Hyperfinite product factors

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1. Introduction

It is an open question whether all hyperfinite factors are *-isomorphic to factors obtained as the infinite tensor product of finite type I factors. In order to study this problem it is necessary to have criteria which tell us when a hyperfinite factor is *-isomorphic to such a product factor. The present paper is devoted to a result of this kind, the criterion being that all, or equivalently, just one normal state is in a sense asymptotically a product state. This result is an intrinsic characterization of product factors in that it is independent of any weakly dense UHF-algebra and also of any tensor product factorization of the underlying Hilbert space.

We first recall some terminology. A UHF-algebra is a C^*-algebra $\mathfrak{A}$ with identity I in which there is an increasing sequence of I_{n_i}-factors M_{n_i} containing I such that $n_i \to \infty$ and $\bigcup_{i=1}^{\infty} M_{n_i}$ is uniformly dense in $\mathfrak{A}$, see [2]. A factor $\mathfrak{R}$ is said to be hyperfinite if there is a UHF-algebra which is weakly dense in $\mathfrak{R}$. More specially $\mathfrak{R}$ is said to be an ITPFI-factor (infinite tensor product of finite type I factors) if there exists an infinite sequence of I_{n_i}-factors M_{n_i} with $n_i \geq 2$ for an infinite number of i's, and a product state $\omega = \otimes_{i=1}^{\infty} \omega_i$ of the C^*-algebraic tensor product $\mathfrak{A} = \otimes_{i=1}^{\infty} M_{n_i}$, such that $\mathfrak{R}$ equals the weak closure of $\pi_{\omega}(\mathfrak{A})$, where π_{ω} is the representation of $\mathfrak{A}$ induced by ω. It was shown by Murray and von Neumann, see [1, Théorème 3, p. 280], that all hyperfinite II_1-factors are *-isomorphic, and hence *-isomorphic to ITPFI-factors. It is not known whether all hyperfinite factors of types II_{∞} or III are *-isomorphic to ITPFI-factors. We refer the reader to the book of Dixmier [1] for the theory of von Neumann algebras and to the paper of Guichardet [3] for that of infinite tensor products.

The author is indebted to J. Tomiyama for pointing out a gap in an early version of the paper. In this version there was also a rather long proof of the implication

(iii) → (i) in the theorem. I am grateful to G. Elliott, O. A. Nielsen, and J. Woods for communicating to me the short proof of this implication, which they have kindly let me publish here. G. Elliott has also pointed out that the theorem can be generalized from hyperfinite factors to factors generated by a commuting (not necessarily countable) family of finite type I subfactors.

2. Product factors

Definition. Let $\mathfrak{R}$ be a factor and ω a normal state of $\mathfrak{R}$. We say ω is *asymptotically a product state* if given $\varepsilon > 0$ and a type I_n-factor $M \subset \mathfrak{R}$ then we can find a type I_m-factor $N = N(\varepsilon, M)$ such that $M \subset N \subset \mathfrak{R}$ and such that

$$\|\omega - \omega | N \otimes \omega | N^c\| < \varepsilon ,$$

where $N^c = N' \cap \mathfrak{R}$, and we identify $\mathfrak{R}$ and $N \otimes N^c$. $\mathfrak{R}$ is said to be a *product factor* if every normal state of $\mathfrak{R}$ is asymptotically a product state.

We have here implicitly assumed that both M and N contain the identity I of $\mathfrak{R}$. This will always be done when we write $\mathfrak{A} \subset \mathfrak{B}$ for two C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$ with identities.

Remark. The property of being a product factor is a *-isomorphic invariant. Indeed, if $\mathfrak{R}$ is a product factor and α is a *-isomorphism of $\mathfrak{R}$ onto a factor $\mathfrak{M}$ then the dual map of α carries the normal states of $\mathfrak{M}$ onto those of $\mathfrak{R}$, hence they are all asymptotically product states.

THEOREM. *Let $\mathfrak{R}$ be a hyperfinite factor. Then the following three conditions are equivalent:*

(i) *$\mathfrak{R}$ is a product factor.*

(ii) *There is a normal state on $\mathfrak{R}$ which is asymptotically a product state.*

(iii) *$\mathfrak{R}$ is *-isomorphic to an ITPFI-factor.*

We first prove two lemmas both of which are probably well known.

LEMMA 1. *Let $\mathfrak{R}$ be a countably decomposable infinite factor or the hyperfinite II_1-factor. If N is a type I_n-factor contained in $\mathfrak{R}$ then $\mathfrak{R} \cong \mathfrak{R} \otimes N$.*

Proof. If $\mathfrak{R}$ is the hyperfinite II_1-factor then $\mathfrak{R} \otimes N$ is also hyperfinite and of type II_1, hence $\mathfrak{R} \cong \mathfrak{R} \otimes N$ by the isomorphism theorem for such factors [1, Théorème 3, p. 280]. Assume $\mathfrak{R}$ is an infinite factor. Let $\{e_{ij} : i, j = 1, \ldots, n\}$ be a complete set of matrix units in N. Then the projection e_{11} is infinite, hence equivalent to the identity, since $\mathfrak{R}$ is countably decomposable. Thus $\mathfrak{R} \cong e_{11}\mathfrak{R}e_{11}$. Clearly $\mathfrak{R} \cong e_{11}\mathfrak{R}e_{11} \otimes N$. Hence $\mathfrak{R} \cong \mathfrak{R} \otimes N$.

LEMMA 2. *Let $\mathfrak{R}$ be a hyperfinite factor and N a type I_n-factor contained in $\mathfrak{R}$. Let $N^c = N' \cap \mathfrak{R}$. Then N^c is a hyperfinite factor isomorphic to $\mathfrak{R}$, and $\{N \cup N^c\}'' = \mathfrak{R}$.*

Proof. By Lemma 1 there is a *-isomorphism α of $\mathfrak{R}$ onto $\mathfrak{R} \otimes N$. Let $M = \alpha^{-1}(I \otimes N)$, where we write $I \otimes N$ for the set of operators $I \otimes x$, $x \in N$.

Then M is a type I_n-factor contained in $\mathfrak{R}$, so there is a unitary operator u in $\mathfrak{R}$ such that $uNu^{-1} = M$ [4, Lemma 3.3]. Let $\beta(x) = \alpha(uxu^{-1})$ for $x \in \mathfrak{R}$. Then β is a *-isomorphism of $\mathfrak{R}$ onto $\mathfrak{R} \otimes N$ such that $\beta(N) = \alpha(M) = I \otimes N$. Thus $\beta(N^c) = (I \otimes N)^c = (I \otimes N)' \cap (\mathfrak{R} \otimes N) = (\mathfrak{B}(\mathcal{H}) \otimes I) \cap (\mathfrak{R} \otimes N) = \mathfrak{R} \otimes I \cong \mathfrak{R}$, where $\mathcal{H}$ is the underlying Hilbert space, and $\mathfrak{B}(\mathcal{H})$ all bounded operators on $\mathcal{H}$. Therefore $N^c \cong \mathfrak{R}$. Finally we have

$$\{N \cup N^c\}'' = \beta^{-1}(\{\beta(N) \cup \beta(N^c)\}'') = \beta^{-1}(\{(I \otimes N) \cup (\mathfrak{R} \otimes I)\}'') = \beta^{-1}(\mathfrak{R} \otimes N) = \mathfrak{R}.$$

Proof of theorem. We show (i) $\rightarrow$ (ii) $\rightarrow$ (iii) $\rightarrow$ (i). Clearly (i) $\rightarrow$ (ii). We first show (ii) $\rightarrow$ (iii).

Let $\mathfrak{A}$ be a UHF-algebra which is weakly dense in $\mathfrak{R}$, say $\mathfrak{A}$ is the norm closure of $\bigcup_{i=1}^{\infty} M_{n_i}$ of an increasing sequence of type I_{n_i}-factors M_{n_i}. Let ω be a normal state of $\mathfrak{R}$ which is asymptotically a product state. Let $\{e_i\}_{i \geq 1}$ be a uniformly dense sequence of operators in the unit ball of $\bigcup_{i=1}^{\infty} M_{n_i}$. We shall by induction construct a sequence $\{N_i\}$ of type I_{m_i}-factors contained in $\mathfrak{R}$ such that

$$N_1 \subset N_2 \subset \ldots . \tag{1}$$

$$\|\omega - \omega|N_k \otimes \omega|N_k^c\| < 2^{-k}, \quad k = 1, 2, \ldots . \tag{2}$$

$$\bigcup_{i=1}^{\infty} N_i \text{ is strongly dense in } \mathfrak{R}. \tag{3}$$

(Here and later we identify $N \otimes N^c$ and $\{N \cup N^c\}''$ in order to make notation more explicit). Since ω is a normal state the representation it induces is normal, hence a *-isomorphism of $\mathfrak{R}$ onto a factor. We may thus assume ω is a vector state ω_ξ, where ξ is a cyclic unit vector in the underlying Hilbert space $\mathcal{H}$. Since $\mathfrak{A}$ is separable and ξ cyclic, $\mathcal{H}$ is separable. Let $\{\xi_i\}_{i \geq 1}$ be a dense sequence of vectors in $\mathcal{H}$ with $\xi_1 = \xi$. Since e_1 belongs to some M_{n_i} there is from the assumption that ω is asymptotically a product state a I_{m_1}-factor $N_1 \subset \mathfrak{R}$ such that $e_1 \in N_1$ and

$$\|\omega - \omega|N_1 \otimes \omega|N_1^c\| < \frac{1}{2}.$$

Suppose we have chosen $N_k \supset N_{k-1} \supset \ldots \supset N_1$ and found $a_1^k, \ldots, a_k^k$ in N_k with $\|a_j^k\| \leq 1$ such that

$$\|(a_j^k - e_j)\xi_r\| < 2^{-k}, \quad j, r = 1, \ldots, k, \tag{4}$$

and such that (2) holds. We shall construct $N_{k+1} \supset N_k$ and $a_1^{k+1}, \ldots, a_{k+1}^{k+1}$ in N_{k+1} such that (2) and (4) hold with k replaced by $k+1$.

By Lemma 2 N_k^c is a hyperfinite factor and $\{N_k \cup N_k^c\}'' = \mathfrak{R}$. Let N_k^c be the weak closure of $\bigcup_{i=1}^{\infty} P_i$, where $\{P_i\}$ is an increasing sequence of I_{p_i}-factors. Then we can, using the Kaplansky density theorem [1, Théorème 3, p. 43], find an integer n, operators $a_1^{k+1}, \ldots, a_{k+1}^{k+1}$ in the unit ball of $\{N_k \cup P_n\}''$ such that

$$\|(a_j^{k+1} - e_j)\xi_r\| < 2^{-k-1}\,, \quad j\,, r = 1\,, \ldots\,, k+1\,.$$

Now $\{N_k \cup P_n\}''$ is a finite type I-factor (which is *-isomorphic to $N_k \otimes P_n$). Since ω is asymptotically a product state there is a type $I_{m_{k+1}}$-factor N_{k+1} containing $\{N_k \cup P_n\}''$ such that

$$\|\omega - \omega|N_{k+1} \otimes \omega|N_{k+1}^c\| < 2^{-k-1}\,.$$

Therefore, by induction (2) and (4) hold for all k. Since the argument also defines the sequence $\{N_i\}$ we have also shown (1).

We next show that (3) holds. Let $a \in \mathfrak{R}$. Say $\|a\| \leq 1$. Let $\psi_1\,, \ldots\,, \psi_l \in \mathcal{H}$, and let $\eta > 0$. Then we can find $\xi_{j_1}\,, \ldots\,, \xi_{j_l}$ in the sequence $\{\xi_i\}$ such that $\|\xi_{j_r} - \psi_r\| < \eta/4$, $r = 1\,, \ldots\,, l$. Let k be a positive integer such that $2^{-k} < \eta/4$ and such that $k \geq \max\{j_r : r = 1\,, \ldots\,, l\}$. Since $\{e_i\}$ is a dense sequence in the unit ball in $\mathfrak{A}$, it is strongly dense in the unit ball in $\mathfrak{R}$ by the Kaplansky density theorem [1, Théorème 3, p. 43]. Therefore there exists $e_t \in \{e_i\}$ such that $\|(a - e_t)\psi_r\| < \eta/4$, $r = 1\,, \ldots\,, l$. By (4) we can find N_{k_1}, $k_1 \geq k$, and $b \in N_{k_1}$ with $\|b\| \leq 1$ such that $\|(b - e_t)\xi_s\| < \eta/4$, $s = 1\,, \ldots\,, k_1$. In particular this holds for $s = j_r$, $r = 1\,, \ldots\,, l$. Therefore we have

$$\begin{aligned}\|(b-a)\psi_r\| &\leq \|(b-e_t)\psi_r\| + \|(a-e_t)\psi_r\| \leq \|b-e_t\|\,\|\psi_r - \xi_{j_r}\| + \|(b-e_t)\xi_{j_r}\| + \eta/4 \\ &< 2\eta/4 + \eta/4 + \eta/4 = \eta\,.\end{aligned}$$

Thus $\bigcup_{i=1}^{\infty} N_i$ is strongly dense in $\mathfrak{R}$, and we have shown that (1), (2), (3) all hold as asserted.

Let $\mathfrak{B}$ denote the uniform closure of $\bigcup_{i=1}^{\infty} N_i$. Then $\mathfrak{B}$ is a UHF-algebra which is strongly dense in $\mathfrak{R}$. Let $N_0 = \{\lambda I\}$ and put $M_k = N_k \cap N_{k-1}^c$, $k = 1\,, 2\,, \ldots$. Then we can identify $\mathfrak{B}$ with $\otimes_{k=1}^{\infty} M_k$. Let $\omega_k = \omega|M_k$. By (2) we have that if $1 \leq k \leq l$ then

$$\begin{aligned}&\|\omega_1 \otimes \ldots \otimes \omega_k \otimes \omega|N_k^c - \omega_1 \otimes \ldots \otimes \omega_l \otimes \omega|N_l^c\| \\ &= \|\omega_1 \otimes \ldots \otimes \omega_k \otimes (\omega|N_k^c - \omega_{k+1} \otimes \ldots \otimes \omega_l \otimes \omega|N_l^c)\| \\ &\leq \|\omega|N_k^c - \omega_{k+1} \otimes \ldots \otimes \omega_l \otimes \omega|N_l^c\| \\ &\leq \|\omega|N_k^c - \omega_{k+1} \otimes \omega|N_{k+1}^c\| + \|\omega_{k+1} \otimes (\omega|N_{k+1}^c - \omega_{k+2} \otimes \ldots \otimes \omega_l \otimes \omega|N_l^c)\| \\ &< 2^{-k-1} + \|\omega|N_{k+1}^c - \omega_{k+2} \otimes \ldots \otimes \omega_l \otimes \omega|N_l^c\| \\ &< 2^{-k-1} + 2^{-k-2} + \ldots + 2^{-l} = 2^{-k}(1 - 2^{-(l-k)}) < 2^{-k}\,.\end{aligned}$$

Therefore the normal states $\omega_1 \otimes \ldots \otimes \omega_k \otimes \omega|N_k^c$ form a Cauchy sequence, and accordingly converge uniformly to a normal state $\otimes_{i=1}^{\infty} \omega_i$, which is clearly a product state on $\mathfrak{B}$. If π denotes the representation of $\mathfrak{B}$ induced by $\otimes_{i=1}^{\infty} \omega_i$ then π extends naturally to a *-isomorphism of $\mathfrak{R}$ onto the ITPFI-factor $\pi(\mathfrak{B})''$. Hence we have shown that (ii) $\to$ (iii).

Finally we show (iii) $\rightarrow$ (i). Assume there is a UHF-algebra $\mathfrak{A}$ which is the infinite tensor product of type I_{n_i}-factors M_{n_i}, so $\mathfrak{A} = \otimes_{i=1}^{\infty} M_{n_i}$. Suppose there is a product state $\otimes_{i=1}^{\infty} \omega_i$ of $\mathfrak{A}$, where ω_i is a state of M_{n_i}. Let $\omega_i = \omega_{\xi_i} \circ \pi_i$ be the Gelfand-Segal decomposition of ω_i and $\omega_{\xi} \circ \pi$ that of $\otimes \omega_i$. Say ξ_i and ξ are cyclic vectors in the Hilbert spaces $\mathcal{H}_i$ and $\mathcal{H}$ respectively. Then $\xi = \otimes \xi_i$, $\pi = \otimes {}^{\xi}\pi_i$, and $\mathcal{H} = \otimes {}^{\xi}\mathcal{H}_i$, see [3, Prop. 2.9]. Since the property of being a product factor is an isomorphism invariant we may assume $\mathfrak{R} = \pi(\mathfrak{A})''$.

Suppose that a type I_n-factor M contained in $\mathfrak{R}$, that a normal state ω on $\mathfrak{R}$, and that $\varepsilon > 0$ are given. Then $M \otimes \mathfrak{R}$ acts on $\mathcal{H} \otimes \mathcal{H}$, and by Lemma 1 there is a *-isomorphism α of $\mathfrak{R}$ onto $M \otimes \mathfrak{R}$. As in the proof of Lemma 2 it follows from [4, Lemma 3.3] that there is an inner automorphism of $M \otimes \mathfrak{R}$ carrying $\alpha(M)$ onto $M \otimes I$. Composing this automorphism with α we may assume $\alpha(M) = M \otimes I$. Now since M is a I_n-factor and $\mathcal{H}$ is infinite dimensional there is a unit vector ψ in $\mathcal{H} \otimes \mathcal{H}$ such that $\omega \circ \alpha^{-1} = \omega_{\psi}$ on $M \otimes \mathfrak{R}$, see [1, Corollaire 10, p. 302]. From [3, Ch. 1] or [5, Lemma 3.1] there are an integer $N \geq 2$ and a vector η in $\mathcal{H} \otimes \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_{N-1}$ such that

$$\|\psi - \eta \otimes (\bigotimes_{i=N}^{\infty} \xi_i)\| < \varepsilon/6 ;$$

moreover, η can be taken to be a unit vector. Let

$$N = \alpha^{-1}(M \otimes M_{n_1} \otimes \ldots \otimes M_{n_{N-1}} \otimes I) ,$$

where we identify M_{n_i} with $\pi_i(M_{n_i})$, and I stands for the identity on $\otimes_{i=N}^{\infty\xi} \mathcal{H}_i$. Then N is a finite type I factor, and $M \subset N \subset \mathfrak{R}$. It remains to show that $\|\omega - \omega|N \otimes \omega|N^c\| < \varepsilon$, or equivalently, that $\|\omega_{\psi} - \omega_{\psi}|\alpha(N) \otimes \omega_{\psi}|\alpha(N)^c\| < \varepsilon$. Let $\zeta = \eta \otimes (\otimes_{i=N}^{\infty} \xi_i)$. Then $\omega_{\zeta}|M \otimes \mathfrak{R} = \omega_{\zeta}|\alpha(N) \otimes \omega_{\zeta}|\alpha(N)^c$, and so

$$\begin{aligned}
&\|\omega_{\psi} - \omega_{\psi}|\alpha(N) \otimes \omega_{\psi}|\alpha(N)^c\| \\
&\leq \|\omega_{\psi} - \omega_{\zeta}\| + \|\omega_{\zeta}|\alpha(N) \otimes \omega_{\zeta}|\alpha(N)^c - \omega_{\psi}|\alpha(N) \otimes \omega_{\zeta}|\alpha(N)^c\| \\
&+ \|\omega_{\psi}|\alpha(N) \otimes \omega_{\zeta}|\alpha(N)^c - \omega_{\psi}|\alpha(N) \otimes \omega_{\psi}|\alpha(N)^c\| \\
&< \varepsilon/3 + \|\omega_{\zeta}|\alpha(N) - \omega_{\psi}|\alpha(N)\| + \|\omega_{\zeta}|\alpha(N)^c - \omega_{\psi}|\alpha(N)^c\| \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon .
\end{aligned}$$

Thus ω is asymptotically a product state, hence $\mathfrak{R}$ is a product factor. The proof is complete.

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