractional pseudopower* (see [6] p. 253) $m^{(s/r)}$ as follows: If

$$m=[m_1, m_2, \ldots, m_i]_r$$

is the r-canonical representation of m, then we let

$$m^{(s/r)}=[m_1, m_2, \ldots, m_i]_s \tag{5.4}$$

For $m=0$ we put

$$0^{(s/r)}=0. \tag{5.5}$$

A *binary word* of length n is a sequence of n 0's and 1's. Write

$$a_1 \dots a_n \subseteq b_1 \dots b_n,$$

if $a_i \leqslant b_i$ for $i=1, \dots, n$. A *binary complex* C is a set of binary words of length n, with the property that if b is in C and $a \subseteq b$, then a is in C. The main result in [6] is the following theorem:

Kruskal's theorem

If a binary complex C has exactly m words of weight r, and if $s>r\,(s<r)$, then the maximum (minimum) number of integers of weight s that it can have is $m^{(s/r)}$.

We may think of a binary word as the binary notation of a non-negative integer, so we shall use the words »integer» and »binary word» as synonyms.

The *weight* of an integer is the number of 1's in its binary word. We shall omit the easy proof of the following lemma:

Lemma 2. *Let $m_1>m_2> \dots >m_k$ be non negative integers. Then the number of integers of weight r, which are less than $2^{m_1}+ \dots + 2^{m_k}$, is $[m_1, \dots, m_k]_r$.*

If S is some set of binary words of weight r then we define the set $r(S)$ of *related words* as the set of all words j such that either $j\subseteq i$ for some $i\in S$, or $i\in S$ for each $i\subset j$ (i. e. $i\subseteq j$, $i\neq j$), if i's weight is r.

We shall need the following lemma from which, by the way, Kruskal's theorem follows, if we apply Theorem 1 in [2].

Lemma 3. *Let S be the m first non-negative integers of weight r. Then the number of integers of weight s in $r(S)$ is $m^{(s/r)}$, and $r(S)$ is the set of all non-negative integers, which are less than $2^{m_1}+ \dots +2^{m_k}$, if $m=[m_1, \dots, m_k]_r$ is the r-canonical representation of m.*

Proof. The result follows from Lemma 2, if we show that $r(S)$ is the set of all non-negative integers, which are less than $M=2^{m_1}+ \dots +2^{m_k}$.

We first observe that $i\in r(S)$ implies $i<M$, if the weight of i is less than r, for by the definition of $r(S)$ there is $h\in S$ such that $i\subseteq h$, and $i\leqslant h< M$ by Lemma 2.

Then we assume that $i\in r(S)$ and that the weight of i is s and $s>r$. If $i<M$ were not true, then for some integers $n_1>n_2> \dots n_s>0$, we get

$$i=2^{n_1}+\dots+2^{n_s}\geqslant 2^{m_1}+\dots+2^{m_k}=M.$$

Put $j=2^{n_1}+\dots+2^{n_r}$. Since $j\subset i$, $i\in r(S)$ and the weight of j is r, it follows that $j\in S$. Hence $j<M$ by Lemma 2. From $i\geqslant M$ and $j<M$ it follows that $n_1=m_1$, $n_2=m_2, \dots, n_r=m_r$ and $k>r$. But we have $k\leqslant r$ by the r-canonical representation of m and by (5.2). We have arrived at a contradiction and $i<M$ must be true.

We have proved that $i\in r(S)$ implies $i<M$ and shall now prove the reversed implication.

Let $i < M$ be non-negative of weights $s < r$. Then we have for some index $v < r$ the inequalities

$$2^{m_1} + \ldots + 2^{m_v} \leqslant i < 2^{m_1} + \ldots + 2^{m_{v+1}}, \quad \text{or} \quad i < 2^{m_1}.$$

In the first case we have $m_{v+1} \geqslant r - v \geqslant 1$ by (5.2) and it follows that we can find h of weight r such that $i \subset h$ and

$$2^{m_1} + \ldots + 2^{m_v} < h < 2^{m_1} + \ldots + 2^{m_{v+1}}.$$

Hence $h \in S$ and $i \in r(S)$ follows by the definition of $r(S)$. The second case, when $i < 2^{m_1}$, is proved similarly.

If $i < M$ is non-negative of weight $s > r$ then $h < M$ if $h \subset i$. We conclude that $i \in r(S)$ by the definition of $r(S)$, and the proof of the lemma is completed.

Our next auxiliary function is

$$g(r, i, m) = \sum_{v=0}^{r} \binom{m}{r-v} \binom{i-1}{v} 2^{m-(r-v)}, \tag{5.6}$$

which was introduced by J. B. Kruskal in [7] for the problem we are studying. There is another expression for $g(r, i, m)$ (see (5.7) below), which is more useful in this paper.

With the aid of the well-known relations

$$\binom{v+i-1}{r} = \sum_{j=0}^{i-1} \binom{v}{r-j} \binom{i-1}{j}$$

and

$$\binom{m}{v} \binom{v}{r-j} = \binom{m}{r-j} \binom{m-(r-j)}{v-(r-j)},$$

it is easily proved that

$$g(r, i, m) = \sum_{v=0}^{m} \binom{m}{v} \binom{v+i-1}{r}. \tag{5.7}$$

It is also easy to prove that

$$g(r, i, m) + g(r, i+1, m) = g(r, i, m+1) \tag{5.8}$$

Given an integer $m > 0$, we determine $m_1, m_2, \ldots, m_{\bar{k}}$ such that

$$m \geqslant g(r, 1, m_1) + \ldots + g(r, k, m_k) = m_{(r/r)}, \tag{5.9}$$

where we first choose m_1 as large as possible such that $m \geqslant g(r, 1, m_1)$, then m_2 as large as possible such that $m \geqslant g(r, 1, m_1) + g(r, 2, m_2)$ and so on in analogy with the definition of the r-canonical representation of m. Because of the relation (5.8) we get (cf. the analogous (5.2)):

$$m_1 > m_2 > \ldots > m_k \geqslant r - k + 1. \tag{5.10}$$

Observe that we have equality in (5.1) and that this is not always the case in (5.9). For the sequence defined in (5.9) we shall write

$$g(s, 1, m_1) + \ldots + g(s, k, m_n) = m_{(s/r)}, \tag{5.11}$$

for any non-negative integer s ($g(s, k, m_i)$ is non-zero if and only if $s - i + 1 \leqslant m_i$).

The *lower semipower* $m_{(s/r)}$ is defined in analogy with Kruskal's definition of the (upper) semipower $m^{(s/r)}$.

We can now state our main result on the optimal number of faces in cubical complexes:

Theorem 3. *If a cubical complex C has m faces of dimension r, and if $s > r$ ($s < r$), then the maximum (minimum) number of faces of dimension s that it can have is*

$$m_{(s/r)} + (m - m_{(r/r)})^{(s/r)}.$$

If S is a set of cubical faces of dimension r then we define $r(S)$, the set of related faces w_{uv} such that w_{uv} is either a subfaces of some $w_{ij} \in S$, or $w_{ij} \in S$ for each subface w_{ij} of w_{uv} if the dimension of w_{uv} is r. Hence

$$r(S) = \bigcup_{p=-r}^{\infty} \partial^p(S).$$

The following two lemmas are analogous to Lemma 2 and Lemma 3.

Lemma 4. *Let $m_1 > m_2 > \ldots > m_k$ be non-negative integers. Then the number of r-cubes w_{ij}, with $j < 2^{m_1} + \ldots + 2^{m_k}$,* is

$$g(r, 1, m_1) + \ldots + g(r, k, m_k).$$

Lemma 5. *Assume that $m = m_{(r/r)}$ and let S be the m first r-cubes w_{ij} in our total ordering of cubes. $r(S)$ is then the set of all cubes w_{ij} for which $j < 2^{m_1} + \ldots + 2^{m_k}$. The number of s-cubes in $r(S)$ is $m_{(s/r)}$.*

The proofs of these two lemmas are analogous to the proofs of Lemma 2 and Lemma 3 and will be omitted. After this preparation we can prove our main result.

Proof of Theorem 3. By Corollary 1 to Theorem 1 we can assume that S_r contains the $m = |S_r|$ first r-cubes. If $m = m_{(r/r)}$ then the conclusion in Theorem 3 holds by Lemma 4 and by (5.5).

We can now assume that $m > m_{(r/r)} = g(r, 1, m_1) + \ldots + g(r, k, m_k)$. Put $M = 2^{m_1} + \ldots + 2^{m_k}$. $r(S)$ contains all faces w_{ij} for which $j < M$ and in addition some faces w_{iM}.

The number of r-faces w_{iM} in S is $m - m_{(r/r)}$ by Lemma 4. Observe that $w_{uM} \in r(S)$ if and only if w_{uM} is a subface of some r-face $w_{iM} \in S$ or $w_{iM} \in S$ for each subface w_{iM} of dimension r of w_{uM}. This means that $w_{uM} \in r(S)$ if and only if $M - u \subseteq M - i$ for some $w_{iM} \in S$ or $M_{iM} \in S$ for each $M - i \subset M - u$ if the weight of $M - i$ is r. The $m - m_{(r/r)}$ first r-faces w_{iM} correspond to the $m - m_{(r/r)}$ first integers $M - i$ of weight r. The theorem now follows by Lemma 3.

When $m = m_{(r/r)}$ our result in Theorem 4 agrees with the conjecture by J. B. Kruskal in [7]. But for $m \neq m_{(r/r)}$ we have got an "error-term" not foreseen by Kruskal. We conclude by a simple example. Suppose we have four 2-faces in the 3-cube. The minimum number of edges (1-faces) contained in these 2-faces is 11 by our Theorem 4. From the last statement in [7] it would follow that the minimum is 12.

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Dr. Klaus Leeb has informed me in a letter of July 20, 1970 that he has proved the main result (Theorem 3) in an old paper. Dr. Leeb's paper is probably unpublished. I quote from Dr. Leeb's letter: "Yesterday I read your abstract in AMS Notices. I suppose you not only did complexes over powers of 2, but general for k. I did the same thing when reading about Harper-codes in an old paper."

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