The Ritt theorem in several variables

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§ 1. Formulation of the problem

The motivation for the problem treated in this note arises in the theory of convolution equations. Let U be a locally convex space of functions or distributions. It is assumed that by means of a transformation $\mathcal{F}$ of Fourier type[1]), the space U is isomorphic to a subalgebra $\hat{U}$ of the algebra $\mathcal{A} = \mathcal{A}(\mathbf{C}^n)$ of all entire functions in $\mathbf{C}^n$. Using the inverse transformation $\mathcal{F}^{-1}$ one can transfer the operation of multiplication from $\hat{U}$ to U. The resulting ring multiplication in U is called the *convolution* and is denoted $\phi * \psi$ $(\phi, \psi \in U)$. More generally, let $G \in \mathcal{A}$ be such that the multiplication by the function G is a continuous endomorphism $\mathcal{M}_G$ of $\hat{U}$. If $\mathcal{C}_G$ is the continuous endomorphism of U corresponding to $\mathcal{M}_G$ under the isomorphism $\mathcal{F}$, then $\mathcal{C}_G$ is called a *convolutor* of the space U. Sometimes — but not always — there exists a distribution S (or another "generalized function") such that the Fourier transform of S is the function G, and for each $\phi \in U$, $\mathcal{C}_G(\phi)$ can be interpreted as the convolution $S * \phi$ in some generalized sense.

Given $f \in U$, consider the equation

$$S * u = f \tag{1.1}$$

which is equivalent to the equation $\hat{S}(\zeta)\hat{u}(\zeta) = \hat{f}(\zeta)$. It is natural to ask for necessary and sufficient conditions for the solvability of (1.1) in U. An obvious necessary condition is that $\hat{f}/\hat{S}$ be entire. In order to conclude that $\hat{f}/\hat{S}$ is in $\hat{U}$ one usually

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[1]) $\mathcal{F}$ can be either the classical Fourier-Laplace transformation or the Fourier-Borel transformation or another similar transformation depending upon the nature of the space U. The inverse transformation will be denoted by $\mathcal{F}^{-1}$; instead of $\mathcal{F}(\phi)$ we shall write $\hat{\phi}$. Similarly, $\hat{U}$ stands for $\mathcal{F}(U)$, etc.

has to use additional properties of S and U, and this is often done by combining a theorem of Paley-Wiener type[2]) with the following intermediary step: There is another subalgebra $\mathscr{B}$ of $\mathscr{A}$ containing $\hat{S}$ and $\hat{f}$, and such that

($\mathscr{S}$) *whenever* $F, G \in \mathscr{B}$ *are such that* $F/G \in \mathscr{A}$, *then* $F/G \in \mathscr{B}$.

Obviously, if $\mathscr{B} = \hat{U}$ has property ($\mathscr{S}$), then the above necessary condition for the solvability of (1.1) is also sufficient. However, this is rarely the case, and one usually has to consider a different subalgebra $\mathscr{B}$. In the sequel, subalgebras $\mathscr{B}$ with the property ($\mathscr{S}$) will be called *stable*. Denoting by $F_{\mathscr{B}}$ the quotient field of $\mathscr{B}$, stability of $\mathscr{B}$ means $F_{\mathscr{B}} \cap \mathscr{A} = \mathscr{B}$.

As an illustration of the previous general argument, consider the following two examples:

Example 1. Let $U = \mathscr{A}'$ be the dual of the space $\mathscr{A}$ endowed with the compact-open topology. By means of the Fourier-Borel transform (cf. § 2), U is isomorphic to the algebra $U = \mathrm{Exp}$ = the set of all entire functions of exponential type in $\mathbf{C}^n$. The stability of Exp is a classical result of Lindelöf [18] (cf. also below). Hence a necessary and sufficient condition for the solvability of equation (1.1) in the space of analytic functionals in $\mathbf{C}^n$ is that $\hat{f}/S \in \mathscr{A}$. Non-trivial examples of the use of the stability of Exp in the study of convolution equations can be found in Hörmander [13] and Malgrange [19, 20].

Example 2. Let $U = \mathscr{E}'$ be the space of distributions with compact support in $\mathbf{R}^n$. Consider equation (1.1) with $S, f \in \mathscr{E}'$. Then, although the function $\hat{f}/\hat{S}$ is of exponential type (provided it is entire), in general $\hat{f}/\hat{S} \notin \hat{\mathscr{E}}'$, unless $\hat{S}$ is invertible (cf. [12]). Hence $\hat{\mathscr{E}}'$ is not stable, and it is the non-stability of $\hat{\mathscr{E}}'$ which causes serious difficulties in the study of such convolution equations. In order to overcome these difficulties, one often has to resort to intricate methods (cf. [13, 20]).

It therefore seems interesting to examine the stability of those subalgebras of $\mathscr{A}$ which arise in the study of convolution equations. Below we list some of them:

$\mathscr{P}$: polynomials in $\mathbf{C}^n$;

E: finite *exponential sums*, i.e. entire functions of the form

[2]) For example, the Paley-Wiener-Schwartz theorem for the case of distributions, or the Pólya-Ehrenpreis-Martineau theorem (cf. § 2) when analytic functionals are involved, etc.

$$H(\zeta) = \sum_{j=1}^{s} h_j e^{\langle\theta_j, \zeta\rangle}, \quad (\zeta \in \mathbf{C}^n) \tag{1.2}$$

where h_j are complex numbers, and $\theta_j \in \mathbf{C}^n$, also called *frequencies* of H, are given points in $\mathbf{C}^n$ [3]);

$E_{\mathcal{P}}$: *exponential polynomials* in $\mathbf{C}^n$, i.e. functions of the form (1.2) with $h_j \in \mathcal{P}$;

$\tilde{E}_{\mathcal{P}}$: entire functions of the form H/P for some $H \in E_{\mathcal{P}}$, $P \in \mathcal{P}$;

$\hat{\mathcal{R}}_\omega$: Fourier transforms of distributions $\Phi \in \mathcal{E}'$ such that, for some constants $t \geq 0$, $r > 0$, $c > 0$ and A real (all depending on Φ),

$$\max_{\zeta' \in \Delta(\zeta; r)} |\hat{\phi}(\zeta')| \geq c \exp\left[A\omega(\xi) + h_{[\Phi]}(\eta)\right], \quad (\forall\ \zeta \in \mathbf{C}^n), \tag{1.3}$$

where $\Delta(\zeta; r)$ denotes the polydisk of center ζ and radius r; $\zeta = \xi + i\eta$; $h_{[\Phi]}$ is the supporting function of the support of Φ; and $\omega(\xi) = \ln(2 + |\xi|)$.[4]) (For the properties of such classes, see [3, 4, 5]);

$\hat{\mathcal{E}}'$: Fourier transforms of Schwartz distributions with compact support; or, more generally,

$\hat{\mathcal{E}}'_\omega$: the same for Beurling distributions (cf.[4]));

Exp: entire functions in $\mathbf{C}^n$ of exponential type.

The stability of $\mathcal{P}$ is simple (cf. e.g. [9]). That E is stable for $n = 1$ constitutes the so-called Ritt theorem [23]. For $n > 1$ this was proved by Avanissian and Martineau [1]. $E_{\mathcal{P}}$ is clearly non-stable. Indeed, for $n = 1$,

$$\frac{\sin z}{z} \in \tilde{E}_{\mathcal{P}} \setminus E_{\mathcal{P}} \quad (z \in \mathbf{C}). \tag{1.4}$$

It can be easily established that $\hat{\mathcal{R}}_\omega$ is stable [5]. $\hat{\mathcal{E}}'$, and more generally $\hat{\mathcal{E}}'_\omega$, are not stable (cf. Example 2 above). The stability of Exp was proved by E. Lindelöf [18] for $n = 1$. The extension to arbitrary $n \geq 1$ is due to Ehrenpreis [10] and Malgrange [19].

It remains to be found, first how "unstable" $E_{\mathcal{P}}$ really is [5]), and secondly,

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[3]) In what follows it will always be assumed that the frequencies θ_j are pairwise distinct and the coefficients h_j are all non-zero. Besides, $\langle\ ,\ \rangle$ denotes the bilinear product in $\mathbf{C}^n$: $\langle\theta, \zeta\rangle = \theta_1\zeta_1 + \ldots + \theta_n\zeta_n$.

[4]) Actually, for ω one may take any continuous subadditive function in $\mathbf{R}^n$ satisfying certain growth conditions. Then, instead of $\mathcal{E}'$, one has to take the space $\mathcal{E}'_\omega$ of Beurling distributions (cf. [2]).

[5]) i.e., to describe the structure of entire functions of the form F/G where $F, G \in E_{\mathcal{P}}$ (or, equivalently, $F, G \in \tilde{E}_{\mathcal{P}}$).

whether $\tilde{E}_{\mathcal{P}}$ is stable or not. Our main objective in this note is to show that both questions have a common answer:

MAIN THEOREM. *The subalgebra* $\tilde{E}_{\mathcal{P}}$ *is stable. Hence the most general form of an entire function of the form* F/G, $F, G \in E_{\mathcal{P}}$, (cf.[5])) *is*

$$\frac{F}{G} = \frac{H}{P} \quad (H \in E_{\mathcal{P}}, P \in \mathcal{P}). \tag{1.5}$$

Given an exponential polynomial H, the greatest common divisor of its coefficients,

$$d_H = (h_1, \ldots, h_s), \tag{1.6}$$

will be called *the content of* H. Furthermore, let $f = H/P$ be any element of $\tilde{E}_{\mathcal{P}}$. Then we shall say that H/P is a *reduced form* of f, provided $(P, d_H) = 1$. If this is so and $f = H^*/P^*$, for some $H^* \in E_{\mathcal{P}}$ and $P^* \in \mathcal{P}$, then it is easy to see that $P | P^*$ and $d_H | d_{H^*}$. Hence the following lemma holds:

LEMMA. *Every function in* $\tilde{E}_{\mathcal{P}}$ *has a unique reduced form* [6]). *Furthermore, let* $H, F \in E_{\mathcal{P}}$ *be such that* $Q = H/F \in \mathcal{P}$. *Then* $Q | d_H$.

Then the main theorem can be reformulated as follows:

THEOREM 1. *Let* $F, G \in E_{\mathcal{P}}$ *be such that* $F/G \in \mathcal{A}$. *Then there exist unique* [6]) $H \in E_{\mathcal{P}}$ *and* $P \in \mathcal{P}$ *such that* $F/G = H/P$ *and* $(P, d_H) = 1$.

As an application of Theorem 1 one obtains:

THEOREM 2. *Let* $F, G \in E_{\mathcal{P}}$ *and* $F/G \in \mathcal{A}$. *Let* H/P *be the reduced form of* F/G. *Then* $P | d_G$.

COROLLARY 1. *Let* F, G *be exponential polynomials in* $\mathbf{C}^n$ *such that* F/G *is entire and* $d_G = 1$. *Then* F/G *is also an exponential polynomial.*

COROLLARY 2. E *is stable* $(n \geq 1)$.

As was mentioned above, Corollary 2 for $n = 1$ is the Ritt theorem [23]. Other proofs of Ritt's theorem were given by H. Selberg [24], P. D. Lax [16] and A. Shields [25]. Shields proves that, for $n = 1$, the hypotheses $F \in E_{\mathcal{P}}, G \in E$ and $F/G \in \mathcal{A}$ imply $F/G \in E_{\mathcal{P}}$. He also mentions that, according to an unpublished result of

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[6]) Unique up to a constant multiple of H and P.

W. D. Bouwsma, Corollary 1 holds when $n = 1$. Finally, Corollary 2 is due to V. Avanissian and A. Martineau (unpublished [1]).[7])

The main theorem is established in Section 2. Theorem 2, as well as another application of Theorem 1 are discussed in the concluding Section 3.

Theorems 1 and 2 were announced in our note [7].

For applications of exponential polynomials in one variable see [15].

§ 2. Proof of the main theorem

For $\zeta \in \mathbf{C}^n$, $\bar{\zeta}$ denotes the complex conjugate of ζ, i.e. $\bar{\zeta} = (\bar{\zeta}_1, \ldots, \bar{\zeta}_n)$. When ζ is considered as a point in $\mathbf{R}^{2n}$, the coordinates of ζ are $(\mathrm{Re}\,\zeta_1, \mathrm{Im}\,\zeta_1, \ldots, \mathrm{Re}\,\zeta_n, \mathrm{Im}\,\zeta_n)$. The Euclidean inner product in $\mathbf{R}^m$ will be denoted $\langle\ ,\ \rangle_m$. We recall from § 1 that the bilinear product in $\mathbf{C}^n$, denoted simply by $\langle\ ,\ \rangle$, is

$$\langle z, \zeta\rangle = z_1\zeta_1 + \ldots + z_n\zeta_n, \quad z, \zeta \in \mathbf{C}^n.$$

If F is an exponential polynomial with frequencies $a_1, \ldots, a_p$, i.e.

$$F(\zeta) = \sum_{j=1}^{p} P_j(\zeta) e^{\langle a_j, \zeta\rangle}, \tag{2.1}$$

where P_j are polynomials, we will denote by $[F]$ the convex hull of the points $\bar{a}_1, \ldots, \bar{a}_p$ in $\mathbf{R}^{2n}$. Let $h_{[F]}$ be the supporting function of the set $[F]$, i.e. for each $\zeta \in \mathbf{C}^n = \mathbf{R}^{2n}$

$$h_{[F]}(\zeta) = \max_{x \in [F]} \langle x, \zeta\rangle_{2n} = \max_{1 \le j \le p} \langle \bar{a}_j, \zeta\rangle_{2n} = \max_{1 \le j \le p} \mathrm{Re}\,\langle a_j, \zeta\rangle. \tag{2.2}$$

We shall need a few simple facts about analytic functionals, i.e. elements of $\mathscr{A}'$ (cf. [14, 17, 21, 26]). The Fourier-Borel transform $\hat{\mu}(\zeta) = \mu(e^{\langle \cdot, \zeta\rangle})$ establishes an isomorphism of the spaces $\mathscr{A}'$ and Exp. Given $\mu \in \mathscr{A}'$, the *indicator* p_μ of μ is defined by

$$p_\mu(\zeta) = \overline{\lim_{t\to\infty}} \frac{\log|\hat{\mu}(t\zeta)|}{t}$$

and its upper-semicontinuous regularization $\bar{p}_\mu$,

$$\bar{p}_\mu(\zeta) = \overline{\lim_{\zeta'\to\zeta}}\, p_\mu(\zeta'),$$

is plurisubharmonic.

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[7]) Professor H. S. Shapiro has kindly informed us that several years ago he proved Corollary 2 by means of an inductive argument. His proof has not been published. *Added in proof:* In the meantime K. Kitagawa [27] published a proof of Corollary 1. His proof is rather sketchy. A complete proof of Corollary 1 was recently announced by V. Avanissian and R. Gay in [28, 29].

A *carrier* of an analytic functional μ is a compact subset K of $\mathbf{C}^n$ such that for every neighborhood U of K there is a constant C such that

$$|\mu(\phi)| \leq C \sup_{z \in U} |\phi(z)|,$$

for all $\phi \in \mathscr{A}$. A compact convex set K is called a *convex support* of μ if K is a minimal compact convex carrier of μ, i.e. K is a carrier of μ such that if L is another carrier of μ and $L \subseteq K$, then $\text{ch.}\, L = K$, where $\text{ch.}\, L$ denotes the convex hull of L.

In the sequel the Pólya-Ehrenpreis-Martineau theorem will be used in the following formulation (cf. [14], Th. 5.2, Cor. 5.3):

THEOREM I. *Let* $\mu \in \mathscr{A}'$, *then*

$$\bar{p}_\mu(\zeta) \equiv \inf\{h_K(\zeta) \colon K \text{ carries } \mu\}. \tag{2.3}$$

Hence μ *has a unique convex support if and only if* $\bar{p}_\mu$ *is a convex function.*

In [4, 6] we established the following lower estimate for exponential polynomials:

THEOREM II. *Let* $P \in E_{\mathscr{P}}$, *i.e.* P *is an exponential polynomial. Then, for each* $\varepsilon > 0$, *there exists a constant* $C = C(\varepsilon, P)$ *such that if* f *is an analytic function in the polydisk* $\Delta(\zeta, \varepsilon) = \Delta$,

$$\Delta = \{\zeta' \in \mathbf{C}^n \colon \max_j |\zeta_j' - \zeta_j| \leq \varepsilon\}, \tag{2.4}$$

then

$$|f(\zeta)| e^{h_{[P]}(\zeta)} \leq C \max_{z \in \Delta} |f(z) P(z)|. \tag{2.5}$$

To every exponential polynomial there corresponds, via the Fourier-Borel transform, a unique $\mu_P \in \mathscr{A}'$ such that $\hat{\mu}_P(\zeta) = P(\zeta)$. Hence we have

COROLLARY 1. *For every* $P \in E_{\mathscr{P}}$, $[P]$ *is the unique convex support of* μ_P *and*

$$\bar{p}_{\mu_P}(\zeta) \equiv p_{\mu_P}(\zeta) \equiv h_{[P]}(\zeta). \tag{2.6}$$

(Indeed, it suffices to set $f \equiv 1$ in Theorem II and apply Theorem I).

Let A be a compact convex subset of $\mathbf{R}^m$. For an arbitrary $\theta \in \mathbf{R}^m$, set $A^\theta = \{x \in A \colon \langle x, \theta \rangle_m = h_A(\theta)\}$. If A^θ consists of one point only, θ is called a *regular direction* of A. The set of all regular directions will be denoted $\text{reg}\, A$. The set of all extremal points will be denoted $\text{ext}\, A$.

LEMMA 1 (cf. [8]). *Let* A *and* B *be compact convex sets in* $\mathbf{R}^m$. *Then for each* θ,

$$A^\theta \cap \operatorname{ext} A = \operatorname{ext}(A^\theta), \tag{2.7}$$

$$(A + B)^\theta = A^\theta + B^\theta, \tag{2.8}$$

$$\operatorname{ext}(A + B) \subseteq \operatorname{ext} A + \operatorname{ext} B. \tag{2.9}$$

Moreover, every z *in* $\operatorname{ext}(A + B)$ *has a unique decomposition* $z = z_1 + z_2$, $z_1 \in \operatorname{ext} A$, $z_2 \in \operatorname{ext} B$. *Although the inclusion in* (2.9) *cannot be replaced by equality, one has for every* $\theta \in \operatorname{reg} A \cup \operatorname{reg} B$,

$$(\operatorname{ext}(A + B))^\theta = (\operatorname{ext} A)^\theta + (\operatorname{ext} B)^\theta. \tag{2.10}$$

(Relations (2.7)—(2.9) are obvious. The uniqueness of the decomposition $z = z_1 + z_2$ was proved in [11]. Equation (2.10) follows from (2.7) and (2.8)).

Finally, a simple lemma on piecewise linear functions in $\mathbf{R}^m$ will be necessary. Given $x_0 \in \mathbf{R}^m$ and $\varrho > 0$, set

$$B_m(x_0; \varrho) = \{x \in \mathbf{R}^m : \|x - x_0\| = \max_{1 \le i \le m} |x_i - x_{0,i}| < \varrho\}.$$

We shall write $B_m(\varrho)$ for $B_m(0; \varrho)$. Let $\mathscr{L}$ be the class of all continuous functions on $B_m(1)$ with the following property: for each $\phi \in \mathscr{L}$ there exist N distinct vectors $\theta_j \in \mathbf{R}^m$ $(j = 1, \ldots, N)$ such that for each $x \in B_m(1)$, $\phi(x) - \phi(0) = \langle x, \theta_j \rangle_m$ for some j. Given a function f on an open convex set $G \subseteq \mathbf{R}^m$, f will be called *piecewise linear on* G if for each $x_0 \in G$ there exist a $\varrho > 0$ and an affine mapping χ of $\mathbf{R}^m$ (i.e. $\chi(z) = Az + x_0$ for some non-singular $m \times m$ matrix A) such that $\chi(0) = x_0$, $\chi(B_m(1)) = B_m(x_0; \varrho) \subseteq G$, and if ϕ denotes the restriction of $f \circ \chi$ to $B_m(1)$, then ϕ is in $\mathscr{L}$.

LEMMA 2. *Let* f *be a piecewise linear function defined on an open convex set* $G \subseteq \mathbf{R}^m$. *Then* f *is convex if (and only if)* f *is subharmonic.*

Proof. In view of the local character of convexity the lemma will follow if we show that any subharmonic function ϕ, $\phi \in \mathscr{L}$, is convex in $B_m(\varrho)$ for some $\varrho \le 1$. We can assume $\phi(0) = 0$. Let $N(\phi)$ be the number N corresponding to ϕ by definition. Our claim is trivial when either $N(\phi) = 1$ or $m = 1$. Assume that it has been proved for all integers $1, 2, \ldots, N - 1$ and arbitrary m $(N \ge 2)$. Fix ϕ to be any function in $\mathscr{L}$ for which $N(\phi) = N$. Let $\theta_1, \ldots, \theta_N$ be the corresponding vectors. Set $V = \{x \in \mathbf{R}^m : \langle x, \theta_i - \theta_j \rangle = 0, \forall\, i, j,\ i \ne j\}$ and $d = \dim V$. Since $N \ge 2$ and θ_i are distinct vectors, $d < m$, i.e. $0 \le d \le m - 1$. Consider first the case $d = 0$, i.e. $V = \{0\}$. If $x_0 \in B_m(1)$ is arbitrary, $x_0 \ne 0$, let N_0 be the total number of θ_j's for which $\phi(x_0) = \langle x_0, \theta_j \rangle_m$. Then $N_0 < N$, because $d = 0$. By continuity, in some $B_m(x_0; \delta) \subseteq B_m(1)$, one needs only N_0 linear functions to define ϕ. By the induction hypothesis ϕ is convex in some $B_m(x_0; \delta_0)$, $0 < \delta_0 \le \delta$. Hence, it remains to be shown that ϕ is convex at the origin. Let x_1, x_2 be any two distinct points in $B_m(1)$ such that $x_1 = \alpha x_2$ for some $\alpha \leq 0$. One has to show

$$\phi(\lambda x_1 + (1-\lambda)x_2) \leq \lambda\phi(x_1) + (1-\lambda)\phi(x_2), \quad 0 \leq \lambda \leq 1. \tag{2.11}$$

For $m = 1$, this is trivial because subharmonicity coincides with convexity. If $m > 1$, there will be a vector $y \in B_m(1)$ linearly independent of x_1 such that $x_i + y \in B_m(1)$ $(i = 1, 2)$. Then for any $k = 1, 2, \ldots$ the segment with end-points $X_{i,k} = x_i + k^{-1}y$ does not contain the origin and by the local convexity of ϕ in $B_m(1) \setminus \{0\}$, the inequality (2.11) is satisfied for $X_{1,k}, X_{2,k}$ instead of x_1, x_2. Letting $k \to \infty$, (2.11) follows by continuity. Finally, for the case $d \geq 1$, one can assume that

$$V = \{x \in \mathbf{R}^m : x_i = 0 \text{ for } i > d\}.$$

For $x \in \mathbf{R}^m$, set $\tilde{x} = (x_{d+1}, \ldots, x_m)$ and $\tilde{\phi}(\tilde{x}) = \phi(x)$. It suffices to prove the convexity of $\tilde{\phi}$ in $B_{m-d}(0; \varrho)$ for some $\varrho \leq 1$. However, since

$$\dim \{\tilde{x} \in \mathbf{R}^{m-d} : \langle \tilde{x}, \tilde{\theta}_i - \tilde{\theta}_j \rangle_{m-d} = 0 \ \ \forall\, j, i \ \ i \neq j\} = 0,$$

we are in the preceding case.

Corollary 2. *Let f be a plurisubharmonic function in $\mathbf{C}^n$ which is piecewise linear in $\mathbf{R}^{2n} = \mathbf{C}^n$. Then f is convex in $\mathbf{R}^{2n}$.*

Proof of Theorem 1. Let

$$F(\zeta) = \sum_{j=1}^{p} P_j(\zeta) e^{\langle a_j, \zeta \rangle}, \quad (P_j \in \mathcal{P}) \tag{2.12}$$

$$G(\zeta) = \sum_{j=1}^{q} Q_j(\zeta) e^{\langle b_j, \zeta \rangle}, \quad (Q_j \in \mathcal{P}) \tag{2.13}$$

be such that

$$K = \frac{F}{G} \in \mathcal{A}. \tag{2.14}$$

Then $K \in \mathrm{Exp}$. Let $\nu_0, \mu, \mu_0 \in \mathcal{A}'$ be such that $K = \hat{\nu}_0$, $G = \hat{\mu}$, $F = \hat{\mu}_0$. Obviously, $p_{\mu_0} \leq p_\mu + p_{\nu_0}$. Since $p_{\mu_0} = \bar{p}_{\mu_0} = h_{[F]}$, $p_\mu = \bar{p}_\mu = h_{[G]}$ (cf. (2.6)), we obtain from Theorems I, II that for every $\varepsilon > 0$, there are constants C_1, C_2 depending only on ε, F and G such that for every $\zeta \in \mathbf{C}^n$,

$$e^{p_\mu(\zeta)} |K(\zeta)| \leq C_1 \max_{z \in \Delta(\zeta; \varepsilon)} |F(z)| \leq C_2 e^{p_{\mu_0}(\zeta) + \varepsilon|\zeta|}. \tag{2.15}$$

This shows that $p_{\nu_0} \leq p_{\mu_0} - p_\mu$, hence by (2.6)

$$p_{\nu_0} = \bar{p}_{\nu_0} = h_{[F]} - h_{[G]}. \tag{2.16}$$

By Theorem 2, p_{ν_0} is a convex function. Since p_{ν_0} is also positively homogeneous of order 1, there exists a compact convex set $[K] \subseteq \mathbf{R}^{2n}$ such that $p_{\nu_0} = h_{[K]}$, hence by (2.16),

$$[F] = [G] + [K]. \tag{2.17}$$

(By Theorem I of this section, the set $[K]$ is obviously the unique convex support of the functional ν_0).

Let $\mathcal{V}$ be the family of all linear varieties in $\mathbf{R}^{2n}$ of dimension $2n - 1$, each of which contains at least two different points of the form

$$z = \sum_{i=1}^{p} l_i \bar{a}_i + \sum_{j=1}^{q} m_j \bar{b}_j, \tag{2.18}$$

where all the coefficients l_i, m_j are integers. Then the set $\mathcal{N} = \{\theta \in S^{2n-1}: \theta \perp \Lambda$, for some $\Lambda \in \mathcal{V}\}$ has measure zero in S^{2n-1}, the unit sphere in $\mathbf{R}^{2n}$. (Indeed, fix arbitrary $z_1, \ldots, z_T$ $(T \geq 2)$ of the form (2.18); then the normal vectors to all $\Lambda \in \mathcal{V}$ such that $z_i \in \Lambda$ $(i = 1, \ldots, T)$, define an algebraic subvariety of S^{2n-1} of dimension $\leq 2n - 2$.)

Obviously, one can assume that the compact sets $[F]$, $[G]$, $[K]$ lie in $\mathbf{R}^{2n}_{+}$, the positive orthant in $\mathbf{R}^{2n}$. The set $\mathcal{N}$ being of measure zero in S^{2n-1}, one can find $\nu \in (S^{2n-1} \setminus \mathcal{N}) \cap \mathbf{R}^{2n}_{+}$. Since $\nu \notin \mathcal{N}$, ν is a regular direction for both $[F]$ and $[G]$. Hence $h_{[F]}(\nu) = \langle \bar{a}_j, \nu \rangle_{2n}$ for exactly one $\bar{a}_j$. Renumbering the a_i's, if necessary, and using the fact that $\nu \notin \mathcal{N}$, one can assume that

$$h_{[F]}(\nu) = \langle \bar{a}_1, \nu \rangle_{2n} > \langle \bar{a}_2, \nu \rangle_{2n} > \ldots > \langle \bar{a}_p, \nu \rangle_{2n} > 0 \tag{2.19}$$

Similarly,

$$h_{[G]}(\nu) = \langle \bar{b}_1, \nu \rangle_{2n} > \ldots > \langle \bar{b}_q, \nu \rangle_{2n} > 0. \tag{2.20}$$

Set $k_1 = a_1 - b_1$. By Lemma 1, $\bar{k}_1 \in \operatorname{Ext}[K]$, and $\bar{k}_1$ is the only point of $[K]$ for which $h_{[K]}(\nu) = \langle \bar{k}_1, \nu \rangle_{2n}$.

Set

$$d = \langle \bar{b}_1 - \bar{b}_2, \nu \rangle_{2n}, \quad H^+ = \{x \in \mathbf{R}^{2n}: \langle x, \nu \rangle_{2n} > \langle \bar{a}_1, \nu \rangle_{2n} - d\}, \quad H^- = \mathbf{R}^{2n} \setminus H^+.$$

Let r be such that $\bar{a}_i \in H^+$ for $i = 1, \ldots, r$ and $\bar{a}_i \in H^-$ for $i = r + 1, \ldots, p$. Using the notation (2.12)—(2.14), set

$$\begin{cases} f_1(\zeta) = F(\zeta) - P_1(\zeta) e^{\langle a_1, \zeta \rangle}, \\ g_1(\zeta) = G(\zeta) - Q_1(\zeta) e^{\langle b_1, \zeta \rangle}, \\ K_1(\zeta) = Q_1(\zeta) K(\zeta) - P_1(\zeta) e^{\langle k_1, \zeta \rangle}, \\ F_1(\zeta) = Q_1(\zeta) f_1(\zeta) - P_1(\zeta) g_1(\zeta) e^{\langle k_1, \zeta \rangle}. \end{cases} \tag{2.21}$$

Then f_1, F_1 are exponential polynomials, K_1 is entire, and

$$F_1 = K_1 G. \tag{2.22}$$

Next we claim that

$$\bar{k}_1 + \bar{b}_j \in H^- \quad (j = 2, \ldots, q). \tag{2.23}$$

Indeed, by (2.19) and (2.20), $\langle \bar{k}_1 + \bar{b}_j, \nu\rangle_{2n} = \langle \bar{a}_1, \nu\rangle_{2n} + \langle -\bar{b}_1 + \bar{b}_j, \nu\rangle_{2n} \leq \langle \bar{a}_1, \nu\rangle_{2n} + \langle -\bar{b}_1 + \bar{b}_2, \nu\rangle_{2n} = \langle \bar{a}_1, \nu\rangle_{2n} - d$.

Since $\bar{a}_2, \ldots, \bar{a}_r \in H^+$, it follows from (2.23) that none of the terms with frequencies a_j $(2 \leq j \leq r)$ can be cancelled in F_1 by a term coming from $P_1(\zeta)g_1(\zeta)e^{<k_1, \zeta>}$. Moreover, it also shows that a_1 cannot be a frequency of F_1. Hence, if $x \in [F_1]$, $x \neq \bar{a}_2$, and

$$h_{[F_1]}(\nu) = \langle \bar{a}_2, \nu\rangle_{2n} > \langle x, \nu\rangle_{2n}. \tag{2.24}$$

Thus, the frequencies of F_1 *are* $a_2, \ldots, a_r, a'_{r+1}, \ldots, a'_{p_1}$, *where* $\{\bar{a}'_{r+1}, \ldots, \bar{a}'_{p_1}\}$ *is a subset of* $\{\bar{a}_{r+1}, \ldots, \bar{a}_p, \bar{a}_1 - \bar{b}_1 + \bar{b}_2, \ldots, \bar{a}_1 - \bar{b}_1 + \bar{b}_q\} \subseteq H^-$ *and* $[F_1] \subseteq [F]$. Indeed, $[F_1] \subseteq \text{ch.}\,([f_1] \cup \{\bar{k}_1 + \bar{b}_j\}_{j \geq 2}) \subset [F] \cup ([K] + [G]) = [F]$.

Next we proceed with F_1, G, K_1 in the same fashion as above with F, G, K. Hence there is a $\nu_1 \in \mathscr{A}'$ such that $\hat{\nu}_1 = K_1$ and ν_1 has a unique convex support $[K_1]$, and $[F_1] = [K_1] + [G]$. In particular, by (2.24) and (2.20),

$$\langle \bar{a}_2, \nu\rangle_{2n} = h_{[K_1]}(\nu) + h_{[G]}(\nu) = h_{[K_1]}(\nu) + \langle \bar{b}_1, \nu\rangle_{2n}.$$

Hence $h_{[K_1]}(\nu) = \langle \bar{k}_2, \nu\rangle_{2n}$ for a unique $\bar{k}_2 \in [K_1]$. On the other hand, by Lemma 1, $k_2 = a_2 - b_1$. Set

$$\begin{cases} f_2(\zeta) = F_1(\zeta) - Q_1(\zeta)P_2(\zeta)e^{<a_2, \zeta>} \\ g_2(\zeta) = g_1(\zeta) \\ K_2(\zeta) = K_1(\zeta) - P_2(\zeta)e^{<k_2, \zeta>} \\ F_2(\zeta) = f_2(\zeta) - P_2(\zeta)g_1(\zeta)e^{<k_2, \zeta>}. \end{cases} \tag{2.25}$$

Then $f_2, F_2 \in E_{\bar{p}}$, $K_2 \in \mathscr{A}$ and

$$F_2 = K_2 G, \tag{2.26}$$

a_2 is not a frequency of F_2, but each a_i, $i = 3, \ldots, r$ is. The remaining frequencies $a''_{r+1}, \ldots, a''_{p_2}$ form a subset of

$$\{a_{r+1}, \ldots, a_p, a_1 - b_1 + b_2, \ldots, a_1 - b_1 + b_q, a_2 - b_1 + b_2, \ldots, a_2 - b_1 + b_q\}.$$

Hence $\{\bar{a}''_{r+1}, \ldots, \bar{a}''_{p_2}\} \subseteq H^-$. Moreover, $[F_2] \subseteq [F_1]$, because

$$[F_2] \subseteq \text{ch}\,([f_2] \cup \{\bar{k}_2 + b_j\}_{j \geq 2}) \subseteq [F_1] \cup ([K_1] + [G]) = [F_1].$$

Continuing in the same fashion, one finally constructs $F_r \in E_{\bar{p}}$, and $K_r \in \mathscr{A}$ such that (i) $F_r = K_r G$, (ii) $[F_r] \subseteq H^- \cap [F]$. Since $\nu \notin {}^{c}\mathcal{N}$, the frequencies $a_j^{(r)}$ of F_r can be numbered so that $\bar{a}_1^{(r)}$ is the only point in $[F_r]$ for which $\langle \bar{a}_1^{(r)}, \nu\rangle_{2n} = h_{[F_r]}(\nu)$ and $\langle \bar{a}_1^{(r)}, \nu\rangle_{2n} > \langle \bar{a}_2^{(r)}, \nu\rangle_{2n} > \ldots \langle \bar{a}_{p_r}^{(r)}, \nu\rangle_{2n} > 0$. Set $H_1^+ = \{x \in \mathbf{R}^{2n}: \langle x, \nu\rangle_{2n} > \langle \bar{a}_1^{(r)}, \nu\rangle_{2n} - d\}$, $H_1^- = \mathbf{R}^{2n} \setminus H_1^+$, and let $r_1 \geq 1$ be such that $\bar{a}_i^{(r)} \in H_1^+$ for $i = 1, \ldots, r_1$ and $\bar{a}_j^{(r)} \in H_1^-$ for $j > r_1$. It is now clear that we can repeat the same procedure indefinitely. If at some point we obtain $\mathcal{F} = F_{r+r_1+\cdots+r_N} = 0$, the theorem follows. However, this must actually happen when N is sufficiently large. For, let N be so large that $H_N^- \cap \mathbf{R}_+^{2n} = \emptyset$, hence $[\mathcal{F}] \subseteq [F] \cap H_N^- = \emptyset$ and $\mathcal{F} \equiv 0$.

§ 3. Applications

By Theorem 1, if F and G are exponential polynomials such that the quotient $K = F/G$ is entire, we can write K in the reduced form, $K = H/P$, which is uniquely determined (cf. § 1 and[6]). Now the question arises when $P \equiv 1$. The next theorem gives a simple sufficient condition.

THEOREM 2. *Let* $F, G \in E_{\mathcal{P}}$ *be such that* $F/G \in \mathcal{A}$. *Let* H/P *be the reduced form of* F/G. *Then* P *divides* d_G. *In particular,* $P \equiv 1$ *whenever* $d_G = 1$.

Proof. Set

$$\begin{aligned} G(z) &= \sum_{j=1}^{p} a_j(z) e^{\langle \alpha_j, z \rangle} \\ H(z) &= \sum_{j=1}^{q} b_j(z) e^{\langle \beta_j, z \rangle}. \end{aligned} \tag{3.1}$$

First we shall prove the following special case by induction on p.

(A) (i) P *is irreducible* (ii) $d_G = 1$. *Then* $P \equiv 1$.

If $p = 1$, then by (ii), $a_1(z)$ is a constant, $a_1 \neq 0$. Hence H/P is the reduced form of the exponential polynomial $(1/a_1) e^{-\langle \alpha_1, z \rangle} F(z)$. In view of the uniqueness of the reduced form, P must be constant.

Suppose now that (A) holds whenever G has at most $p - 1$ frequencies, $p \geq 1$. There are two possible cases: either $P | b_j$ for all $j = 1, \ldots, q$ or $P \nmid b_j$ for some j. In the first case, $P \equiv 1$ by definition of reduced form. Hence it suffices to consider the second case when, after rearranging the β_j's if necessary, there is a $q_0 \geq 1$ such that $P \nmid b_j$, $j = 1, \ldots, q_0$ and $b_j = b_j^* P$, $b_j^* \in \mathcal{P}$, for $j = q_0 + 1, \ldots, q$. We claim that it suffices to consider the case $q_0 = q$. Indeed, if $q_0 < q$, set

$$F^*(z) = F(z) - G(z) \sum_{j > q_0} b_j^*(z) e^{\langle \beta_j, z \rangle}, \quad H^*(z) = \sum_{j=1}^{q_0} b_j(z) e^{\langle \beta_j, z \rangle}.$$

Then F^*/G is entire and H^*/P is its reduced form. Therefore we shall assume

$$P \nmid b_j \quad (\forall j). \tag{3.2}$$

It will be shown that (3.2) leads to contradiction if $P \neq$ constant, and this will prove (A). It follows from § 2 that $[PF] = [H] + [G]$, and

$$[PF] = \operatorname{ch}\{\bar{\alpha}_i + \bar{\beta}_j : i = 1, \ldots, p, \ j = 1, \ldots, q\}.$$

Let γ be a fixed extreme point of the polyhedron $[PF]$. By Lemma 1, $\gamma = \bar{\alpha}_{i_0} + \bar{\beta}_{j_0}$ for exactly one i_0 and j_0. Renumbering the frequencies one can assume that $i_0 = p$, i.e.

$$\bar{\alpha}_p + \bar{\beta}_{j_0} \neq \bar{\alpha}_i + \bar{\beta}_j \quad (i < p, \forall j). \tag{3.3}$$

Consider all j_0's for which (3.3) holds. Renumbering the b_j's one can assume that there is some J, $1 \leq J \leq q$, such that (3.3) holds for all $j_0 \geq J$, but does not hold for $j_0 < J$. Hence each of the frequencies $\alpha_p + \beta_j$, $j \geq J$, appears in the product $HG = PF$ exactly once. By the lemma in § 1, this means that $P | a_p b_j$ for $j \geq J$, thus by (3.2) and (i),

$$a_p = \tilde{a}_p P \quad \text{for some} \quad \tilde{a}_p \in \mathcal{P}. \tag{3.4}$$

Set

$$\begin{cases} G^*(z) = G(z) - a_p(z) e^{\langle \alpha_p, z \rangle}; \\ \tilde{G}(z) = G^*(z)/d_{G^*}(z), \\ \tilde{F}(z) = F(z) - \tilde{a}_P(z) e^{\langle \alpha_p, z \rangle} H(z), \\ \tilde{H}(z) = H(z) d_{G^*}(z). \end{cases} \tag{3.5}$$

Then

$$\tilde{H}/P \text{ is the reduced form of } \tilde{F}/\tilde{G}. \tag{3.6}$$

Indeed, by (3.5), $\tilde{F}/\tilde{G} = \tilde{H}/P$, and since $(d_H, P) = 1$, $(d_{\tilde{H}}, P) = 1$ means by (i) that $P \nmid d_{G^*}$, where $d_{G^*} = (a_1, \ldots, a_{p-1})$. However this follows from (ii) and (3.4). Since $\tilde{G}$ has $p - 1$ terms, the induction hypothesis shows that P is constant, which contradicts (3.2).

(B) Next assume that P is irreducible and d_G arbitrary. Assume that $P \nmid d_G$. In particular $P \neq$ constant. Writing $G = d_G G_1$, one can apply (A) to $F/G_1 = H d_G/P$. Hence P is a constant, a contradiction.

(C) Finally, let P and d_G be arbitrary. If $P = P_1 \cdots P_r$ is the factorization of P into irreducible factors, then the theorem follows by applying (B) to each of the equations

$$(F \prod_{i \neq j} P_i)/G = H/P_j \quad (j = 1, \ldots, r).$$

Another application of Theorem 1 is the following statement (Theorem 3), which gives a simple necessary condition for a quotient of two exponential polynomials to be entire.

Given arbitrary finite sets $B = \{\beta_1, \ldots, \beta_q\}$, $C = \{\gamma_1, \ldots, \gamma_r\}$ of points in $\mathbf{R}^m$ we shall say that the β_j's are *rational affine combinations* of the γ_k's, if for some j_0, k_0 and all j

$$\beta_j - \beta_{j_0} = \sum_{k=1}^{r} w_{jk}(\gamma_k - \gamma_{k_0}), \quad w_{jk} \in \mathbf{Q}. \tag{3.7}$$

It is clear that if (3.7) holds for some j_0, k_0, it holds with suitable rationals w_{jk} for any other pair j_0, k_0. The next statement is an easy consequence of Theorem 1.

THEOREM 3. *Let F, G be exponential polynomials such that F/G is entire. Then the frequencies of G are rational affine combinations of the frequencies of F.*

The proof follows along similar lines as the proof of the theorem in Section 1 of [22].

References

1. AVANISSIAN, V., Oral communication, 1970.
2. BERENSTEIN, C. A. & DOSTAL, M. A., Analytically uniform spaces and their applications to convolution equations, *Lecture Notes in Math.*, Vol. 256, Springer-Verlag (1972).
3. —»— Sur une classe de functions entières, *C. R. Acad. Sci. Paris*, 274 (1972), 1149—1152.
4. —»— Some remarks on convolution equations, *Ann. Inst. Fourier (Grenoble)*, 23 (1973), 55—74.
5. —»— On convolution equations I, "L'Analyse harmonique dans le domaine complexe", *Lecture Notes in Math.*, Vol. 336, Springer-Verlag (1973), 79—94.
6. —»— On convolution equations II, to appear in "*Proceedings of the Conference on Analysis, Rio de Janeiro 1972*", (Hermann et Cie, Publ.).
7. —»— A lower estimate for exponential sums, *Bull. Amer. Math. Soc.* 80 (1974), 687—691; cf. also Prelim. Rep. 73T-B237 in *Notices A.M.S.* (August, 1973) p. A-492.
8. DOSTAL, M. A., An analogue of a theorem of Vladimir Bernstein and its applications to singular supports, *Proc. London Math. Soc.* 19 (1969), 553—576.
9. EHRENPREIS, L., Mean-periodic functions I, *Amer. J. Math.* 77 (1955), 293—328.
10. —»— *Fourier analysis in several complex variables*, Wiley-Interscience, New York, 1970.
11. FUJIWARA, M., Über den Mittelkörper zweier konvexen Körper, *Sci. Rep. Res. Inst. Tôhoku Univ.* 5 (1916), 275—283.
12. HÖRMANDER, L., On the range of convolution operators, *Ann. of Math.* 76 (1962), 148—170.
13. —»— Convolution equations in convex domains, *Invent. Math.* 4 (1968), 306—317.
14. KISELMAN, C. O., On entire functions of exponential type and indicators of analytic functionals, *Acta Math.* 117 (1967), 1—35.
15. LAIRD, P. G., Some properties of mean periodic functions, *J. Austral. Math. Soc.* 14 (1972), 424—432.
16. LAX, P. D., The quotient of exponential polynomials, *Duke Math. J.* 15 (1948), 967—970.
17. LELONG, P., *Fonctionnelles analytiques et fonctions entières (n variables)*, Les Presses de l'Univ. de Montréal, 1968.

18. Lindelöf, E., Sur les fonctions entières d'ordre entier, *Ann. Sci. Ecole Norm. Sup.* 22 (1905), 369—395.
19. Malgrange, B., Existence et approximation des solutions des équations aux dérivées partielles et des équations de convolution, *Ann. Inst. Fourier (Grenoble)*, 6 (1956), 271—335.
20. —»— Sur les équations de convolution, *Rend. Sem. Mat. Univ. e Politec. Torino* 19 (1959/60), 19—27.
21. Martineau, A., Sur les fonctionnelles analytiques, *J. Analyse Math.* 11 (1963), 1—164.
22. Ritt, J. F., A factorization theory for functions $\Sigma a_i e^{\alpha_i x}$, *Trans. Amer. Math. Soc.* 29 (1927), 584—596.
23. —»— On the zeros of exponential polynomials, *Trans. Amer. Math. Soc.* 31 (1929), 680—686.
24. Selberg, H., Über einige transzendente Gleichungen, *Avh. Norske Vid. Akad. Oslo*, I/10 (1931), 1—8.
25. Shields, A., On quotients of exponential polynomials, *Comm. Pure Appl. Math.* 16 (1963), 27—31.
26. Trèves, F., *Linear partial differential equations with constant coefficients*, Gordon & Breach, 1966.

Added in proof (cf. footnote 7):

27. Kitagawa, K., Sur les polynômes exponentiels, *J. Math. Kyoto Univ.* 13 (1973), 489—496.
28. Avanissian, V. & Gay, R., Sur une transformation des fonctionnelles analytiques portables par des convexes compacts de $\mathbf{C}^d$, et la convolution d'Hadamard, *C. R. Acad. Sci., Paris*, 279 (1974), 133—136.
29. —»— Sur les fonctions entières arithmétiques de type exponentiel et le quotient d'exponentielle-polynômes de plusieures variables (*ibidem*).

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