# On the spectral synthesis problem for $(n-1)$-dimensional subsets of $\mathbf{R}^n$, $n \geq 2$

YNGVE DOMAR

University of Uppsala, Sweden

## 1. Introduction

Let $E$ be a closed subset of $\mathbf{R}^n$ and $K(E)$ the space of all functions in $\mathcal{D}(\mathbf{R}^n)$, vanishing in some neighborhood of $E$. $\mathcal{F}L^1(\mathbf{R}^n)$ is the Banach space of Fourier transforms of functions in $L^1(\mathbf{R}^n)$. $\mathcal{D}(\mathbf{R}^n) \subset \mathcal{F}L^1(\mathbf{R}^n)$, and we denote by $\overline{K(E)}$ the closure of $K(E)$ in $\mathcal{F}L^1(\mathbf{R}^n)$. The well-known concept of sets of spectral synthesis can be defined as follows: $E$ is a set of spectral synthesis if $\overline{K(E)}$ contains every element in $\mathcal{F}L^1(\mathbf{R}^n)$ that vanishes on $E$.

C. Herz [3] has proved that $S^1 \subset \mathbf{R}^2$ is a set of spectral synthesis. His proof can unfortunately not be extended to obtain the corresponding result for more general curves. It is however possible to use a different approach to get the desired extension of the result of Herz (cf. [2]). We shall here apply basically the same method to investigate a still more general problem.

As was discovered by L. Schwartz [9], the sphere $S^{n-1} \subset \mathbf{R}^n$ is not of spectral synthesis, if $n \geq 3$. N. Th. Varopoulos [10] has investigated this question in more detail, using methods related to the Herz method for $n = 2$. Let us denote, for any closed set $E$ and any positive integer $m$, by $J_m(E)$ the space of functions in $\mathcal{D}(\mathbf{R}^n)$, $n \geq 2$, vanishing on $E$ together with all their partial derivatives of order $\leq m - 1$. Taking closures in $\mathcal{F}L^1(\mathbf{R}^n)$, we have then

$$\overline{J_1(S^{n-1})} \supset \overline{J_2(S^{n-1})} \supset \ldots \supset \overline{J_{[(n+1)/2]}(S^{n-1})} = \overline{K(S^{n-1})}\,, \tag{1.1}$$

where all inclusions are strict. It is very easy to understand from this why there is a fundamental difference between the case $n = 2$ and the case $n \geq 3$ in this context.

The cited paper of Varopoulos does however contain a considerably more precise description of the situation than the one given above. Let us by $B_m(S^{n-1})$, $m \geq 1$, denote the linear space spanned by all measures on $S^{n-1}$ with infinitely differentiable density function and by all partial derivatives of order $\leq m - 1$ of those measures when considered as distributions on $\mathbf{R}^n$. It is possible to show that $B_m(S^{n-1})$ is a subspace of the dual of $\mathcal{F}L^1(\mathbf{R}^n)$, if $m \leq (n+1)/2$, and in that case we define $I_m(S^{n-1})$ as the annihilator of $B_m(S^{n-1})$ in $\mathcal{F}L^1(\mathbf{R}^n)$. It is illuminating to regard $I_m(S^{n-1})$ as a subspace of $\mathcal{F}L^1(\mathbf{R}^n)$ characterized by the vanishing on $S^{n-1}$, in a generalized sense, of the elements together with their partial derivatives of order $\leq m - 1$. This notion of vanishing is then the same as ordinary vanishing, if the element belongs to $C^m$ in a neighborhood of $S^{n-1}$. It follows from Theorem 3 in [10], that

$$I_m(S^{n-1}) = \overline{J_m(S^{n-1})}\,, \tag{1.2}$$

for $1 \leq m \leq (n+1)/2$. In the case $n = 2$, (1.1) and (1.2) together imply the result of Herz.

Our aim is to generalize (1.1) and (1.2). The generalization is two-fold. In the first place we replace $L^1(\mathbf{R}^n)$ by the more general space $L^1_\alpha(\mathbf{R}^n)$, $n \geq 2$, $\alpha$ real, of Lebesgue measurable functions $\hat{f}$ with the norm

$$\int_{\mathbf{R}^n} |\hat{f}(\xi)|\,(1 + |\xi|)^\alpha\, d\xi\,.$$

Secondly, and this is more important since it creates the need for a method different from the one developed by Herz and Varopoulos, we consider sets $E$ on an arbitrary $(n-1)$-dimensional manifold $M$ in $\mathbf{R}^n$, infinitely differentiable, without multiple points and with non-vanishing Gaussian curvature. The sets $E$ are assumed compact and, in the main theorem, satisfying the restricted cone property (Definition 3.3). $K(E)$ and $J_m(E)$ are then well defined, and for $1 \leq m \leq \alpha + (n+1)/2$, spaces $I_m(E)$ can be defined as in the case $E = S^{n-1}$ (Definition 2.5 and Definition 2.8). Our results are formulated in Theorem 2.9 and Theorem 3.4, of which the second theorem is the most important.

For sets $E$ satisfying the restricted cone property it is possible to express some consequences of our results by generalizing some concepts from the theory of spectral synthesis. Thus it is natural to say that $E$ is of spectral synthesis with respect to $L^1_\alpha(\mathbf{R}^n)$, if $I_1(E) = \overline{K(E)}$. By our theorems this is true, if $[\alpha + (n+1)/2] = 1$, that is to say if $1 \leq 2\alpha + n < 3$. Adopting a notion introduced by Herz [4] one can say that $E$ is a smooth set with respect to $L^1_\alpha(\mathbf{R}^n)$, if $I_1(E) = \overline{J_1(E)}$). This is always true if $2\alpha + n \geq 1$.

There are various possibilities for further generalizations. The spaces $L^1_\alpha(\mathbf{R}^n)$, $n \geq 2$, can thus be replaced by spaces $L^p_\alpha(\mathbf{R}^n)$, $1 \leq p < \infty$, defined by the norm

$$\left[\int_{\mathbf{R}^n} |\hat{f}(\xi)|^p\,(1 + |\xi|)^{\alpha p} d\xi\right]^{1/p},$$

and similar results hold for these spaces. The infinite differentiability of the manifold $M$ can be exchanged to differentiability up to a certain order, as was the assumption made in the study [2]. In the cases when the Gaussian curvature of $M$ vanishes in some subset of $E$, or when the manifold $M$ is of lower dimension than $n-1$, the corresponding problems can be stated but are in general still open.

## 2. Preliminaries

The following lemma of van der Corput type is of fundamental importance for our investigation. The lemma is essentially due to W. Littman [5].

LEMMA 2.1. *Let $\varphi \in \mathcal{D}(\mathbf{R}^m)$, $m \geq 1$, be a function with its support contained in an open set $B$. Let $\psi$ be real-valued and infinitely differentiable in $B$ and such that the inverse of the Hessian determinant $\left|\dfrac{\partial^2\psi}{\partial y_i \partial y_j}\right|$ exists and is bounded in $B$.*

*Then there exists, for every real number $A$, a positive constant $D$ such that*

$$\left|\int_{\mathbf{R}^m} e^{i(\langle y,\eta\rangle+\psi(y)\zeta)}\,\varphi(y)dy\right| \leq D(1+|\zeta|)^{-m/2}\left(1+\frac{|\eta|}{1+|\zeta|}\right)^{-A}, \tag{2.1}$$

*for every $\eta \in \mathbf{R}^m$, $\zeta \in \mathbf{R}$. For fixed $m$, $\varphi$, $B$ and $A$, the same constant $D$ can be chosen for all functions $\psi$ for which we have uniform bounds on the absolute values of the functions, on the absolute values of each of their partial derivatives and on the inverse of the Hessian of $\psi$.*

*Proof of Lemma* 2.1. It is possible to write $\varphi$ as a finite sum of functions in $\mathcal{D}(\mathbf{R}^m)$, each of them with its support included in some closed sphere included in $B$. Hence it is allowed to assume from the beginning that $B$ is a sphere.

For any set of $(\eta, \zeta)$ such that $|\eta|/(1+|\zeta|)$ is uniformly bounded, the inequality follows from the above-mentioned paper of Littman. The only thing that needs to be checked, since it is not explicitely stated by Littman, is the claimed uniformity property of the constant $D$. An examination shows, however, that this is a direct consequence of his proof.

Hence we can restrict our attention to the case when

$$|\eta| > 2(1+|\zeta|)(1+\sup|\mathrm{grad}\,\psi|).$$

We can then integrate partially $p$ times in the direction $t$ for which $\dfrac{\partial}{\partial t}\langle y,\eta\rangle = |\eta|$, and this procedure gives that the left hand member of (2.1) is dominated by

$$D'(|\eta|+|\zeta|)^p\,(|\eta|-|\zeta|(\sup|\mathrm{grad}\,\psi|))^{-2p} \leq D''|\eta|^{-p},$$

where $D'$ and $D''$ are constants with the same uniformity properties as those claimed for $D$. Choosing $p \geq m/2 + A$, we obtain (2.1).

We are going to study distributions on a fixed space $\mathbf{R}^n$. Later on we assume that $n \geq 2$, but the case $n = 1$ can as well be accepted in this introductory discussion. Functions and distributions on the dual $\mathbf{R}^n$ are denoted by $\hat{f}$, $\hat{\mu}$, etc., in order to distinguish them from functions and distributions $f$, $\mu$, etc. on the original $\mathbf{R}^n$.

*Definition* 2.2. For every real $\alpha$, $L^1_\alpha(\mathbf{R}^n)$ denotes the Banach space of all Lebesgue measurable functions $\hat{f}$ on (the dual) $\mathbf{R}^n$ with a finite norm

$$\|\hat{f}\|^1_\alpha = \int_{\mathbf{R}^n} |\hat{f}(\xi)|(1 + |\xi|)^\alpha \, d\xi ,$$

and $L^\infty_\alpha(\mathbf{R}^n)$ denotes the Banach space of all Lebesgue measurable functions $\hat{f}$ on (the dual) $\mathbf{R}^n$ with a finite norm

$$\|\hat{f}\|^\infty_\alpha = \operatorname*{ess.\,sup.}_{\xi \in \mathbf{R}^n} |\hat{f}(\xi)|(1 + |\xi|)^{-\alpha} .$$

We observe that any function $\hat{f}$ in $L^1_\alpha(\mathbf{R}^n)$ or $L^\infty_\alpha(\mathbf{R}^n)$ can be considered as a distribution in the space $\mathcal{S}'(\mathbf{R}^n)$. Hence such a function $\hat{f}$ has, in ordinary distribution sense, a Fourier transform $f$, and we prefer to normalize the Fourier transformation in a way that corresponds to the formal relation

$$f(x) = \int_{\mathbf{R}^n} e^{-i\langle x, \xi\rangle} \hat{f}(\xi) d\xi ,$$

$x \in \mathbf{R}^n$. We adopt the convention that whenever pairs of functions or distributions $f$, $\hat{f}$ or $\mu$, $\hat{\mu}$ etc. are mentioned in the same context they denote pairs of Fourier transforms.

*Definition* 2.3. $\mathcal{F}L^1_\alpha(\mathbf{R}^n)$ and $\mathcal{F}L^\infty_\alpha(\mathbf{R}^n)$ denote the Banach spaces of Fourier transforms $f$ of elements $\hat{f}$ in $L^1_\alpha(\mathbf{R}^n)$ and $L^\infty_\alpha(\mathbf{R}^n)$ with the norms $\|f\|^1_\alpha$ and $\|f\|^\infty_\alpha$, respectively, defined by

$$\|f\|^1_\alpha = \|\hat{f}\|^1_\alpha , \|f\|^\infty_\alpha = \|\hat{f}\|^\infty_\alpha .$$

$L^\infty_\alpha(\mathbf{R}^n)$ can be considered as the Banach space of bounded linear functionals on $L^1_\alpha(\mathbf{R}^n)$, the corresponding is thus true for $\mathcal{F}L^\infty_\alpha(\mathbf{R}^n)$ and $\mathcal{F}L^1_\alpha(\mathbf{R}^n)$, and we define

$$(f, g) = (\hat{f}, \hat{g}) = \int_{\mathbf{R}^n} \hat{f}(\xi)\, \hat{g}(-\xi) d\xi ,$$

whenever $f \in \mathcal{F}L^1_\alpha(\mathbf{R}^n)$, $g \in \mathcal{F}L^\infty_\alpha(\mathbf{R}^n)$. It should be observed that $\mathcal{S}(\mathbf{R}^n)$ is a subspace of $L^1_\alpha(\mathbf{R}^n)$ and $L^\infty_\alpha(\mathbf{R}^n)$ as well as of the transform spaces, and that, by our definition

$$(f, g) = (2\pi)^{-n}\langle f, g\rangle, \tag{2.2}$$

whenever $f$ or $g$ belongs to $\mathcal{S}(\mathbf{R}^n)$, where $\langle\,,\rangle$ has the usual distribution meaning.

We need some simple properties of the space $\mathcal{F}L^1_\alpha(\mathbf{R}^n)$ and they are collected in the following lemma. All that is stated in it is known, but we shall give the proofs in order to avoid too many trivial references.

LEMMA 2.4.

1°. *Let* $g \in \mathcal{F}L^\infty_\alpha(\mathbf{R}^n)$ *and suppose that* $(f, g) = 0$ *for some family of* $f \in \mathcal{S}(\mathbf{R}^n)$, *such that the family of* $\hat{f}$ *is translation invariant. Then the support of* $g$ *is contained in the set of common zeros of the functions* $f$.

2°. $\mathcal{D}(\mathbf{R}^n)$ *is a dense subspace of* $\mathcal{F}L^1_\alpha(\mathbf{R}^n)$.

3°. *Multiplication with a fixed* $\varphi \in \mathcal{S}(\mathbf{R}^n)$ *is a bounded linear transformation from* $L^1_\alpha(\mathbf{R}^n)$ *to itself and from* $L^\infty_\alpha(\mathbf{R}^n)$ *to itself.*

4°. $(f\varphi, g) = (f, \varphi g)$, *if* $f \in \mathcal{F}L^1_\alpha(\mathbf{R}^n)$, $g \in \mathcal{F}L^\infty_\alpha(\mathbf{R}^n)$, $\varphi \in \mathcal{S}(\mathbf{R}^n)$.

5°. *Every* $f \in \mathcal{F}L^1_\alpha(\mathbf{R}^n)$ *can be approximated arbitrarily closely in* $\mathcal{F}L^1_\alpha(\mathbf{R}^n)$ *by elements of the form* $\varphi f$, *where* $\varphi \in \mathcal{D}(\mathbf{R}^n)$.

*Proof of Lemma* 2.4. 1° follows from (2.2) and elementary distribution theory. Applying 1° to the family $\mathcal{D}(\mathbf{R}^n)$ we find that $g$ has empty support, hence $g = 0$. Thus 2° holds by the Hahn-Banach theorem.

To prove 3° we observe that if $f \in \mathcal{F}L^1_\alpha(\mathbf{R}^n)$, $\varphi \in \mathcal{S}(\mathbf{R}^n)$, then

$$\widehat{f\varphi}(\xi) = \int_{\mathbf{R}^n} \hat{f}(\xi - \xi_0)\hat{\varphi}(\xi_0)d\xi_0,$$

$\xi \in \mathbf{R}^n$. Hence by an easy application of Fubini's theorem

$$\|\widehat{f\varphi}\|^1_\alpha \leq \|\hat{f}\|^1_\alpha \|\hat{\varphi}\|^1_{|\alpha|} \tag{2.3}$$

and

$$\|\widehat{f\varphi}\|^\infty_\alpha \leq \|\hat{f}\|^\infty_\alpha \|\hat{\varphi}\|^\infty_{|\alpha|},$$

and 3° follows.

4° is a direct consequence of 3° and Fubini's theorem.

For the proof of 5° we choose a function $\varphi \in \mathcal{D}(\mathbf{R}^n)$ such that $\varphi(x) = 1$, when $x$ belongs to some open set which contains $x = 0$. Then we define $\varphi_\varepsilon \in \mathcal{D}(\mathbf{R}^n)$, $\varepsilon > 0$, by the relation $\varphi_\varepsilon(x) = \varphi(\varepsilon x)$, $x \in \mathbf{R}^n$, and observe that $\|f\varphi_\varepsilon - f\|^1_\alpha \to 0$, as $\varepsilon \to 0$, for every $f \in \mathcal{D}(\mathbf{R}^n)$. By 2° it is thus enough to show that the operator norm of $\varphi_\varepsilon$, when this function is considered as multiplier on $\mathcal{F}L^1_\alpha(\mathbf{R}^n)$, is uniformly bounded, as $\varepsilon \to 0$. By (2.3) this norm is

$$\leq \int_{\mathbf{R}^n} |\hat{\varphi}(\xi/\varepsilon)|(1 + |\xi|)^{|\alpha|}\varepsilon^{-n}d\xi \leq \|\hat{\varphi}\|^1_{|\alpha|},$$

if $0 < \varepsilon < 1$. 5° is thus proved.

In the following we assume that $n \geq 2$, and that $M$ is an $(n-1)$-dimensional infinitely differentiable manifold in $\mathbf{R}^n$, without multiple points and with non-vanishing Gaussian curvature. $E$ is a compact subset of $M$ and $E^\circ$ denotes its interior with respect to $M$.

We shall introduce some spaces of distributions supported by $E$. They are needed in order to characterize different degrees of vanishing on $E^\circ$ for elements in $\mathcal{F}L^1_\alpha(\mathbf{R}^n)$.

*Definition* 2.5. $B_1(E)$ denotes the space of all measures on $\mathbf{R}^n$ which can be obtained from the uniform mass distribution on $E^\circ$ by multiplying it with functions in $\mathcal{D}(\mathbf{R}^n)$, vanishing on $E \setminus E^\circ$ together with all their derivatives. For $m \geq 2$, $B_m(E)$ is the linear space spanned by the measures in $B_1(E)$ together with their partial derivatives of order $\leq m-1$.

The elements in $B_1(E)$ are obviously bounded and regular Borel measures. The assumptions on $M$ have moreover the following consequence. For every $x_0 \in M$ there exists a neigborhood $U_{x_0}$ of $x_0$ with respect to $M$ which, denoting one of the coordinates in $\mathbf{R}^n$ by $z$ and the remaining $(n-1)$-dimensional coordinate vector by $y$, can be written in the form

$$\{(y,z)\,|\,z = \psi(y)\ ,\ \ y \in V\}\ ,$$

where $V$ is a closed ball in $\mathbf{R}^{n-1}$, and where $\psi$ is real and infinitely differentiable with non-vanishing Hessian in $V$. Hence, if $\mu \in B_1(E)$ has its support in $U^\circ_{x_0}$, its Fourier transform $\hat{\mu}$ can be represented in the form

$$\hat{\mu}(\eta\ ,\ \zeta) = (2\pi)^{-n} \int\limits_{\mathbf{R}^{n-1}} e^{i(\langle y,\eta\rangle + \psi(y)\zeta)}\, \varphi(y) dy\ ,$$

$\eta \in \mathbf{R}^{n-1}$, $\zeta \in \mathbf{R}$, where $\varphi \in \mathcal{D}(\mathbf{R}^{n-1})$ has its support in $V^\circ$. It follows from Lemma 2.1, choosing $2A = m = n-1$ in (2.1) that $|\hat{\mu}(\xi)|(1 + |\xi|)^{(n-1)/2}$ is bounded for $\xi \in \mathbf{R}^n$. Taking an arbitrary $\mu \in B_1(E)$, it can by a standard compactness argument be partitioned into a finite sum of measures of the above type, and we obtain therefore the following lemma.

LEMMA 2.6. *$\hat{\mu}(\xi)(1 + |\xi|)^{(n-1)/2}$ is bounded for $\xi \in \mathbf{R}^n$, if $\mu \in B_1(E)$.*

Let us now assume that $\mu \in B_1(E)$ and that $D\mu$ is a partial derivative of $\mu$ of order $p \geq 0$. Then $\widehat{D\mu} = P\hat{\mu}$, where $P$ is a monomial of degree $p$. Thus

$$|\widehat{D\mu}(\xi)|(1 + |\xi|)^{(n-1)/2-p}$$

is bounded for $\xi \in \mathbf{R}^n$, by Lemma 2.6. For $\mu \in B_m(E)$ we can thus conclude that

$$|\hat{\mu}(\xi)|(1 + |\xi|)^{(n-1)/2-m+1}$$

is bounded, for $\xi \in \mathbf{R}^n$, and this proves the following lemma.

LEMMA 2.7. *$B_m(E) \subset \mathcal{F}L^\infty_\alpha(\mathbf{R}^n)$, if $m \leq \alpha + \dfrac{n+1}{2}$.*

We are now prepared to introduce the subspaces of $L^1_\alpha(\mathbf{R}^n)$ which shall be the objects of our investigation.

*Definition* 2.8.

1°. $K(E)$ is the space of all functions in $\mathcal{D}(\mathbf{R}^n)$, which vanish in a neighborhood of $E$ with respect to $\mathbf{R}^n$.

2°. For every integer $m \geq 1$, $J_m(E)$ is the space of all elements in $\mathcal{D}(\mathbf{R}^n)$, which vanish on $E$ together with all their partial derivatives up to the order $m-1$.

3°. For $\alpha \geq -(n-1)/2$ and for every integer $m$, $1 \leq m \leq \alpha + (n+1)/2$, $I_m(E)$ is the annihilator in $\mathcal{F}L^1_\alpha(\mathbf{R}^n)$ of the subspace $B_m(E)$ of $\mathcal{F}L^\infty_\alpha(\mathbf{R}^n)$ (cf. Lemma 2.7).

4°. For every integer $m \geq 1$, $C_m(E)$ is the annihilator in $\mathcal{F}L^\infty_\alpha(\mathbf{R}^n)$ of the space $J_m(E)$ in $\mathcal{F}L^1_\alpha(\mathbf{R}^n)$.

The following theorem contains some preliminary results on the relations between the spaces introduced in Definition 2.8. The theorem is not entirely new and the methods used in the proof are known. Cf. Reiter [7, pp. 37—39] and Varopoulos [10] for results of a similar kind.

THEOREM 2.9.

1°. $\mathcal{F}L^1_\alpha(\mathbf{R}^n) \supset I_1(E) \supset \ldots \supset I_{[\alpha+(n+1)/2]}(E)$, *if* $\alpha \geq -\dfrac{n-1}{2}$.

2°. $\mathcal{F}L^1_\alpha(\mathbf{R}^n) \supset \overline{J_1(E)} \supset \overline{J_2(E)} \supset \ldots$.

3°. *Let* $\hat{g}(\xi) = o(1+|\xi|)^{-(n-1)/2}$, *as* $|\xi| \to \infty$, *and let* $g$ *have its support included in* $E$. *Then* $g = 0$. *As a consequence* $\alpha < -(n-1)/2$ *implies that* $\overline{K(E)} = \mathcal{F}L^1_\alpha(\mathbf{R}^n)$.

4°. *Let* $\hat{g}(\xi) = o(1+|\xi|)^{m-(n-1)/2}$, *as* $|\xi| \to \infty$, *when* $m$ *is an integer* $\geq 1$, *and let* $g$ *have its support included in* $E$. *Then* $\langle f, g\rangle = 0$ *for every* $f \in J_m(E)$. *As a consequence* $\alpha \geq -(n-1)/2$ *implies that* $\overline{K(E)} = \overline{J_{[\alpha+(n+1)/2]}(E)}$.

5°. *Let* $\alpha \geq -(n-1)/2$, $1 \leq m \leq \alpha + (n+1)/2$, *and suppose that* $f \in \mathcal{F}L^1_\alpha(\mathbf{R}^n) \cap C^{m-1}(\mathbf{R}^n)$. *Then* $f \in I_m(E)$ *if and only if* $f$ *vanishes on* $E^\circ$ *together with its partial derivatives of order* $\leq m-1$.

*Proof of theorem* 2.9. 1° and 2° are immediate consequences of Definition 2.8.

To prove 3° and 4° we shall use a method due to A. Beurling [1] and H. Pollard [6] (cf. also Herz [3]). Let the distribution $g$ have its support contained in $E$, and let $f \in \mathcal{D}(\mathbf{R}^n)$. Choose $\varphi \in \mathcal{D}(\mathbf{R}^n)$ so that it vanishes outside $\{x \mid |x| \leq 1\}$ and such that

$$(2\pi)^{-n}\int_{\mathbf{R}^n} \varphi(x)dx = 1 .$$

For every $\varepsilon > 0$ we define $\varphi_\varepsilon$ by the relation $\varphi(x)_\varepsilon = \varepsilon^{-n}\varphi(x/\varepsilon)$, $x \in \mathbf{R}^n$, and put $f = f_\varepsilon + f'_\varepsilon$, where $f_\varepsilon = f$ on $E_{2\varepsilon} = \{x \mid \text{dist}(x, E) \leq 2\varepsilon\}$ and $f_\varepsilon = 0$ on $\complement E_{2\varepsilon}$. We define $f * \varphi_\varepsilon$ by the relation

$$f * \varphi_\varepsilon(x) = (2\pi)^{-n}\int_{\mathbf{R}^n} f(x - x_0)\varphi_\varepsilon(x_0)dx_0 ,$$

$x \in \mathbf{R}^n$, and define $f_\varepsilon * \varphi_\varepsilon$ and $f'_\varepsilon * \varphi_\varepsilon$ in the corresponding way. We find $f * \varphi_\varepsilon \in \mathcal{D}(\mathbf{R}^n)$, $f_\varepsilon * \varphi_\varepsilon \in \mathcal{D}(\mathbf{R}^n)$ and $f'_\varepsilon * \varphi_\varepsilon \in K(E)$, and form

$$\langle f, g\rangle = \langle f - f * \varphi_\varepsilon , g\rangle + \langle f_\varepsilon * \varphi_\varepsilon , g\rangle + \langle f'_\varepsilon * \varphi_\varepsilon , g\rangle .$$

The last term vanishes and hence

$$(2\pi)^{-n}|\langle f, g\rangle| \le \int_{\mathbf{R}^n} |\hat{f}(\xi)|\,|1 - \hat{\varphi}(\varepsilon\xi)|\,|\hat{g}(-\xi)|d\xi + \int_{\mathbf{R}^n} |\hat{f}_\varepsilon(\xi)|\,|\hat{\varphi}(\varepsilon\xi)|\,|\hat{g}(-\xi)|d\xi .$$

$\hat{f} \in \mathcal{S}(\mathbf{R}^n)$ and hence the first term of the right hand side tends to 0, as $\varepsilon \to 0$, by dominated convergence. The second term is, by Schwarz' inequality, for any real $\beta$,

$$\le \varepsilon^{-n/2}\Big(\int_{\mathbf{R}^n} |\hat{f}_\varepsilon(\xi)|^2 d\xi\Big)^{1/2}\Big(\int_{\mathbf{R}^n} |\hat{\varphi}(\xi)|^2|\hat{g}(-\xi/\varepsilon)|^2 d\xi\Big)^{1/2} \le$$

$$\le \varepsilon^{-n/2-\beta}\Big((2\pi)^{-n}\int_{\mathbf{R}^n} |f_\varepsilon(x)|^2 dx\Big)^{1/2}\Big(\int_{\mathbf{R}^n} |\hat{\varphi}(\xi)|^2(1 + |\xi|)^{2\beta}|\hat{g}(-\xi/\varepsilon)|^2(1 + |\xi/\varepsilon|)^{-2\beta} d\xi\Big)^{1/2} .$$

The second factor of the last expression tends to 0, by bounded convergence, if $|\hat{g}(\xi)| = o(1 + |\xi|)^\beta$, as $|\xi| \to \infty$. Hence the geometric properties of $E_{2\varepsilon}$ show that $\langle f, g\rangle = 0$, if

$$\varliminf_{\varepsilon \to 0} \varepsilon^{-n/2-\beta+1/2} \sup_{x \in E_{2\varepsilon}} |f(x)|$$

is finite.

Choosing $\beta = -(n-1)/2$, we have $\langle f, g\rangle = 0$, for every $f \in \mathcal{D}(\mathbf{R}^n)$, hence $g = 0$. The annihilator of $K(E)$ in $L^\infty_\alpha(\mathbf{R}^n)$ is by Lemma 2.4, 1° and elementary distribution theory the subspace of all $g \in L^\infty_\alpha(\mathbf{R}^n)$ with support included in $E$. If $\alpha < -(n-1)/2$, the only such $g$ is the element $g = 0$. Hence 3° is proved. 4° is proved in the same way.

In order to prove 5° we use a method of L. Schwartz [8]. If $\mu \in B_1(E)$ and if $D$ is a partial derivation operator of order $p$, $0 \le p \le m-1$, we obtain by $p$ partial integrations

$$(f, D\mu) = (2\pi)^{-n}\langle f, D\mu\rangle = (2\pi)^{-n}(-1)^p\langle Df, \mu\rangle .$$

It follows from this that $(f, D\mu)$ vanishes for every $\mu \in B_1(E)$ if and only if $Df$ vanishes on $E^\circ$. Varying $p$ and $D$ we obtain from this the property 5°.

## 3. The main theorem

We assume in the following that $\alpha \geq -(n-1)/2$.

It is a consequence of Theorem 2.9, 5°, that if $E^\circ$ is non-empty, then all $I_m(E)$ are different whenever they are defined. Another consequence is that if $E^\circ$ is non-empty, then $J_m(E)$ is included in $I_m(E)$, but $J_{m-1}(E)$ is not included in $I_m(E)$, whenever the spaces are defined. Since $I_m(E)$ is closed, we can conclude that all $\overline{J_m(E)}$ are different, if $1 \leq m \leq \alpha + (n+1)/2$ and $E^\circ$ is non-empty. As for the inclusion $\overline{J_m(E)} \subset I_m(E)$ it is easy to see that if the boundary of $E$ in $M$ is too irregular, then the inclusion may be strict. This is for instance the case if $\alpha = 0$ and $E$ contains an isolated point. We shall however in this section introduce a regularity condition on $E$ which guarantees that $\overline{J_m(E)} = I_m(E)$, $1 \leq m \leq \alpha + (n+1)/2$. We shall in fact prove a slightly stronger result, namely that every $g \in C_m(E)$ is the limit in $\sigma(\mathcal{F}L^\infty_\alpha(\mathbf{R}^n), \mathcal{F}L^1_\alpha(\mathbf{R}^n))$ of a sequence $\{g_\nu\}_1^\infty$, $g_\nu \in B_m(E)$.

We shall use the following simple localization lemma, where the notion of weak convergence refers to $\sigma(\mathcal{F}L^\infty_\alpha(\mathbf{R}^n), \mathcal{F}L^1_\alpha(\mathbf{R}^n))$.

LEMMA 3.1. *Every* $g \in C_m(E)$, $1 \leq m \leq \alpha + (n+1)/2$, *is the weak limit of a sequence* $\{g_n\}_1^\infty$, $g_n \in B_m(E)$, *if every point* $x \in E$ *has an open neighborhood* $U_x \subset \mathbf{R}^n$ *such that every* $g \in C_m(E \cap \bar{U}_x)$ *is the weak limit of a sequence* $\{g_\nu\}_1^\infty$, $g_\nu \in B_m(E)$.

*Proof of Lemma* 3.1. The neighborhoods $U_x$ cover the compact set $E$ and we can therefore select a finite subcovering $(U_p)_1^q$. There exist functions $\varphi_p \in \mathcal{D}(\mathbf{R}^n)$, $p = 1, \ldots, q$, such that the support of $\varphi_p$ is included in $U_p$ and such that $\Sigma\varphi_p = 1$ in an open set, containing $E$. By Lemma 2.4, 3°, $\varphi_p g \in \mathcal{F}L^\infty_\alpha(\mathbf{R}^n)$ and, by Lemma 2.4, 4°, $\varphi_p g \in C_m(E \cap \bar{U}_x)$. Now $g = \Sigma_1^q \varphi_p g$, and since every $\varphi_p g$ is a weak limit of the desired kind, the same holds for $g$.

The regularity condition is introduced by the following two definitions.

*Definition* 3.2. A closed set $F \subset \mathbf{R}^{n-1}$ is said to have the *restricted cone property at a point* $y_0 \in \mathbf{R}^{n-1}$, if there exists a neighborhood $V_0$ of $y_0$ and a cone $K$ defined by

$$K = \{y \in \mathbf{R}^{n-1} \mid (1-\delta)|y| \leq \langle y, y_1\rangle \leq \delta\},$$

where $0 < \delta < 1$, $y_1 \in \mathbf{R}^{n-1}$, $|y_1| = 1$, such that

$$y - K \subset F, \tag{3.1}$$

for every $y \in \overline{F \cap V_0}$.

*Definition* 3.3. The set $E \subset M$ is said to have the *restricted cone property*, if for every $x \in E$ and every sufficiently small neighborhood $V$ of $x$ with respect to $\mathbf{R}^n$, the orthogonal projection of $E \cap V$ onto the tangent hyperplane at $x$ has the restricted cone property at the point $x$.

We are now in a position to formulate our main theorem.

THEOREM 3.4. *If* $E$ *has the restricted cone property, then every* $g \in C_m(E)$,

$1 \leq m \leq \alpha + \dfrac{n+1}{2}$, *is the limit in* $\sigma(\mathcal{F}L^\infty_\alpha(\mathbf{R}^n), \mathcal{F}L^1_\alpha(\mathbf{R}^n))$ *of a sequence* $(g_\nu)_1^\infty$ *of elements in* $B_m(E)$. *In particular, this implies that* $\overline{J_m(E)} = I_m(E)$.

The projection onto the hyperplane in Definition 3.3 can obviously be substituted by a projection onto a suitably chosen coordinate hyperplane. Using such a modification of Definition 3.3 and applying the localization lemma 3.1, we see that Theorem 3.4 is proved, if we can prove the following proposition.

PROPOSITION 3.5. *Let* $F$, $V_0$ *and* $K$ *be subsets of* $\mathbf{R}^{n-1}$, *satisfying the properties requested in Definition* 3.2, *and let* $V_1$ *and* $V_2$ *be open sets in* $\mathbf{R}^{n-1}$ *such that*

$$\overline{V_0} \subset V_1 \subset \overline{V_1} \subset V_2$$

*and such that*

$$y_0 - K \subset V_1 \,, y_1 - K \subset V_2 \tag{3.2}$$

*for every* $y_0 \in \overline{V_0}$, $y_1 \in \overline{V_1}$.

*Let* $\psi$ *be real and infinitely differentiable with non-vanishing Hessian in* $\overline{V_2}$. *We define in* $\mathbf{R}^n$ *the sets*

$$\begin{aligned} M_0 &= \{(y, z) \mid y \in V_2, \ z = \psi(y)\} \\ E_1 &= \{(y, z) \mid y \in \overline{F \cap V_1}, \ z = \psi(y)\} \\ E_0 &= \{(y, z) \mid y \in \overline{F \cap V_0}, \ z = \psi(y)\}. \end{aligned}$$

$M_0$ *is then an* $(n-1)$-*dimensional manifold with the same properties as those requested for* $M$. $E_0$ *and* $E_1$ *are considered as subsets of* $M_0$.

*Let* $1 \leq m \leq \alpha + \dfrac{n+1}{2}$, $g \in C_m(E_0)$. *Then there exists a sequence* $(g_\nu)_1^\infty$, $g_\nu \in B_m(E_1)$, *which converges in* $\sigma(\mathcal{F}L^\infty_\alpha(\mathbf{R}^n), \mathcal{F}L^1_\alpha(\mathbf{R}^n))$ *to* $g$.

## 4. Proof of Proposition 3.5

This entire section is devoted to the proof of Proposition 3.5, which as we have remarked earlier implies our main result, Theorem 3.4.

We shall first introduce two auxiliary functions $\beta$ and $\gamma$. $\beta \in \mathcal{D}(\mathbf{R}^{n-1})$ is a function with its support contained in $K$, and satisfying

$$(2\pi)^{-(n-1)} \int_{\mathbf{R}^{n-1}} \beta(y) dy = 1 \,. \tag{4.1}$$

$\gamma = \mathcal{D}(\mathbf{R}^{n-1})$ is assumed to take the value 1 in a neighborhood of $\overline{V_0}$ and to vanish outside $V_1$. We use the notation $\gamma$ as well for the function on $\mathbf{R}^n$ with values $\gamma(y)$ for every $(y, z) \in \mathbf{R}^n$.

We put

$$W_0 = \{(y, z) \mid y \in V_0, \quad z \in \mathbf{R}\}$$
$$W_1 = \{(y, z) \mid y \in V_1, \quad z \in \mathbf{R}\}$$
$$W_2 = \{(y, z) \mid y \in V_2, \quad z \in \mathbf{R}\},$$

and introduce a bijection $S$ of $W_2$ onto itself by the definition

$$S(y, z) = (y, z - \psi(y)),$$

$(y, z) \in W_2$. Obviously $S$ and its inverse $S^{-1}$ are infinitely differentiable on $W_2$. For every differentiable function $\varphi$, defined in a subset $Q$ of $W_2$, $\varphi \circ S$ and $\varphi \circ S^{-1}$ are infinitely differentiable functions defined in $S^{-1}(Q)$ and $S(Q)$ respectively. For a distribution $\nu$, supported by a subset $Q$ of $W_2$ we can in an analogous way define distributions $\nu \circ S$ and $\nu \circ S^{-1}$, supported by $S^{-1}(Q)$ and $S(Q)$, respectively.

For any $h$, $0 < h < 1$, we denote by $\beta_h$ the measure supported by $\{(y, z) \mid y \in \mathbf{R}^{n-1}, \ z = 0\}$ and with density $h^{-(n-1)}\beta(y/h)$, $y \in \mathbf{R}^{n-1}$. $\beta_h^*$ denotes the measure supported by the same hyperplane, but with density $h^{-(n-1)}\beta(-y/h)$, $y \in \mathbf{R}^{n-1}$.

Defining convolution between function and measure in the usual way, we see that if $\varphi$ is infinitely differentiable on $W_2$, then by (3.2) there exists a differentiable function on $W_1$, obtainable by convoluting $\varphi$ and $\beta_h$. We call this function $\varphi * \beta_h$. If $\nu$ is a distribution, with support included in $W_0$, then by (3.2) $\nu * \beta_h^*$ has its support included in $W_1$. Using this we can give the following definition, where $g$ is a distribution, satisfying the conditions of Proposition 3.5.

*Definition* 4.1. For every $h$, $0 < h < 1$, we put

$$T_h^* g = ((g \circ S^{-1}) * \beta_h^*) \circ S$$

and, for every $f \in \mathcal{D}(\mathbf{R}^n)$, the function $T_h f$ is defined by

$$T_h f(y, z) = \gamma(y)[((f \circ S^{-1}) * \beta_h) \circ S](y, z), \quad \text{if} \quad (y, z) \in W_1,$$
$$T_h f(y, z) = 0, \quad \text{if} \quad (y, z) \notin W_1.$$

It is easy to see that these definitions make sense. We have use for the following lemma, where we assume $0 < h < 1$:

LEMMA 4.2.

1°. $T_h^* g \in B_m(E_1)$.

2°. *For every* $f \in \mathcal{D}(\mathbf{R}^n)$ *we have* $T_h f \in \mathcal{D}(\mathbf{R}^n)$ *and*

$$\|T_h f - \gamma f\|_\alpha^1 \to 0, \quad as \quad h \to 0.$$

3°. *There exists a constant* $C$, *independent of* $h$, *such that*

$$\|T_h f\|_\alpha^1 \leq C \|f\|_\alpha^1,$$

*if* $f \in \mathcal{D}(\mathbf{R}^n)$.

We postpone the proof of the lemma and show first how the lemma can be used to prove Proposition 3.5.

By Lemma 4.2, 3°, $T_h$ can be extended to a bounded linear operator from $\mathcal{F}L^1_\alpha(\mathbf{R}^n)$ into itself, with operator norm $\leq C$, for every $h$. We call its adjoint $A_h$. For every $f \in \mathcal{D}(\mathbf{R}^n)$, elementary distribution theory gives

$$\begin{aligned}(2\pi)^n(f, A_h g) &= (2\pi)^n(T_h f, g) = \langle \gamma[((f \circ S^{-1}) * \beta_h) \circ S], g\rangle = \\ &= \langle ((f \circ S^{-1}) * \beta_n) \circ S, g\rangle = \langle f, ((g \circ S^{-1}) * \beta_h^*) \circ S\rangle = \\ &= \langle f, T_h^* g\rangle = (2\pi)^n(f, T_h^* g),\end{aligned}$$

where the symbol $\langle\ ,\ \rangle$ refers to distributions on $W_1$. Hence $A_h g = T_h^* g$.

Now it is easy to conclude that $T_h^* g$ tends to $g$ weakly, as $h \to 0$. For the norm of the operator $A_h$, considered as a bounded linear transformation from $\mathcal{F}L^\infty_\alpha(\mathbf{R}^n)$ to itself is $\leq C$, hence the elements $A_h g$ have uniformly bounded norm in $\mathcal{F}L^\infty_\alpha(\mathbf{R}^n)$. Hence it suffices to show, that $(f, A_h g) \to (f, g)$, as $h \to 0$, for every $f \in \mathcal{D}(\mathbf{R}^n)$. But for such elemnts $f$

$$(f, A_h g) - (f, g) = (T_h f - \gamma f, g),$$

and the right hand member tends to 0, by Lemma 4.2, 2°.

And since, by Lemma 4.2, 1°, we have $T_h^* g \in B_m(E_1)$, for every $h$, $0 < h < 1$, Proposition 3.5 is proved.

*Proof of Lemma* 4.2.

1°. We know that the support of $g$ is included in $E_0$, hence $g \circ S^{-1}$ has its support included in the orthogonal projection of $E_0$ into the coordinate hyperplane $\{(y, z) \mid y \in \mathbf{R}^{n-1},\ z = 0\}$, in the following called the $y$-hyperplane. By (3.1) and (3.2) and by the definitions of $E_0$ and $E_1$ we find that $(g \circ S^{-1}) * \beta_h^*$, for every $h$ with $0 < h < 1$, has its support included in the orthogonal projection of $E_1$ into the $y$-hyperplane. Hence the supports of the distributions $T_h^* g$ are included in $E_1$.

It is wellknown (Schwartz [8], p. 101) that we have for some $q$ a representation

$$g \circ S^{-1} = \sum_{p=0}^{q-1} \mu_p \otimes \delta^{(p)}$$

where $\mu_p$ are distributions on $\mathbf{R}^{n-1}$, with compact support and where $\delta^{(p)}$ is the derivatives of order $p$ of the Dirac measure on $\mathbf{R}$. This has then to be interpreted in the sense that the representation of $\mathbf{R}^n$ as $\mathbf{R}^{n-1} \times \mathbf{R}$ corresponds to the coordinate representation $(y, z)$, $y \in \mathbf{R}^{n-1}$, $z \in \mathbf{R}$. From this it is easy to understand that we have a representation

$$(g \circ S^{-1}) * \beta_h^* = \sum_{p=0}^{q-1} \varphi_p \otimes \delta^{(p)},$$

where $\varphi_p$ now belong to $\mathcal{D}(\mathbf{R}^{n-1})$. It is seen from this that the support of every

$\varphi_p \otimes \delta^{(p)}$ is included in the support of $(g \circ S^{-1}) * \beta_h^*$, and from this we see at once, by applying the mapping $S$ to the two members, that $T_h^* g \in B_q(E_1)$.

It remains to show that $\varphi_{q-1} \neq 0$ implies that $q \leq m$. If $\varphi_{q-1} \neq 0$, then it is possible to find a function $\varphi \in \mathcal{D}(\mathbf{R}^n)$ with support included in $W_1$, vanishing on the $y$-hyperplane together with all derivatives of order $\leq q-2$, but such that

$$\langle \varphi , (g \circ S^{-1}) * \beta_h^* \rangle \neq 0 .$$

Hence

$$\langle (\varphi * \beta_h) \circ S , g \rangle \neq 0 ,$$

which shows that there is a function in $J_{q-1}(E_0)$, which is not annihilated by $g$. By the assumption $g \in C_m(E_0)$, and thus $q \leq m$.

2°. The infinite differentiability of $T_h f$, for $f \in \mathcal{D}(\mathbf{R}^n)$, is a consequence of the infinite differentiability of $(f \circ S^{-1}) * \beta_h$ in $W_1$ and of the assumption on the support of $\gamma$. The compactness of the support is evident. Hence $T_h f \in \mathcal{D}(\mathbf{R}^n)$. The supports of $T_h f$ for $f$ fixed, $h$ variable, are in fact included in a fixed compact subset of $W_2$. Hence it suffices to show, due to well-known estimates, that all partial derivatives of $T_h f$ converge uniformly to the partial derivative of $\gamma f$. It is evident that it suffices to show that the partial derivatives of $k * \beta_h$ converge uniformly to $k$ on every compact subset of $W_1$, if $k \in \mathcal{D}(\mathbf{R}^n)$, which is immediate.

3°. Let us agree in the following to interpret the product of $\gamma$ and any complex-valued function defined in $W_1$, as a function on $\mathbf{R}^n$ with values determined by the product in $W_1$, and with the value 0 outside $W_1$. Then for $f \in \mathcal{D}(\mathbf{R}^n)$, with the changes in the order of the integration motivated by absolute convergence, we have, for $(y , z) \in \mathbf{R}^n$,

$$T_h f(y , z) = \gamma(y) \int_{\mathbf{R}^{n-1}} f(y - y_0 , z + \psi(y - y_0) - \psi(y)) h^{-(n-1)} \beta(y_0/h) dy_0 =$$

$$= \gamma(y) \int_{\mathbf{R}^{n-1}} f(y - h\sigma , z + \psi(y - h\sigma) - \psi(y)) \beta(\sigma) d\sigma =$$

$$= \gamma(y) \int_{\mathbf{R}} d\zeta e^{-iz\zeta} \int_{\mathbf{R}^{n-1}} \int_{\mathbf{R}^{n-1}} e^{-i(\langle y-h\sigma, \eta_0\rangle + (\psi(y-h\sigma)-\psi(y))\zeta)} \hat{f}(\eta_0 , \zeta) \beta(\sigma) d\eta_0 d\sigma .$$

Hence, by Fourier's inversion formula, we have for every $(y , \zeta) \in \mathbf{R}^n$

$$(2\pi)^{-1} \int_{\mathbf{R}} T_h f(y , z) e^{iz\zeta} dz =$$

$$= \gamma(y) \int_{\mathbf{R}^{n-1}} \int_{\mathbf{R}^{n-1}} e^{-i(\langle y-h\sigma, \eta_0\rangle + (\psi(y-h\sigma)-\psi(y))\zeta)} \hat{f}(\eta_0 , \zeta) \beta(\sigma) d\eta_0 d\sigma ,$$

and forming the Fourier transform of $T_h f$ we thus obtain

$$(2\pi)^{n-1}\widehat{T_hf}(\eta\,,\,\zeta) = (2\pi)^{-1}\int\limits_{\mathbf{R}^{n-1}}\int\limits_{\mathbf{R}} T_hf(y\,,\,z)e^{iz\zeta+i\langle y\,,\,\eta\rangle}\,dzdy = \tag{4.2}$$

$$= \int\limits_{\mathbf{R}^{n-1}} d\eta_0 \hat{f}(\eta_0\,,\,\zeta)E(\eta\,,\,\eta_0\,,\,\zeta\,,\,h)\,,$$

where $(\eta\,,\,\zeta) \in \mathbf{R}^n$ and where, for every $(\eta\,,\,\eta_0\,,\,\zeta) \in \mathbf{R}^{2n-1}$

$$E(\eta\,,\,\eta_0\,,\,\zeta\,,\,h) = \int\limits_{\mathbf{R}^{n-1}}\int\limits_{\mathbf{R}^{n-1}} e^{i(\langle y\,,\,\eta-\eta_0\rangle+h\langle\sigma\,,\,\eta_0\rangle+\zeta(\psi(y-h\sigma)-\psi(y)))}\gamma(y)\beta(\sigma)dyd\sigma\,.$$

The function $(y\,,\,\sigma) \to (\psi(y - h\sigma) - \psi(y))/h$ is infinitely differentiable on the support of $(x\,,\,y) \to \gamma(y)\beta(\sigma)$ because of (3.2), and each partial derivative has a bound, uniform in $h$. Furthermore, the Hessian of the function is $-H(y - h\sigma)H(y)$, where $H$ is the Hessian of $\psi$, hence its inverse has bounds, which are uniform in $h$. Applying Lemma 2.1 with $m = 2n - 2$ and $A = n + |\alpha|$, we obtain for every $(\eta\,,\,\eta_0\,,\,\zeta) \in \mathbf{R}^{2n-1}$,

$$|E(\eta\,,\,\eta_0\,,\,\zeta\,,\,h)| \leq D(1 + |\zeta h|)^{-(n-1)}\left(1 + \frac{|\eta - \eta_0|}{1 + |\zeta h|}\right)^{-n-|\alpha|}\,,$$

where $D$ is a constant, independent of $h$, and by (4.2) we obtain

$$\|T_hf\|_\alpha^1 = \int\limits_{\mathbf{R}} d\zeta \int\limits_{\mathbf{R}^{n-1}} |\widehat{T_hf}(\eta\,,\,\zeta)|(1 + |(\eta\,,\,\zeta)|)^\alpha d\eta \leq \tag{4.3}$$

$$\leq (2\pi)^{1-n}\int\limits_{\mathbf{R}} d\zeta \int\limits_{\mathbf{R}^{n-1}} d\eta_0|\hat{f}(\eta_0\,,\,\zeta)| \int\limits_{\mathbf{R}^{n-1}} E(\eta\,,\,\eta_0\,,\,\zeta\,,\,h)|(1 + |(\eta\,,\,\zeta)|)^\alpha d\eta\,.$$

But elementary inequalities show that

$$(1 + |(\eta\,,\,\zeta|)^\alpha \leq (1 + |(\eta_0\,,\,\zeta)|)^\alpha\cdot\left(1 + \frac{|\eta - \eta_0|}{1 + |\zeta|}\right)^{|\alpha|} \leq (1 + |(\eta_0\,,\zeta)|)^\alpha\left(1 + \frac{|\eta - \eta_0|}{1 + |h\zeta|}\right)^{|\alpha|}\,,$$

for every $(\eta\,,\,\eta_0\,,\,\zeta) \in \mathbf{R}^{2n-1}$ and hence

$$\int\limits_{\mathbf{R}^{n-1}} |E(\eta\,,\,\eta_0\,,\,\zeta\,,\,h)|(1 + |(\eta\,,\,\zeta)|)^\alpha d\eta \leq$$

$$\leq D(1 + |(\eta_0\,,\,\zeta)|)^\alpha \int\limits_{\mathbf{R}^{n-1}} (1 + |\zeta h|)^{-(n-1)}\left(1 + \frac{|\eta - \eta_0|}{1 + |\zeta h|}\right)^{-n} d\eta =$$

$$= D_1(1 + |(\eta_0\,,\,\zeta)|)^\alpha\,,$$

for every $(\eta_0\,,\,\zeta) \in \mathbf{R}^n$, where $D_1$ is independent of $h$. Hence by (4.3)

$$\|T_hf\|_\alpha^1 \leq (2\pi)^{1-n}\,D_1\|f\|_\alpha^1\,,$$

and 3° is proved.

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Yngve Domar,
Department of Mathematics,
Sysslomansgatan 8,
S-752 23 Uppsala,
Sweden