On a theorem of A. C. Offord and its analogue for Fourier series

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1. Let l_n $(n = 1, 2, \ldots)$ be a sequence of natural numbers such that $l_n = o(n)$ and let

$$\delta_n = \frac{2l_n\pi}{2n+1}. \tag{1}$$

Let further, f be a Lebesgue integrable function of period 2π. Let us denote by $f_n(x)$ $(n = 1, 2, \ldots)$ the average of f over the interval $(x - \delta_n, x + \delta_n)$, that is

$$f_n(x) = \frac{1}{2\delta_n} \int_{x-\delta_n}^{x+\delta_n} f(t)dt. \tag{2}$$

The function f_n is continuous and we may consider the uniquely determined trigonometric interpolating polynomial $\tilde{S}_n(x, f_n)$ of degree at most n which coincides with the function f_n at the nodes

$$x_k \equiv x_{kn} = \frac{2k\pi}{2n+1} \quad (k = 0, \pm 1, \pm 2, \ldots).$$

It is well known that (see A. Zygmund [11]),

$$\tilde{S}_n(x, f_n) = \frac{2}{2n+1} \sum_{k=-n}^{n} f_n(x_{kn}) D_n(x - x_{kn}), \tag{3}$$

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where $D_n(t)$ is the n-th Dirichlet kernel defined by

$$D_n(t) = \frac{\sin\,(n+\frac{1}{2})t}{2\sin\frac{t}{2}}. \tag{4}$$

In his paper [9], A. C. Offord claimed that the following statement is valid:

THEOREM 1. *If* x *is a Lebesgue point of* $f \in L^1_{2\pi}$ *then*

$$\tilde{S}_n(x, f_n) \to f(x) \quad \text{as} \quad n \to \infty. \tag{5}$$

At first sight this result seems to be very surprising since there is a well known classical result that there exists a continuous function f_1 of period 2π such that $\tilde{S}_n(x, f_1)$ diverges almost everywhere. (See J. Marcinkiewicz [7] and for a very similar result G. GRÜNWALD [4]. Actually, f_1 may be chosen so that $\tilde{S}_n(x. f_1)$ diverges for all x except for $x = 0$ (see A. Zygmund [11])). But a deeper analysis shows, however, that in case $\overline{\lim}_{n\to\infty} l_n < \infty$ (5) is almost obvious for continuous functions.

Indeed, let for simplicity $l_n = 1$ $(n = 1, 2, \ldots)$. It is easy to check that

$$\begin{aligned} \tilde{S}_n(x, f_n) - f(x) = \tfrac{1}{2}\left[\tilde{S}_n(x, f_n^* - f) + \tilde{S}_n\left(x - \frac{2\pi}{2n+1}, f_n^* - f\right)\right] + \\ + \left\{\tfrac{1}{2}\left[\tilde{S}_n(x, f) + \tilde{S}_n\left(x - \frac{2\pi}{2n+1}, f\right)\right] - f(x)\right\}, \end{aligned} \tag{6}$$

where f_n^* is defined by

$$f_n^*(x) = \frac{2n+1}{2\pi}\int_x^{x+\frac{2\pi}{2n+1}} f(t)dt. \tag{7}$$

Since the inequality

$$\tfrac{1}{2}\,\frac{2}{2n+1}\sum_{k=-n}^{n}|D_n(x-x_k) + D_n(x-x_{k+1})| \leqq \frac{4}{\pi} \tag{8}$$

holds independently of n (see S. Bernstein [1]), we obtain from (6) and (7) that

$$|\tilde{S}_n(x, f_n) - f(x)| \leqq \frac{4}{\pi}\,\omega\left(f; \frac{2\pi}{2n+1}\right) + \left|\tfrac{1}{2}\left[\tilde{S}_n(x, f) + \tilde{S}_n\left(x - \frac{2\pi}{2n+1}, f\right)\right] - f(x)\right|,$$

where the expression between the absolute value signs tends uniformly to 0 as $n \to \infty$ because of S. Bernstein's and W. Rogosinski's results. (See e.g. A. Zygmund [11]).

Offord's theorem is less trivial for arbitrary integrable functions f and sequences l_n. Unfortunately, the proof of this theorem given by A. C. Offord turned out to be false. This fact was observed by G. Róna, who has given another proof of this theorem. (See G. Róna [10]). It can be easily checked that this second proof given by G. Róna is also false. For this reason now we turn to Offord's theorem and give a really correct proof of it. To prove (5) we need two lemmas.

LEMMA 1. *Let* n_1, n_2 *be arbitrary integers such that* $n_1 \leqq n_2$, $|n_1 - n_2| \leqq 2n$. *We have*

$$\left| \frac{2}{2n+1} \sum_{k=n_1}^{n_2} D_n(x - x_k) \right| \leqq C_1 \quad (n = 1, 2, \ldots), \tag{9}$$

where C_1 *is an absolute constant.*

Proof. This lemma is known and plays an important role in proving convergence theorems in the theory of trigonometric interpolation. Here we give an extremely simple proof of it based on inequality (8). Let first $n_2 - n_1 + 1$ be even. In this case we have

$$\frac{2}{2n+1} \sum_{k=n_1}^{n_2} D_n(x - x_k) = \frac{2}{2n+1} \sum_{k=n_1\, n_1+2, \ldots,\, n_2-1} [D_n(x - x_k) + D_n(x - x_{k+1})]$$

and consequently

$$\left| \frac{2}{2n+1} \sum_{k=n_1}^{n_2} D_n(x - x_k) \right| \leqq \frac{2}{2n+1} \sum_{k=-n}^{n} |D_n(x - x_k) + D_n(x - x_{k+1})|.$$

Hence by virtue of (8) we obtain

$$\left| \frac{2}{2n+1} \sum_{k=n_1}^{n_2} D_n(x - x_k) \right| \leqq \frac{8}{\pi}. \tag{10}$$

Let now $n_2 - n_1 + 1$ be odd. If $n_1 = n_2$ then (9) holds because for every k

$$\frac{2}{2n+1} |D_n(x - x_k)| \leqq 1. \tag{11}$$

If $n_1 < n_2$ then

$$\frac{2}{2n+1} \sum_{k=n_1}^{n_2} D_n(x - x_k) = \frac{2}{2n+1} \sum_{k=n_1}^{n_2-1} D_n(x - x_k) + \frac{2}{2n+1} D_n(x - x_{n_2}).$$

Since $n_2 - n_1$ is even, we have by virtue of (10) and (11)

$$\left| \frac{2}{2n+1} \sum_{k=n_1}^{n_2} D_n(x - x_k) \right| \leqq \frac{8}{\pi} + 1.$$

LEMMA 2. *Let* l *and* n *be natural numbers. If* $0 < l < n/3$ *and* $\pi \geqq |x| \geqq 6l\pi/(2n+1)$ *then the inequality*

$$\left| \frac{2}{2n+1} \sum_{j=-l+1}^{l} D_n(x - x_j) \right| \leqq C_2 \frac{l}{n^2 x^2}$$

holds with an absolute constant C_2.

Proof. We obviously have

$$\frac{2}{2n+1} \sum_{j=-l+1}^{l} D_n(x - x_j) = \frac{2}{2n+1} \sum_{j=0}^{l-1} [D_n(x - x_{-l+1+2j}) + D_n(x - x_{-l+2+2j})],$$

so we obtain from (4) that

$$\left| \frac{2}{2n+1} \sum_{j=-l+1}^{l} D_n(x - x_j) \right| \leqq$$
$$\leqq \frac{|\sin (n + \frac{1}{2})x|}{2n+1} \sum_{j=0}^{l-1} \left| \operatorname{cosec} \frac{x - x_{-l+1+2j}}{2} - \operatorname{cosec} \frac{x - x_{-l+2+2j}}{2} \right|.$$

Hence we conclude that there exists an absolute constant C_3 such that

$$\left| \frac{2}{2n+1} \sum_{j=-l+1}^{l} D_n(x - x_j) \right| \leqq \frac{C_3}{n^2} \sum_{j=0}^{l-1} \left| \frac{1}{|x - x_{-l+1+2j}|\,|x - x_{-l+2+2j}|} \right| =$$
$$= \frac{C_3}{n^2 x^2} \sum_{j=0}^{l-1} \frac{|x|}{|x - x_{-l+1+2j}|} \frac{|x|}{|x - x_{-l+2+2j}|}.$$

Furthermore we have

$$|x - x_{-l+1+2j}| \geqq |x| - |x_{-l+1+2j}| \geqq \frac{|x|}{2} + \frac{3l\pi}{2n+1} - |x_{-l+1+2j}| \geqq$$
$$\geqq \frac{|x|}{2} + \frac{3l\pi}{2n+1} - \frac{2l\pi}{2n+1} > \frac{|x|}{2}$$

for $j = 0, 1, \ldots, l-1$ and similarly

$$|x - x_{-l+2+2j}| > \frac{|x|}{2}$$

for $j = 0, 1, \ldots, l-1$. Hence

$$\left| \frac{2}{2n+1} \sum_{j=-l+1}^{l} D_n(x - x_j) \right| \leqq \frac{4C_3 l}{n^2 x^2}.$$

Now we are able to prove Offord's theorem. So let $f \in L^1_{2\pi}$ and let x be a Lebesgue point of the function f. We can easily get from the formulas (2) and (3) that

$$\tilde{S}_n(x, f_n) - f(x) = \frac{1}{2\delta_n} \frac{2}{2n+1} \sum_{k=-n}^{n} \int_{x_k}^{x_{k+1}} [f(t) - f(x)]dt \sum_{j=-l_n+1}^{l_n} D_n(x - x_{k+j}).$$

Let us fix now a positive number ε less than π and suppose that n is so large that $3\delta_n < \varepsilon$. Then we have

$$\tilde{S}_n(x, f_n) - f(x) = \frac{1}{2\delta_n} \frac{2}{2n+1} \Big\{ \sum_{\varepsilon \leqq |x-x_k| \leqq \pi} + \sum_{3\delta_n < |x-x_k| < \varepsilon} + \sum_{|x-x_k| \leqq 3\delta_n} \Big\}$$
$$\int_{x_k}^{x_{k+1}} [f(t) - f(x)]dt \sum_{j=-l_n+1}^{l_n} D_n(x - x_{k+j}) = \Sigma_1 + \Sigma_2 + \Sigma_3. \tag{12}$$

First let us estimate Σ_1. Using Lemma 2 we obtain

$$|\Sigma_1| \leqq \frac{C_2 l_n}{2\delta_n n^2} \sum_{\varepsilon \leqq |x-x_k| \leqq \pi} \int_{x_k}^{x_{k+1}} |f(t) - f(x)|dt \cdot \frac{1}{|x - x_k|^2} \leqq \frac{C_2 l_n}{2\delta_n n^2 \varepsilon^2} \int_{-\pi}^{\pi} |f(t) - f(x)|dt.$$

Thus

$$|\Sigma_1| \leqq C_4 \frac{1}{n} \qquad (n = 1, 2, \ldots), \tag{13}$$

where C_4 does not depend on n. To estimate Σ_2 we split the terms contained in Σ_2 into two parts so that

$$\Sigma_2 = \sum_{x-\varepsilon < x_k < x-3\delta_n} + \sum_{x+3\delta_n < x_k < x+\varepsilon} = \Sigma_{21} + \Sigma_{22}.$$

We shall estimate Σ_{22}. Let α and β be integers such that

$$x_{\alpha-1} \leqq x + 3\delta_n < x_\alpha \leqq x_\beta < x + \varepsilon \leqq x_{\beta+1}.$$

Since ε is assumed to be fixed, α and β exists if n is large enough. We have

$$\Sigma_{22} = \frac{1}{2\delta_n} \frac{2}{2n+1} \sum_{k=\alpha}^{\beta} \int_{x_k}^{x_{k+1}} [f(t) - f(x)]dt \sum_{j=-l_n+1}^{l_n} D_n(x - x_{k+j})$$

and by virtue of Lemma 2 we get

$$|\Sigma_{22}| \leqq \frac{C_2 l_n}{2\delta_n n^2} \sum_{k=\alpha}^{\beta} \int_{x_k}^{x_{k+1}} |f(t) - f(x)|dt \frac{1}{|x - x_k|^2}.$$

By Abel's transformation we obtain

$$|\Sigma_{22}| \leqq \frac{C_2 l_n}{2\delta_n n^2} \sum_{k=\alpha}^{\beta-1} \int_{x_\alpha}^{x_{k+1}} |f(t) - f(x)| dt \left| \frac{1}{|x - x_k|^2} - \frac{1}{|x - x_{k+1}|^2} \right| + \\ + \frac{C_2 l_n}{2\delta_n n^2} \int_{x_\alpha}^{x_{\beta+1}} |f(t) - f(x)| dt \frac{1}{|x - x_\beta|^2} = \Sigma_{221} + \Sigma_{222}. \tag{14}$$

It is quite obvious that there exists an absolute constant C_5 such that

$$\Sigma_{221} \leqq \frac{C_5}{n^2} \sum_{k=\alpha}^{\beta-1} \frac{1}{x_{k+1} - x} \int_x^{x_{k+1}} |f(t) - f(x)| dt \cdot \frac{1}{(x - x_k)^2}.$$

Let now μ be an arbitrary small positive number and let us choose an ε such that

$$\frac{1}{x_{k+1} - x} \int_x^{x_{k+1}} |f(t) - f(x)| dt \leqq \mu \quad (k = \alpha, \alpha + 1, \ldots, \beta - 1)$$

in case n is large enough $(n \geqq n_0)$. Then we have

$$\Sigma_{221} \leqq \frac{C_5 \mu}{n^2} \sum_{k=\alpha}^{\beta-1} \frac{1}{(x - x_k)^2} \quad (n \geqq n_0)$$

and consequently we can choose an absolute constant C_6 such that

$$\Sigma_{221} \leqq C_6 \mu \quad (n \geqq n_0) \tag{15}$$

since $x_\alpha > x + 3\delta_n$. Further

$$\Sigma_{222} \leqq \frac{C_2 l_n}{2\delta_n n^2 \varepsilon^2} \int_x^{x+2\varepsilon} |f(t) - f(x)| dt \leqq C_7 \frac{1}{n} \quad (n \geqq n_0), \tag{16}$$

where C_7 does not depend on n. It follows from (14), (15) and (16) that

$$|\Sigma_{22}| \leqq C_6 \mu + C_7 \frac{1}{n} \quad (n \geqq n_0)$$

and we may obtain similarly

$$|\Sigma_{21}| \leqq C_6 \mu + C_7 \frac{1}{n} \quad (n \geqq n_0).$$

Hence

$$|\Sigma_2| \leqq 2C_6 \mu + 2C_7 \frac{1}{n} \quad (n \geqq n_0). \tag{17}$$

To obtain a useful estimate for Σ_3 in (12) we have to apply Lemma 1. By virtue of it we have

$$|\Sigma_3| \leqq \frac{C_1}{2\delta_n} \sum_{|x-x_k| \leqq 3\delta_n} \int_{x_k}^{x_{k+1}} |f(t) - f(x)| dt \leqq \frac{C_1}{2\delta_n} \int_{x-4\delta_n}^{x+4\delta_n} |f(t) - f(x)| dt$$

and so if n is large enough $(n \geqq n_1)$ then

$$|\Sigma_3| \leqq \mu \quad (n \geqq n_1) \tag{18}$$

We infer now from (12), (13), (17), and (18) that for an arbitrary small positive ν and for arbitrary large $n \geqq n_2(f, \nu)$ we have

$$|\tilde{S}_n(x, f_n) - f(x)| \leqq \nu + \frac{C_8}{n} \quad (n \geqq n_2)$$

with a constant C_8 independent of n. Thus (5) holds at every Lebesgue point of the function f, and this was to be proved.

2. The purpose of this section is to discuss the following problem. Let λ_n $(n = 1, 2, \ldots)$ be an arbitrary sequence of real numbers such that $\lambda_n \to 0$ as $n \to \infty$. For $f \in L^1_{2\pi}$ let us define $f_n(x)$ as

$$f_n(x) \equiv f_{n, \lambda_n}(x) = \begin{cases} \dfrac{1}{2\lambda_n} \displaystyle\int_{x-\lambda_n}^{x+\lambda_n} f(t) & \text{if } \lambda_n \neq 0, \\ f(x) & \text{if } \lambda_n = 0, \end{cases}$$

and let us consider the n-th partial sum $S_n(x, f_n)$ of the Fourier series of the function f_n, that is

$$S_n(x, f_n) = \frac{1}{\pi} \int_{-\pi}^{\pi} f_n(t) D_n(x - t) dt.$$

THEOREM 2. *Let* $f \in L^1_{2\pi}$ *and* $\lambda_n \to 0$ *as* $n \to \infty$. *Then*

$$S_n(x, f_n) \to f(x) \text{ a.e. as } n \to \infty$$

if and only if

$$\frac{\sin n\lambda_n}{n\lambda_n} [S_n(x, f) - f(x)] \to 0 \text{ a.e. as } n \to \infty,$$

more precisely

$$[S_n(x, f_n) - f(x)] - \frac{\sin n\lambda_n}{n\lambda_n} [S_n(x, f) - f(x)] \to 0 \text{ a.e. as } n \to \infty.$$

To prove Theorem 2 we need the following result which can be easily proved by using a classical result of D. K. Faddeev [3]. (A very particular case of Lemma 3 when $l_n \equiv 1$ and $f \in C_{2\pi}$ was a proved by C. Lanczos [6]).

LEMMA 3. *If* $\lambda_n \equiv \delta_n$, *where* δ_n *is defined in* (1) *and* x *is a Lebesgue point of* $f \in L^1_{2\pi}$ *then*

$$S(x, f_{n,\delta_n}) \to f(x) \quad \text{as} \quad n \to \infty.$$

Proof of Theorem 2. We may obviously assume that $\lambda_n \geqq 0$ $(n = 1, 2, \ldots)$ since $f_{n,\lambda_n} \equiv f_{n,|\lambda_n|}$. Further, for $\lambda_n = 0$

$$S_n(x, f_n) - f(x) = \frac{\sin n\lambda_n}{n\lambda_n} [S_n(x, f) - f(x)]$$

therefore without loss of generality we may suppose $\lambda_n > 0$ $(n = 1, 2, \ldots)$. Let us now represent λ_n in the form

$$\lambda_n = \frac{2l_n\pi}{2n+1} + \frac{2\Theta_n\pi}{2n+1} = \delta_n + \varepsilon_n,$$

where l_n is a non-negative integer and $0 < \Theta_n \leqq 1$. We have

$$S_n(x, f_{n,\lambda_n}) - f(x) = \frac{\delta_n}{\lambda_n} [S_n(x, f_{n,\delta_n}) - f(x)] +$$

$$+ \frac{1}{2\lambda_n} \int\limits_{x-\lambda_n}^{x-\delta_n} [S_n(t, f) - f(x)]dt + \frac{1}{2\lambda_n} \int\limits_{x+\delta_n}^{x+\lambda_n} [S_n(t, f) - f(x)]dt. \tag{19}$$

Let $n_1, n_2, \ldots$ be the indices n for which $\delta_n \neq 0$ $(n = n_1, n_2, \ldots)$. By virtue of Lemma 3 we have

$$S_{n_i}(x, f_{n_i,\delta_{n_i}}) - f(x) \to 0 \quad \text{a.e. as} \quad i \to \infty,$$

therefore

$$\frac{\delta_n}{\lambda_n} [S_n(x, f_{n,\delta_n}) - f(x)] \to 0 \text{ a.e. as } n \to \infty. \tag{20}$$

To estimate the second expression on the rightside of (19), let us notice that

$$\frac{1}{2\lambda_n} \int\limits_{x-\lambda_n}^{x-\delta_n} [S_n(t, f) - f(x)]dt + \frac{1}{2\lambda_n} \int\limits_{x+\delta_n}^{x+\lambda_n} [S_n(t, f) - f(x)]dt =$$

$$= \frac{\varepsilon_n}{2\pi\lambda_n}\left[1 - \frac{\sin\frac{\varepsilon_n}{2}}{\frac{\varepsilon_n}{2}}\cos\left(\delta_n + \frac{\varepsilon_n}{2}\right)\right]\int_{-\pi}^{\pi}[f(t) - f(x)]dt +$$

$$+ \frac{\varepsilon_n}{\lambda_n}\sum_{k=1}^{n-1}[S_k(x,f) - f(x)]\left[\frac{\sin\frac{k\varepsilon_n}{2}}{\frac{k\varepsilon_n}{2}} - \frac{\sin\frac{(k+1)\varepsilon_n}{2}}{\frac{(k+1)\varepsilon_n}{2}}\right]\cos k\left(\delta_n + \frac{\varepsilon_n}{2}\right) + \tag{21}$$

$$+ \frac{\varepsilon_n}{\lambda_n}\sum_{k=1}^{n-1}[S_k(x,f) - f(x)]\frac{\sin\frac{(k+1)\varepsilon_n}{2}}{\frac{(k+1)\varepsilon_n}{2}}\left[\cos k\left(\delta_n + \frac{\varepsilon_n}{2}\right) - \cos(k+1)\left(\delta_n + \frac{\varepsilon_n}{2}\right)\right] +$$

$$+ \frac{\sin\frac{n\varepsilon_n}{2}}{\frac{n\lambda_n}{2}}\cos n\left(\delta_n + \frac{\varepsilon_n}{2}\right)[S_n(x,f) - f(x)] = A_1 + A_2 + A_3 + A_4.$$

It is very easy to estimate A_1. Since $\lambda_n \to 0$ as $n \to \infty$, it follows that $\delta_n \to 0$ and $\varepsilon_n \to 0$ as $n \to \infty$. Therefore

$$\frac{\varepsilon_n}{2\pi\lambda_n}\left[1 - \frac{\sin\frac{\varepsilon_n}{2}}{\frac{\varepsilon_n}{2}}\cos\left(\delta_n + \frac{\varepsilon_n}{2}\right)\right] \to 0 \quad \text{as} \quad n \to \infty$$

and consequently

$$A_1 \to 0 \quad \text{a.e. as} \quad n \to \infty. \tag{22}$$

Now we recall that the function $z \curvearrowright \sin z/z$ together with its first derivative is bounded on the real line. Hence

$$|A_2| \leqq C_9\frac{(\varepsilon_n)^2}{\lambda_n}\sum_{k=1}^{n-1}|S_k(x,f) - f(x)| \leqq C_9\pi\frac{1}{n}\sum_{k=1}^{n-1}|S_k(x,f) - f(x)|$$

and

$$|A_3| \leqq \frac{\varepsilon_n\left(\delta_n + \frac{\varepsilon_n}{2}\right)}{\lambda_n}\sum_{k=1}^{n-1}|S_k(x,f) - f(x)| \leqq \pi\frac{1}{n}\sum_{k=1}^{n-1}|S_k(x,f) - f(x)|,$$

where C_9 is the upper bound for $|(\sin z/z)'|$. By a theorem of J. Marcinkiewicz [8][2]) we have for every $f \in L^1_{2\pi}$

$$\frac{1}{n}\sum_{k=1}^{n-1}|S_k(x,f)-f(x)| \to 0 \quad \text{a.e. as} \quad n \to \infty,$$

that is,

$$A_2, A_3 \to 0 \quad \text{a.e. as} \quad n \to \infty. \tag{23}$$

Let us turn to the expression A_4. Since $\lambda_n = \delta_n + \varepsilon_n$ we have

$$A_4 = \frac{\sin n\lambda_n}{n\lambda_n}[S_n(x,f)-f(x)] + \frac{\sin n\delta_n}{n\lambda_n}[S_n(x,f)-f(x)].$$

Further,

$$\frac{\sin n\delta_n}{n\lambda_n} = \frac{\sin[(n+\frac{1}{2})\delta_n - \frac{1}{2}\delta_n]}{n\lambda_n} = (-1)^{l_n+1}\frac{\sin\dfrac{\delta_n}{2}}{n\lambda_n}.$$

Hence we obtain

$$\left|\frac{\sin n\delta_n}{n\lambda_n}\right| \leqq \frac{\delta_n}{2n\lambda_n} \leqq \frac{1}{2n}.$$

Therefore

$$\frac{n}{\log n}\frac{\sin n\delta_n}{n\lambda_n}[S_n(x,f)-f(x)] \to 0 \quad \text{a.e. as} \quad n \to \infty,$$

since it is a well known classical result, that for every $f \in L^1_{2\pi}$ (See e.g. A. Zygmund [11].)

$$\frac{1}{\log n}|S_n(x,f)-f(x)| \to 0 \quad \text{a.e. as} \quad n \to \infty. \tag{24}$$

Therefore we conclude that

$$A_4 - \frac{\sin n\lambda_n}{n\lambda_n}[S_n(x,f)-f(x)] \to 0 \quad \text{a.e. as} \quad n \to \infty. \tag{25}$$

By virtue of formulas (19)—(23) and (25) we have

$$[S_n(x,f_n)-f(x)] - \frac{\sin n\lambda_n}{n\lambda_n}[S_n(x,f)-f(x)] \to 0 \quad \text{a.e. as} \quad n \to \infty$$

which was to be proved.

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[2]) Actually, J. Marcinkiewicz proved only that $1/n\,\Sigma_{k=1}^{n-1}[S_k(x,f)-f(x)]^2 \to 0$ a.e. as $n \to \infty$, but using the Hölder inequality one can obtain the result we need.

COROLLARY 1. *Let* $\lambda_n \to 0$ *as* $n \to \infty$. *If*

$$\overline{\lim_{n\to\infty}} \left| \frac{\sin n\lambda_n}{n\lambda_n} \right| \log n < \infty$$

then for every $f \in L^1_{2\pi}$ *we have*

$$\frac{1}{2\lambda_n} \int_{x-\lambda_n}^{x+\lambda_n} S_n(t, f)dt \to f(x) \quad \text{a.e. as} \quad n \to \infty.$$

This follows immediately from Theorem 2 and (24). Let us remark that (26) is necessarily fulfilled if $|\lambda_n| > C_{10} \log n/n$ $(C_{10} > 0)$ or

$$\lambda_n = \frac{l_n \pi}{n} + O\left(\frac{1}{n \log n}\right),$$

where l_n are integers such that $l_n \neq 0$.

COROLLARY 2. *Let* $\lambda_n \to 0$ *as* $n \to \infty$. *If*

$$\overline{\lim_{n\to\infty}} \left| \frac{\sin n\lambda_n}{n\lambda_n} \right| > 0 \tag{27}$$

then there exists a function f *belonging to* $L^1_{2\pi}$ *such that*

$$\overline{\lim_{n\to\infty}} \left| \frac{1}{2\lambda_n} \int_{x-\lambda_n}^{x+\lambda_n} S_n(t, f)dt \right| = \infty \quad \text{a.e.} \tag{28}$$

Indeed, let us choose a sequence $n_1 < n_2 < \ldots < n_i < \ldots$ so that

$$\lim_{i\to\infty} \frac{\sin n_i\lambda_{n_i}}{n_i\lambda_{n_i}} > 0.$$

By a result of Kolmogorov [5] there is a function $f \in L^1_{2\pi}$ such that[3])

$$\overline{\lim_{i\to\infty}} |S_{n_i}(x, f)| = \infty \quad \text{a.e.}$$

Therefore we have for this function f,

$$\overline{\lim_{n\to\infty}} \left| \frac{\sin n\lambda_n}{n\lambda_n} [S_n(x, f) - f(x)] \right| = \infty \quad \text{a.e.}$$

[3]) Kolmogorov proved only that there exists a function f belonging to $L^1_{2\pi}$ such that $S_n(x, f)$ diverges everywhere as $n \to \infty$, but the result we need also follows from the arguments used in the course of the proof of this theorem given by A. N. Kolmogorov. This fact was noticed by A. Zygmund (see [11], vol. I, p. 314).

Hence, applying Theorem 2, we obtain (28).

Let us observe that (27) holds whenever we can choose a sequence $n_1 < n_2 < \ldots < n_i < \ldots$ such that $n_i \lambda_{n_i}$ converges as $i \to \infty$ and

$$\lim_{i \to \infty} n_i \lambda_{n_i} \neq k\pi \quad (k = \pm 1, \pm 2, \ldots).$$

Corollary 3. *Let* $\lambda_n \to 0$ *as* $n \to \infty$. *If*

$$\lim_{n \to \infty} \left| \frac{\sin n\lambda_n}{n\lambda_n} \operatorname{loglog} n \right| = \infty \tag{29}$$

then one can find a function $f \in L^1_{2\pi}$ *such that* (28) *holds.*

To prove this let us write

$$\left| \frac{\sin n\lambda_n}{n\lambda_n} \right| S_n(x, f) = \frac{1}{\alpha_n} \cdot \frac{S_n(x, f)}{\alpha_n \operatorname{loglog} n},$$

where

$$\alpha_n = \left(\left| \frac{\sin n\lambda_n}{n\lambda_n} \right| \operatorname{loglog} n \right)^{-\frac{1}{2}}.$$

Since $\alpha_n \to 0$ as $n \to \infty$, by virtue of a theorem of Y.-M. Chen [2] there exists a function $f \in L^1_{2\pi}$ such that

$$\overline{\lim_{n \to \infty}} \frac{S_n(x, f)}{\alpha_n \operatorname{loglog} n} > 1 \quad \text{a.e.},$$

that is,

$$\overline{\lim_{n \to \infty}} \left| \frac{\sin n\lambda_n}{n\lambda_n} S_n(x, f) \right| = \infty \quad \text{a.e.}$$

Using Theorem 2 we obtain (28).

We remark that (29) holds if for example $n\lambda_n = 0(1)$ and

$$\min_{k = \pm 1, \pm 2, \ldots} |n\lambda_n - k\pi| > C_{11} (\operatorname{loglog} n)^{-\varepsilon} \quad (0 \leqq \varepsilon < 1).$$

3. To study the behaviour of the polynomials $\tilde{S}_n(x, f_{n,\lambda_n})$ for arbitrary sequences λ_n and functions $f \in L^1_{2\pi}$ seems to be much more difficult than that of $S_n(x, f_{n,\lambda_n})$. For the time being we have no deep results concerning the convergence or divergence of $\tilde{S}_n(x, f_{n,\lambda_n})$. In particular, we do not know what to expect in the case $\lambda_n = 1/\sqrt{n}$ which is a »good» parameter for the Fourier sums as it has been proved in Theorem 2. At the same time, however, we think that the following conjectures are true:

a) *Let* $\lambda_n = \pi/2n$ $(n = 1, 2, \ldots)$. *Then there exists a function* $f \in C_{2\pi}$ *such that* $\tilde{S}_n(x, f_{n,\lambda_n})$ *diverges almost everywhere.*

b) *Let* $\lambda_n = \pi/n$ $(n = 1, 2, \ldots)$. *Then there exists a function* $f \in L^1_{2\pi}$ *such that* $\tilde{S}_n(x, f_{n,\lambda_n})$ *diverges almost everywhere.*

We hope to return soon to these questions.

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