ARKIV FÖR MATEMATIK Band 8 nr 26

1.70 10 1 Communicated 3 June 1970 by OTTO FROSTMAN

# Translation invariant convex metrics

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## 1. Introduction

In this paper we shall consider metric linear topological spaces with a metric that is translation invariant and convex. A metric $d$ on a set of points $M$ will be called convex if for all points $x, y \in M$ and for all $\delta_1 \geqslant 0$, $\delta_2 \geqslant 0$ such that $\delta_1 + \delta_2 = d(x, y)$ there exists a point $z \in M$ with $d(x, z) = \delta_1$ and $d(z, y) = \delta_2$.

The concept of a convex metric was first introduced by Menger [3]. According to his definition a metric space $M$ is convex if for all points $x, y \in M$ there exists a metric midpoint (i.e. $\delta_1 = \delta_2$ above).

Only slight modifications are needed to make our proofs valid under the weaker assumption of midpoint convexity. If the space is complete it follows from the theorems of Menger that the definitions are equivalent.

It can be proved that an invariant convex metric defining the topology of a linear topological space is a norm if and only if the spheres around the origin are convex. On the other hand such a space might have spheres which are not convex even if it is locally convex. Rådström [5] has given an example of this in $L^1$ [0, 1]. However, assuming local convexity we shall show that the space is isomorphic to a normed space. We also give an explicit expression of an equivalent norm in the convex metric. Furthermore we shall show that if the space is reflexive and separable it is isometric to a Banach space.

## 2. Auxiliary theorems on convex metrics

If $A$ and $B$ are subsets of a linear space $X$ we write $A + B = \{x \in X;\ x = a + b,\ a \in A, b \in B\}$. A family of subsets of $X$ will form a commutative semigroup if it is closed under this addition. Such a semigroup will be called a one-parameter semigroup if there is an application $\delta \to A(\delta)$ from the positive real numbers onto the semigroup satisfying

$$A(\delta_1 + \delta_2) = A(\delta_1) + A(\delta_2) \tag{1}$$

The definition of a one-parameter semigroup usually contains some assumption of continuity, but this will not be needed here and is omitted.

Let $X$ be a metric linear topological space with a metric $d$ that is translation invariant. Denote the open sphere of radius $\delta$ around the origin by $S(\delta)$. It then follows (see [4]) that the metric is convex if and only if the sets $\overline{S(\delta)}$ form a one-parameter semigroup, the bar denoting the closure. We need the "only if" part of this statement for the open spheres.

**Lemma 1.** *If $Z$ is a metric linear topological space, whose metric is translation invariant and convex, its open spheres around the origin form a one-parameter semigroup.*

*Proof.* $S(\delta_1+\delta_2)\supset S(\delta_1)+S(\delta_2)$ follows immediately from the triangle inequality. Suppose now that $x\in S(\delta_1+\delta_2)$. Choose $r_1$ and $r_2$ such that $r_1<\delta_1$, $r_2<\delta_2$ and $r_1+r_2=|x|$, where $|x|$ denotes $d(x, 0)$. Because of the convexity of the metric we can find $x_1$ and $x_2$ such that $|x_1|=r_1$, $|x_2|=r_2$ and $x_1+x_2=x$. But then $x_1\in S(\delta_1)$ and $x_2\in S(\delta_2)$, which proves the lemma.

If we apply the theorems of [4] to a linear space $X$ with the above properties it follows that the convex metric is a norm if and only if the spheres $S(\delta)$ are convex. Here we give a simple direct proof of this statement.

Suppose that the spheres are convex and let $x\in X$. As the metric is convex there exist points $y, z\in X$ such that $x=y+z$ and $|y|=|z|=\frac{1}{2}|x|$. Consequently $\frac{1}{2}x=\frac{1}{2}y+\frac{1}{2}z$ and $\frac{1}{2}x\in\overline{S(\frac{1}{2}|x|)}$ since the closure of a convex set is convex. This implies $|\frac{1}{2}x|\leqslant\frac{1}{2}|x|$. On the other hand the triangle inequality gives $|\frac{1}{2}x|\geqslant\frac{1}{2}|x|$ that is $|\frac{1}{2}x|=\frac{1}{2}|x|$ for all $x\in X$.

Then $2^{-n}|x|=|2^{-n}x|$ for all $n\geqslant 0$. Moreover $|x|=|2^{-n}2^nx|=2^{-n}|2^nx|$ that is $2^n|x|=|2^nx|$. Now if $k\geqslant 1$ and if $|kx|=k|x|$ then $k|x|=|kx|\leqslant|(k-1)x|+|x|\leqslant(k-1)\,|x|+|x|=k\,|x|$ so $(k-1)\,|x|=|(k-1)x|$. Hence $|kx|=k|x|$ for all $k\geqslant 0$ by induction. From this and the fact that $|-x|=|x|$ we infer that $|rx|=|r|\,|x|$ for all rational numbers of the form $r=m2^{-n}$. The continuity of $\alpha\to|\alpha x|$ then gives that $|\alpha x|=|\alpha|\,|x|$ for all real $\alpha$, that is the metric is a norm. The converse is trivial. Incidentally it follows from this that if the metric is not a norm there exists a neighbourhood basis consisting of spheres which are not convex. Because if we put $r=\inf\{\delta;\ \delta\geqslant 0 \text{ and } S(\delta) \text{ is not convex}\}$ and suppose that $r>0$, we can choose $\alpha$ such that $0<\alpha<2r$ and $S(\alpha)$ is not convex. Now by (1) $S(\alpha)=S(\alpha/2)+S(\alpha/2)$ and $S(\alpha/2)$ is convex. But the sum of convex sets is convex, which contradicts the choice of $\alpha$.

It is obvious that this proof is based only on the existence of a metric midpoint. Moreover, a similar reasoning can be applied to prove Lemma 1 using Menger's notion of convexity. From this observation and the fact that subsequent arguments referring to convexity will be based only on the existence of midpoints, it follows that the theorems are true whichever definition of convexity is used.

If $x$ and $y$ are two points in $X$, a segment joining $x$ and $y$ is a set $\{z(\alpha)\in X;\ 0\leqslant\alpha\leqslant|x-y|,\ z(0)=x,\ z(|y-x|)=y\}$ satisfying $|z(\alpha_1)-z(\alpha_2)|=|\alpha_1-\alpha_2|$.

One of the theorems of Menger states that if the space is complete any two points can be joined by a segment. It can be verified that a set of points obtained by partitioning and reordering a segment is also a segment.

In a normed space the straight lines are segments. If the space is linear and if the metric is not a norm the follwing propositions show that there are segments of an entirely different type.

**Proposition 1.** *For any point $x\in X$ the distance between a segment joining* 0 *and $x$ and the point $\frac{1}{2}x$ is not less than $|\frac{1}{2}x|-\frac{1}{2}|x|$.*

*Proof.* Suppose that $y$ is a point on a segment joining 0 and $x$. Then $|y|+|y-x|=|x|$. The triangle inequality gives $|\frac{1}{2}x-y|\geqslant|\frac{1}{2}x|-|y|$ and $|\frac{1}{2}x-y|=|\frac{1}{2}x-(x-y)|\geqslant|\frac{1}{2}x|-|x-y|$. Adding these inequalities we obtain $2|\frac{1}{2}x-y|\geqslant 2|\frac{1}{2}x|-|y|-|x-y|$, that is $|\frac{1}{2}x-y|\geqslant|\frac{1}{2}|x$.

**Proposition 2.** *If the metric is not a norm there exist a point $x$ and a number $\delta>0$ such that the distance between any segment joining $0$ and $x$ and the point $\frac{1}{2}x$ is not less than $\delta$.*

*Proof.* We have seen that if the metric is not a norm then there is a point $x$ such that $|\frac{1}{2}x|-\frac{1}{2}|x|>0$. With $\delta=|\frac{1}{2}x|-\frac{1}{2}|x|$ the statement follows from Proposition 1.

## 3. The normability theorem

For any subset $A$ of a linear space $X$, $H(A)$ denotes the convex hull of $A$. It is well-known that $H(A)$ is the set of all convex combinations $\sum_{i=1}^{n}\lambda_i x_i$ of elements $x_i\in A$. A consequence of this definition is the following lemma, proved in a different way in [6].

**Lemma 2.** *For arbitrary sets $A$ and $B$ in a linear space*

$$H(A+B)=H(A)+H(B) \tag{2}$$

*Proof.* If $x\in H(A+B)$, $x$ is of the form $x=\sum_{i=1}^{n}\lambda_i(a_i+b_i)$ where $a_i\in A$, $b_i\in B$ and the combination is convex. Hence $x=\sum_{i=1}^{n}\lambda_i a_i+\sum_{i=1}^{n}\lambda_i b_i\in H(A)+H(B)$. If on the other hand $x\in H(A)+H(B)$ then for some convex combinations $x=\sum_{i=1}^{m}\alpha_i a_i+\sum_{i=1}^{n}\beta_i b_i$. Using that $\sum_{i=1}^{m}\alpha_i=\sum_{i=1}^{n}\beta_i=1$ we write $x=\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_i\beta_j a_i+\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_i\beta_j b_j=\sum_{k=1}^{mn}\gamma_k(c_k+d_k)$, where $\gamma_k=\alpha_i\beta_j$, $c_k=a_i\in A$, $d_k=b_j\in B$, that is $x\in H(A+B)$.

**Theorem 1.** *Let $X$ be a metric linear topological space with a metric that is convex and translation invariant and suppose that $X$ is locally convex. Then $X$ is normable and the convex hulls of the metric spheres form a family of normed spheres.*

*Proof.* Put $T(\delta)=H(S(\delta))$, the convex hull of the open metric sphere around the origin of radius $\delta$. It is well-known that the convex hull of an open bounded set is open and bounded. This implies that the sets $T(\delta)$ are neighbourhoods of the origin and that the gauge function $p$ of $T(1)$ is a norm. Put $p(x)=\|x\|$. From Lemma 1 and Lemma 2 we conclude that $T(\delta_1+\delta_2)=T(\delta_1)+T(\delta_2)$. As the sets $T(\delta)$ are convex this implies that $T(\delta)=\delta T(1)$, so $\{x;\ \|x\|<\delta\}=T(\delta)$.

It remains to prove that the metric and the normed topologies coincide. Suppose $\delta>0$. Evidently $S(\delta)\subset T(\delta)$. Thus the normed topology is coarser. On the other hand, as $X$ is locally convex, to every $\delta>0$ there is a convex neighbourhood $U$ of the origin, such that $U\subset S(\delta)$. But then we can find $\delta_1>0$ such that $S(\delta_1)\subset U$, which implies $T(\delta_1)\subset U$, as $U$ is convex. Consequently $T(\delta_1)\subset S(\delta)$ and the normed topology is finer, which proves the theorem.

## 4. The norm expressed in the metric

We shall now consider the mappings $x\to|x|_n=2^{-n}|2^n x|$ where $x\in X$ and $n$ is integer. The convexity of $|\cdot|$ used on the point $2^n x$ shows that $|\cdot|_n$ is also a convex metric on $X$. This metric makes $X$ a linear topological space with the same topology as that given by $|\cdot|$. To see this we put $S_n(\delta)=\{x;\ |x|_n<\delta\}$. The triangle inequality implies that $|x|_{n+1}\leqslant|x|_n$, so $S_n(\delta)\subset S_{n+1}(\delta)$. The fact that each $n$-sphere contains an $(n+1)$-sphere is a corollary to the following theorem which shows, that in the limit, the sequence $S_n(\delta)$ increases to the convex sphere $T(\delta)$.

**Theorem 2.** *If* $\|\cdot\|$ *is the norm defined above then*

$$\|x\| = \lim_{n\to+\infty} 2^{-n}|2^n x|$$

*Proof.* The decreasing sequence $|x|_n$ has a limit $q(x)\geqslant 0$. We show that $q$ is a seminorm. It is evidently subadditive. Furthermore $q(m2^{-k}x)=\lim_{n\to+\infty}2^{-n}\,|2^{n-k}mx|$ $=\lim_{n\to+\infty}2^{-n-k}\,|2^n mx| = 2^{-k}\,q(mx)\leqslant m2^{-k}q(x)$. Hence we have $q(\lambda x)\leqslant|\lambda|\;q(x)$ for all real $\lambda$ and all $x\in X$. But this implies $q(\lambda x)=|\lambda|q(x)$, so that $q$ is homogeneous. Write $P(\delta)=\{x;\,q(x)<\delta\}$. Then $S(1)\subset P(1)$ because $q(x)\leqslant|x|$ and since $P(1)$ is convex $T(1)=H(S(1))\subset P(1)$. If on the other hand $x\in P(1)$, there is an $N$ such that $n>N$ implies $2^{-n}|2^n x|<1$. Now since $|\cdot|$ is convex we can find $x_i\in S(1)$, $i=1,\dots,2^{-n}$, such that $\sum_{i=1}^{2^n}x_i=2^n x$. From this it follows that $x=1/2^n\,\sum_{i=1}^{2^n}x_i$ belongs to $H(S(1))=T(1)$, that is $P(1)=T(1)$ and $q(x)=\|x\|$.

**Corollary.** *The metrics* $|\cdot|_n$ *define the same topology.*

*Proof.* Reasoning as in the proof of Theorem 1, for all $n$ and for all $\delta>0$ we can find $\delta_1>0$ such that $S_{n+1}(\delta_1)\subset T(\delta_1)\subset S_n(\delta)$, since $S_{n+1}(\delta_1)\subset T(\delta_1)$ by Theorem 2.

## 5. Reflexive spaces

The relation between the spheres $S_n(\delta)$ and $S_{n+1}(\delta)$ can be written

$$S_{n+1}(\delta)=\tfrac{1}{2}(S_n(\delta)+S_n(\delta)) \tag{3}$$

Actually since $|\cdot|_n$ is a convex metric, Lemma 1 implies that $\frac{1}{2}(S_n(\delta)+S_n(\delta))=\frac{1}{2}S_n(2\delta)$. Moreover $x\in\frac{1}{2}S_n(2\delta)$ if and only if $2x\in S_n(2\delta)$ or $2^{-n}|2^{n+1}x|<2\delta$ that is $x\in S_{n+1}(\delta)$. The equality (3) states that $S_{n+1}(\delta)$ is obtained from $S_n(\delta)$ by adding all the midpoints of points in $S_n(\delta)$.

Analogously, because the statement of Lemma 1 is true also when the spheres are closed and since it follows from the above argument that $\overline{\frac{1}{2}S_n(2\delta)}=\overline{S_{n+1}(\delta)}$ we have

$$\overline{S_{n+1}(\delta)}=\tfrac{1}{2}(\overline{S_n(\delta)}+\overline{S_n(\delta)}) \tag{4}$$

Assume now that $x$ is an extremal point of $\overline{T(1)}$ and that $x\in\overline{S_n(1)}$ for some $n$. We shall show that this implies that $|\alpha x|=|\alpha|\,|x|$ for all real $\alpha$ or in other words that the norm and the metric coincide on the subspace spanned by $x$. First, if $x\in\overline{S_n(1)}$ but $x\notin\overline{S_{n-1}(1)}$, according to (4) there exist points $y,\ z\in\overline{S_{n-1}(1)}\subset\overline{T(1)}$ such that $x=\frac{1}{2}(y+z)$. Then, since $x$ is extremal in $\overline{T(1)}$, we conclude $x=y=z$, contradicting that $x\notin\overline{S_{n-1}(1)}$. Consequently $x\in\overline{S_n(1)}$ for all $n$, that is $|2^n x|=2^n|x|$ for all $n$. Secondly, since $|2^p 2^n x|=2^{p+n}|x|=2^p|2^n x|$ for all $n,p$, we can argue as in section 2 to show that $|m2^{-k}x|=|m2^{-k}|\,|x|$ for all $m$, which by continuity implies $|\alpha x|=|\alpha|\,|x|$.

The next theorem will show that if the space is reflexive and separable then the normed and the metric spheres will have enough extremal points in common to span the whole space. In order to prove this theorem we shall consider the strongly exposed points of the normed unit sphere.

Let $C$ be a convex set in a Banach space $X$. A strongly exposed point of $C$ is a point $x \in C$ such that there is an $f \in X^*$, the topological dual of $X$, for which (i) $f(y) < f(x)$ for all $y \in C$, $y \neq x$; and (ii) $f(x_n) \to f(x)$ and $\{x_n\}_{n=1}^{\infty} \subset C$ imply $\|x_n - x\| \to 0$. If the condition (i) is satisfied but not necessarily (ii) the point $x$ is said to be an exposed point of $C$.

The concept of an exposed point was introduced by Straszewicz [6] who showed that if $C$ is a compact convex subset of a finite dimensional euclidean space then $C$ is the closed convex hull of its exposed points. This and similar notions have attracted some attention lately and several theorems of content analogous to that of Straszewicz have been showed. We shall use the following theorem of Lindenstrauss [2]: Every weakly compact convex subset of a separable Banach space is the closed convex hull of its strongly exposed points.

We now give our theorem which is of a converse character to the example in $L^1$ given by Rådström [5] of a convex metric that is not a norm.

**Theorem 3.** *Let $X$ be a metric linear topological space with a metric that is convex and translation invariant and suppose that $X$ is locally convex. Then, if $X$ is reflexive and separable, the metric is a norm, that is $X$ is isometric to a Banach space.*

*Proof.* For simplicity we write $T = \overline{T(1)}$ and $S = \overline{S_0(1)}$. Being normable, complete and reflexive, $X$ considered as a normed space will be a reflexive space and hence complete. It then follows by the Alaoglu theorem [1 p. 425] that $T$ is weakly compact.

Suppose now that $x$ is a strongly exposed point of $T$ and that $x \notin S$. Since $S$ is closed there is an $\varepsilon > 0$ such that $U = \{y;\ \|y - z\| < \varepsilon\}$ does not intersect $S$. If we choose $f \in X^*$ according to the definition of $x$ being strongly exposed, there is an $n$ such that $y \in T$ and $f(y) > f(x) - (1/n)$ imply $y \in U$. Then $S \subset V = \{y;\ f(y) \leqslant f(x) - (1/n)\}$, but since $V$ is closed and convex it follows that $T \subset V$, contradicting the fact that $x \notin V$. Hence all the strongly exposed points of $T$ belong to $S$.

We finally prove that this implies $T = S$. Assume that $x \in T$. As $T$ is weakly compact and convex the theorem of Lindenstrauss states that $T$ is the closed convex hull of its strongly exposed points. Since the metric and the normed topologies coincide, it follows that for all $\varepsilon > 0$ there is a convex combination $\sum_{i=1}^{n} \lambda_i x_i$ of strongly exposed points of $T$ such that $|x - \sum_{i=1}^{n} \lambda_i x_i| < \varepsilon$. Using that $|\lambda_i x_i| = |\lambda_i|\,|x_i|$ and $|x_i| = 1$ we have $|x| = |x - \sum_{i=1}^{n} \lambda_i x_i + \sum_{i=1}^{n} \lambda_i x_i| \leqslant |x - \sum_{i=1}^{n} \lambda_i x_i| + |\sum_{i=1}^{n} \lambda_i x_i| < \varepsilon + \sum_{i=1}^{n} |\lambda_i|\,|x_i| = 1 + \varepsilon$ that is $|x| \leqslant 1$. Hence $T \subset S$. But $S \subset T$ by definition so $S = T$ and the metric is a norm.

## Acknowledgement

I wish to express my gratitude to Hans Rådström, who suggested the problem and who has given me much valuable advice during my work.

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Tryckt den 18 februari 1971

Uppsala 1971. Almqvist & Wiksells Boktryckeri AB