On some classes of homogeneous ternary cubic diophantine equations

Erik Dofs

1. Introduction

The general homogeneous cubic equation in three variables contains ten arbitrary coefficients. If it has a non-vanishing discriminant it can be transformed into canonical form:

$$ax^3+by^3+cz^3 = dxyz; \tag{1}$$

by means of a real transformation which in general is non-rational. The difficulty of proving the non-solubility in (rational) integers of a canonical equation with integral coefficients is easily seen when considering that this, in one of the seemingly simplest cases:

$$x^3+y^3+z^3 = nxyz; \tag{2}$$

has been done only when $n=-6, 0, 1$.

When discussing (1) it is often convenient to work within $K(\sqrt{-3})$. However, as Hans Riesel pointed out to me it might be delicate to work exclusively within this quadratic field as for example $x^3+y^3+z^3=2xyz$ has a solution $x=2$, $x=2-\varrho$ and $z=2-\bar{\varrho}$ in $K(\sqrt{-3})$ but lacks solutions in $K(1)$, as is shown in part 4 of this paper. Consequently I have chosen a method that predominantly works within $K(1)$, but at one crucial step uses theorems valid for $K(\sqrt{-3})$.

Of greatest interest is to know which equations can be brought to canonical form by means of rational transformations. A theorem stated by Sylvester concerning equations, symmetric in x, y and z is generalized to equations invariant under cyclic transformation of x, y and z.

The notion 'non-trivial solution' or simply 'solution' when no misunderstanding may occur, refers to a solution where $xyz\neq 0$.

2. On $a(x^3+y^3+z^3)+b(x^2y+y^2z+z^2x)+c(xy^2+yz^2+zx^2)+dxyz=0$

Consider the equation:

$$a(x^3+y^3+z^3)+b(x^2y+y^2z+z^2x)+c(xy^2+yz^2+zx^2)+dxyz=0; \tag{3}$$

It has the same number of arbitrary coefficients as:

$$Ax^3+By^3+Cz^3=Dxyz; \tag{4}$$

Thus there might exist a rational transformation bringing (3) into (4), or conversely. Sylvester [1] stated without a proof:

If $b=c$, then (3) may always be transformed so as to 'depend upon' (Sylvesters words) the equation:

$$fu^3+gv^3+hw^3=(6a-d)uvw; \tag{5}$$

where

$$fgh=ad^2-(b^2+3a^2)d+9a^3-3ab^2+2b^3. \tag{6}$$

Noting that the expression for fgh can be written:

$$fgh=\frac{1}{3^3}[(6a-d)^3+(3a+d)^3+(3b)^3+(3b)^3-3(3a+d)\cdot 3b\cdot 3b];$$

the generalization when $b\neq c$ is plausible; namely replace one of the expressions $(3b)$ by $(3c)$ in two places.

Selmer [2] gave a proof of the theorem when $b=c$, but to see the connection with the generalized theorem I give it with Sylvesters notation (Sylvester did not state the limitations of the transformation).

Theorem 1.

$$a(x^3+y^3+z^3)+b(x^2y+y^2z+z^2x+xy^2+yz^2+zx^2)+dxyz=0; \tag{7}$$

can be transformed into

$$fgh(U+V+W)^3=(6a-d)^3UVW;$$

where fgh is given by (6) *above, provided that* $(6a-d)(3a-3b+d)\neq 0$.

Proof. Use the transformation

$$\left.\begin{aligned} U&=(-d+2b)x+(3a-b)y+(3a-b)z;\\ V&=(3a-b)x+(-d+2b)y+(3a-b)z;\\ W&=(3a-b)x+(3a-b)y+(-d+2b)z;\end{aligned}\right\} \tag{8}$$

having the determinant $=(6a-d)(3a-3b+d)^2$.

Elementary but lenghty calculations give Theorem 1. QED

If $6a-d=0$, then the transformation

$$\left.\begin{array}{l} u = x+y+z; \\ v = x+\varrho y+\bar{\varrho} z; \\ w = x+\bar{\varrho} y+\varrho z; \end{array}\right\}, \quad u, v, w \in K(\sqrt{-3}) \quad \text{and} \quad \varrho = \frac{-1+\sqrt{-3}}{2} \tag{9}$$

gives:

$$(3a+2b)u^3+(3a-b)(v^3+w^3) = 0;$$

In particular if $b=0$: $u^3+v^3+w^3=0$, which means that $x^3+y^3+z^3+6xyz=0$ can be transformed into $u^3+v^3+w^3=0$ where $u, v, w \in K(\sqrt{-3})$ if x, y and z are rational. This was known by Sylvester.

If $3a-3b+d=0$, then the l.h.s. can be factorized:

$$(x+y+z)[a(x+y+z)^2-(3a-b)(xy+yz+zx)] = 0$$

Furthermore, the l.h.s. of (7) can be factorized into three real linear factors (U, V and W according to (8)) if $fgh=0$ but the determinant$\neq 0$.

If $3a+6b+d=0$ then the curve represented by (7) is unicursal with the singular point (1, 1, 1).

I now prove the more general:

Theorem 2. *If*

$$a(x^3+y^3+z^3)+b(x^2y+y^2z+z^2x)+c(xy^2+yz^2+zx^2)+dxyz = 0;$$

then in general

$$X^3+Y^3+CZ^3 = (6a-d)XYZ; \tag{10}$$

where X, Y and Z are cubics in x, y and z and

$$C = \frac{1}{3^3}[(6a-d)^3+(3a+d)^3+(3b)^3+(3c)^3-3(3a+d)\cdot 3b\cdot 3c];$$

Proof. First use the transformation (9) to get

$$u^3(3a+d+3b+3c)+v^3(3a+d+3b\bar{\varrho}+3c\varrho)+w^3(3a+d+3b\varrho+3c\bar{\varrho})+$$
$$+3uvw(6a-d) = 0; \tag{11}$$

u_0, v_0 and w_0 are defined by

$$\left.\begin{array}{r} u_0 = u^3(3a+d+3b+3c); \\ v_0 = v^3(3a+d+3b\bar{\varrho}+3c\varrho); \\ w_0 = w^3(3a+d+3b\varrho+3c\bar{\varrho}); \end{array}\right\} \quad \text{where} \quad v_0 = \bar{w}_0$$

and (11) gives:

$$[(3a+d)^3+(3b)^3+(3c)^3-3(3a+d)\cdot 3b\cdot 3c](u_0+v_0+w_0)^3+3^3(6a-d)^3u_0v_0w_0=0. \tag{12}$$

Now calculate $u_0+v_0+w_0$, $u_0+\bar{\varrho}v_0+\varrho w_0$ and $u+\varrho v_0+\bar{\varrho}w_0$!

$$u_0+v_0+w_0 = 3(3a+d)(x^3+y^3+z^3)+27b(x^2y+y^2z+z^2x)+$$

$$+27c(xy^2+yz^2+zx^2)+18(3a+d)xyz = -3(6a-d)(x^3+y^3+z^3-3xyz);$$

(After substitution from (3))

$$u_0+\bar{\varrho}v_0+\varrho w_0 = 9[c(x^3+y^3+z^3)+(3a+d)(x^2y+y^2z+z^2x)+$$

$$+3b(xy^2+yz^2+zx^2)+6cxyz];$$

$$u_0+\varrho v_0+\bar{\varrho}w_0 = 9[b(x^3+y^3+z^3)+3c(x^2y+y^2z+z^2x)+$$

$$+(3a+d)(xy^2+yz^2+zx^2)+6bxyz];$$

X, Y and Z are defined by

$$\left.\begin{aligned} 9X &= u_0+\bar{\varrho}v_0+\varrho w_0;\\ 9Y &= u_0+\varrho v_0+\bar{\varrho}w_0;\\ 3(6a-d)Z &= u_0+v_0+w_0; \end{aligned}\right\}. \quad X,\,Y,\,Z\in K(1)$$

which gives

$$u_0v_0w_0 = (3X)^3+(3Y)^3+(6a-d)^3Z^3-3\cdot 3X\cdot 3Y\cdot(6a-d)Z;$$

but (12) also gives

$$[(3a+d)^3+(3b)^3+(3c)^3-3(3a+d)\cdot 3b\cdot 3c]Z^3 = -u_0v_0w_0 =$$

$$= -[(3X)^3+(3Y)^3+(6a-d)^3Z^3-3\cdot 3X\cdot 3Y\cdot(6a-d)Z]; \tag{13}$$

Finally (13) gives:

$$(3X)^3+(3Y)^3+[(6a-d)^3+(3a+d)^3+(3b)^3+(3c)^3-$$

$$-3(3a+d)\cdot 3b\cdot 3c]Z^3 = 3^3(6a-d)XYZ;$$

but the coefficient of Z^3 is divisible by 3^3.

$$\because X^3+Y^3+CZ^3 = (6a-d)XYZ;$$

where

$$C = \frac{1}{3^3}[(6a-d)^3+(3a+d)^3+(3b)^3+(3c)^3-3(3a+d)\cdot 3b\cdot 3c];$$

The condition that the Jacobian$\neq 0$, gives when $b\neq c$ (at points where $x^3+y^3+z^3-3xyz\neq 0$):

$$3a+d+3b+3c \neq 0, \quad 6a-d \neq 0.$$

If $x+y+z=0$, then (3) gives at once:

$$x|b-c, \quad y|b-c, \quad (x,y)=1;$$

If $x=y=z=1$ is a point on the curve then $3a+d+3b+3c=0$, which means that the corresponding cubic curve is unicursal. QED

Equation (10) can be written as:

$$(3X)^3+(3Y)^3+(6a-d)^3Z^3-3(3X)(3Y)(6a-d)Z =$$

$$= -Z^3((3a+d)^3+(3b)^3+(3c)^3-3(3a+d)\cdot 3b\cdot 3c),$$

i.e. it is of the form

$$x^3+y^3+z^3-3xyz = u^3+v^3+w^3-3uvw; \tag{14}$$

discussed by Carmichael [3] who gave a solution of (14) in four parameters. Carmichael [4] has also proved theorems that give a possibility when given C and D (in $X^3+Y^3+CZ^3=DXYZ$) to determine a, b, c and d thus:

$$C = \frac{1}{3^3}((6a-d)^3+(3a+d)^3+(3b)^3+(3c)^3-3(3a+d)\cdot 3b\cdot 3c);$$

$$D = 6a-d;$$

In fact, he discusses representation of natural numbers by $x^3+y^3+z^3-3xyz$, of which form D^3-27C is. His theorems do not give all possible a, b, c, d which he shows with counterexamples, but his results are easily extended so as to give these.

In part 5, I return to the question of conditions on a, b, c, d to be able to decide the solubility of (10).

3. On $Ax^3+By^3+Cz^3=Dxyz$

Consider the equation:

$$ax^3+by^3+cz^3 = dxyz; \tag{15}$$

Mordell [5] writes in his excellent book:

'It is very difficult to prove the non-existence of integer solutions of the general equation (15). Sometimes progress can be made with the special case $a=b$ and in particular with

$$x^3+y^3+cz^3 = dxyz'; \tag{16}$$

He also gives an incomplete proof* that a descent arises if $3\nmid d$, c a prime, d^3-27c square-free and has only prime factors$\equiv 2 (\mathrm{mod} 3)$.

Sylvester [6, 7, 8, 9] stated several far-reaching theorems concerning (15) without proof, the truth of which Mordell [10] partly doubted. Sylvester in his last theses [9] on this matter states that (16) is insoluble if

a) $d^3-27c=\Delta^3\Delta'$ where Δ' is cube-free and

b) Δ' even and contains no factor of the form f^2+3g^2 and

c) c a prime number,

except in the cases when $\sqrt{-d/c}$ is an integer.

I have found a counterexample to Sylvesters statement: $x^3+y^3+17z^3=-9xyz$ $(\Delta'=2^2\cdot 11)$, which has the solution $x=3$, $y=5$ and $z=-1$. This proves that Sylvester was lacking some conditions necessary to decide the solubility of (16).

Mordell [10] also doubts Sylvesters statement [7] concerning (15), that if $a:b$ is a rational cube then under certain conditions (15) can be made to 'depend upon'

$$AX^3+BY^3+CZ^3=dXYZ, \text{ where } ABC=abc \text{ and } XYZ/z.$$

But this turns out to be correct with some limitations (Lemma 5, 8 and 9 below). In this investigation I generally assume $d\neq 0$ if not otherwise stated, as my methods demand this. The case $d=0$ has been throughly discussed a. o. by Selmer [11], using different methods. Hurwitz, in an elegant paper [12], has shown theorems concerning the number of solutions of (15), but gives no advice if a particular equation has solutions or not. Hurwitz made use of Desboves [13] formulae, which give new solutions of (15) if some solution x_0, y_0, z_0 is known:

$$\left.\begin{aligned} x' &= x_0(by_0^3-cz_0^3); \\ y' &= y_0(cz_0^3-ax_0^3); \\ z' &= z_0(ax_0^3-by_0^3); \end{aligned}\right\}$$

or

$$\left.\begin{aligned} x' &= x_1^2y_0z_0-x_0^2y_1z_1; \\ y' &= y_1^2z_0x_0-y_0^2z_1x_1; \\ z' &= z_1^2x_0y_0-z_0^2x_1y_1; \end{aligned}\right\}$$

---

* This is realized by testing the proof with $x^3+y^3+3z^3=2xyz$, which has the solution $x=-4$, $y=-11$, $z=9$.

Also note the misprint $d\equiv 0$ (mod 3); should read $d\not\equiv 0$ (mod 3), p. 131.

if x_0, y_0, z_0 and x_1, y_1, z_1 are known solutions. It seems as if the only (published) strict proofs of nonsolubility of an equation of the type (15) (except for those proven by congruences) having $abc \neq 1$. $abcd \neq 0$ are the proof for $x^3+y^3+5z^3=5xyz$ given by Ward [14] and the proofs for

$$x^3+y^3+13z^3 = 7xyz \quad \text{and} \quad x^3+y^3+9z^3 = 6xyz,$$

which can be deduced from the proofs for $x^3+y^3+z^3=xyz$ and $x^3+y^3+z^3=0$ respectively. (See Part 4 below.)

*

Assume that in equation (15):
$abc=f^{\omega}$. f a prime. $\omega \not\equiv 0$ (mod 3).
δ is defined by: $f^{\delta} \| d$; i.e. $f^{\delta} | d$ but $f^{\delta+1} \nmid d$.
Then if $\delta > 0$, generally a reduction of (15) can be accomplished according to

Lemma 1. *If* $\omega > 3\delta$, *then* (15) *can be reduced to*

$$f^{\omega_1}x^3+f^{\omega_2}y^3+f^{\omega_3}z^3 = Dxyz; \tag{17}$$

where

$$\omega_1+\omega_2+\omega_3 = \omega-3\delta. \quad D = d/f^{\delta}.$$

If $\omega < 3\delta$, *then* (15) *can be reduced to the form* (17) *again, but now*

$$\omega_1+\omega_2+\omega_3 = \omega-3[\omega/3], \quad D = d/f^{[\omega/3]};$$

i.e. $\omega_1+\omega_2+\omega_3=1, 2$ *depending on* ω (mod3).

Proof. The lemma is shown by repeated inclusion of the factor f into x, y or z. If $\omega > 6$ then plainly ω_1, ω_2 or ω_3 is greater than 3 and an inclusion can be made if $\delta > 0$ (if $\delta=0$ there is nothing to prove). When making an inclusion, $\omega_1+\omega_2+\omega_3$ is subtracted by 3 and δ by 1. i.e. $\omega_n=\omega-3n$ and $\delta_n=\delta-n$ after n inclusions. The process stops as soon as $\delta_n=0$ or (non-exclusive) $\omega_n=4, 5$. Which of these cases occurs first obviously depends on $\omega-3\delta \gtrless 0$. If $\delta_n=0$, then δ inclusions have been made and $\omega_n=\omega-3\delta$, i.e. the first case of the lemma has occured. If $\omega_n=4, 5$ it is easily shown that another inclusion can be made (if still $\delta > 0$) and then $[\omega/3]$ inclusions have been made and the second case occurs. QED

The condition $\omega \not\equiv 0$ (mod3) implies the simple but important:

Lemma 2. *If* $\omega_1+\omega_2+\omega_3 \not\equiv 0$ (mod3), *then* $(\omega_1-\omega_2)(\omega_2-\omega_3)(\omega_3-\omega_1) \equiv 0$ (mod3) *and precisely one of these factors is divisible by 3.*

Proof. Assume on the contrary

$$\omega_1 - \omega_2 \equiv \pm 1 \pmod 3;$$
$$\omega_2 - \omega_3 \equiv \pm 1 \pmod 3;$$
$$\omega_3 - \omega_1 \equiv \pm 1 \pmod 3.$$

An addition gives: $0 \equiv \pm 1 \pm 1 \pm 1 \pmod 3$; i.e. upper/lower signs are valid simultaneously. However, this gives $0 \equiv (\omega_1 - \omega_2) - (\omega_2 - \omega_3) \equiv \omega_1 + \omega_2 + \omega_3 \pmod 3$, a contradiction. Thus $(\omega_1 - \omega_2)(\omega_2 - \omega_3)(\omega_3 - \omega_1) \equiv 0 \pmod{3}$. Also, precisely one of the factors is divisible by 3, as otherwise the same contradiction would occur. QED

Corollary. *If* $a = f^{\omega_1}, b = f^{\omega_2}$ *and* $c = f^{\omega_3}$, *where* $\omega_1 + \omega_2 + \omega_3 \not\equiv 0 \pmod{3}$, *then precisely one of* $a{:}b$, $b{:}c$ *or* $c{:}a$ *can be written as* $S^3 : T^3$ *where* S *or* $T = 1$.

*

If the equation $f^{\omega_1}x^3 + f^{\omega_2}y^3 + f^{\omega_3}z^3 = dxyz$ has been reduced according to Lemma 1, then it is possible to prove

Lemma 3. $f^{\omega_1}x^3 + f^{\omega_2}y^3 + f^{\omega_3}z^3 = dxyz$ *has a non-trivial solution if and only if there is a non-trivial solution of at least one equation:*

$$f^{\omega_{10}}x^3 + f^{\omega_{20}}y^3 + f^{\omega_{30}}z^3 = dxyz;$$

where

$$\omega_1 + \omega_2 + \omega_3 = \omega_{10} + \omega_{20} + \omega_{30} \not\equiv 0 \pmod{3}.$$

Proof. Plainly for every M:

$$f^{\omega_{10}+M}x^3 + f^{\omega_{20}+M}y^3 + f^{\omega_{30}+M}z^3 = f^M dxyz.$$

Put $m_1 + m_2 + m_3 = M$. Then after inclusion

$$f^{\omega_1}(f^{m_1}x)^3 + f^{\omega_2}(f^{m_2}y)^3 + f^{\omega_3}(f^{m_3}z)^3 = d \cdot f^{m_1}x \cdot f^{m_2}y \cdot f^{m_3}z;$$

where:

$$\left.\begin{aligned} \omega_1 &= \omega_{10} - 2m_1 + m_2 + m_3; \\ \omega_2 &= \omega_{20} + m_1 - 2m_2 + m_3; \\ \omega_3 &= \omega_{30} + m_1 + m_2 - 2m_3. \end{aligned}\right\} \tag{18}$$

Given ω_{10}, ω_{20} and ω_{30} it is sufficient to prove that m_1, m_2 and m_3 can be chosen so that the arbitrary combination ω_1, ω_2 and ω_3 is given by (18) above. According to Lemma 2: $(\omega_{10} - \omega_{20}) \cdot (\omega_{20} - \omega_{30}) \cdot (\omega_{30} - \omega_{10}) \equiv 0 \pmod{3}$. Suppose $\omega_{10} - \omega_{20} \equiv 0 \pmod{3}$.

The same is valid for $\omega_1, \omega_2, \omega_3$ and choose $\omega_1 - \omega_2 \equiv 0 \pmod{3}$. Put $\Delta_1 = \omega_1 - \omega_{10}$ and $\Delta_2 = \omega_2 - \omega_{20}$. Then $2\Delta_1 + \Delta_2 \equiv \Delta_1 + 2\Delta_2 \equiv 0 \pmod{3}$. The three equations for m_1, m_2 and m_3 are linearly dependent. The first two give

$$\left.\begin{aligned} m_1 &= m_3 - \frac{2\Delta_1+\Delta_2}{3}; \\ m_2 &= m_3 - \frac{\Delta_1+2\Delta_2}{3}; \end{aligned}\right\} \quad (m_1, m_2 \text{ and } m_3 \text{ integers}).$$

Now choose m_3 big enough so that m_1, m_2 and m_3 all $\geqq 0$. Thus the construction of a non-trivial solution of $f^{\omega_1}x^3+f^{\omega_2}y^3+f^{\omega_3}x^3=dxyz$ is complete. QED

The following theorem proved by Sylvester [7] is wellknown:

If there exists a solution of $ax^3+by^3+cz^3=dxyz$, where a, b, c arbitrary but $abc \neq 0$ then there generally exists a solution of $X^3+Y^3+abcZ^3=dXYZ$.

The 'Theorem of Derivation' given by Sylvester is deduced as follows:

Let $ax^3+by^3+cz^3=dxyz$;

Write $F=ax^3$, $G=by^3$, $H=cz^3$;

Define X, Y and Z by:

$$\left.\begin{aligned} X &= F^2G+G^2H+H^2F-3FGH; \\ Y &= FG^2+GH^2+HF^2-3FGH; \\ Z &= (F^3+G^3+H^3-3FGH)/d = \tfrac{1}{2}\cdot xyz((F-G)^2+(G-H)^2+(H-F)^2); \end{aligned}\right\}$$

Then $X^3+Y^3+abcZ^3=dXYZ$;

However, a non-trivial solution x, y, z of $ax^3+by^3+cz^3=dxyz$ does not always give a non-trivial solution of $X^3+Y^3+abcZ^3=dXYZ$. An example of this is given by $4x^3+2y^3+z^3=-3xyz$ which has the solutions $x, y, z=1, -1, 1$ & $1, 2, -2$ & $1, -1, -2$ but for this values $XYZ=0$. It is easily seen that this complication can occur only if abc is a cube and the example above is in fact the only problematic case of this kind when abc is a prime power (i.e. $abc=f^\omega$. $\omega\equiv 0 \pmod 3$), except for the cases where $F=G=H$.

The converse theorem is not valid generally (See Selmer [11] for example), but Lemma 3 in fact makes it valid if $abc=f^\omega$. $\omega \not\equiv 0$ (mod3). f a prime.

The condition $\omega \not\equiv 0$ (mod3) is necessary:

$$x^3+y^3+8z^3 = 6xyz \quad \text{has a solution} \quad x=1, \quad y=1 \quad \text{and} \quad z=-1,$$

but $x^3+2y^3+4z^3=6xyz$ has no solution (end of part 3).

Assume

$$f^{\omega_1}x^3+f^{\omega_2}y^3+f^{\omega_3}z^3=dxyz \tag{19}$$

is reduced according to Lemma 1. F is defined by: $F=d^3-3^3f^\omega$ and G by: $G=(3^2d^3, F)$. Also suppose $d\neq 0$.

Lemma 4. *All values of G are given by the classification below.*

						G	Case
$f=3$	$3\nmid d$					1	M_0
	$3\nmid d$	$3^2\nmid d$	$\omega=1$			3^3	M_{31}
			$\omega=2$				M_{32}
		$3^2\mid d$	$\omega=1$			3^4	M_4
			$\omega=2$			3^5	M_5
$f\neq 3$	$f\nmid d$	$3\nmid d$				1	N_0
		$3\mid d$	$3^3\Vert F$			3^3	N_3
			$3^4\Vert F$			3^4	N_4
			$3^5\mid F$			3^5	N_5
	$f\mid d$	$3\nmid d$	$\omega=1$			f	N_{01}
			$\omega=2$			f^2	N_{02}
		$3\mid d$	$3^3\Vert F$	$\omega=1$		3^3f	N_{31}
				$\omega=2$		3^3f^2	N_{32}
			$3^4\Vert F$	$\omega=1$		3^4f	N_{41}
				$\omega=2$		3^4f^2	N_{42}
			$3^5\mid F$	$\omega=1$		3^5f	N_{51}
				$\omega=2$		3^5f^2	N_{52}

Proof. Obvious. The reason to separate the case $f=3$ will be seen below. It is not absolutely necessary to distinguish the cases where $f|G$, but as Theorem 3 can be sharpened considerable in these cases I have chosen this classification. QED

Some further definitions:

H is defined by $H=F/3^h$ where $3^h\Vert F$ i.e. H is the '3-free' part of F.

T is the set of $t\in Z$ (rational integers) such that $|t|=\Pi_i p_i^{\alpha_i}$ where $\forall i$: $(p_i\equiv 2 \pmod 3)$. p_i prime).

U is a subset of T such that $\forall i$: $\alpha_i\equiv 1 \pmod 2$. i.e. all p_i's have an odd exponent. Also let: $\pm 1\in T$ and U. Note that U is wider than the set of square-free $t\in T$.

I have proved the following.

Theorem 3. *If $H\in T$ and the conditions for Lemma* 1 *are fulfilled then in the different cases (Lemma* 4*) the following can be stated:*

M_0 and N_0: The equation $f^{\omega_1}x^3+f^{\omega_2}y^3+f^{\omega_3}z^3=dxyz$; has a non-trivial solution if and only if there exists a non-trivial solution U_0, V_0, W_0 of some equation $f^{\omega_{10}}U^3+f^{\omega_{20}}V^3+f^{\omega_{30}}W^3=dUVW$; where $\omega_{10}+\omega_{20}+\omega_{30}=\omega$. $|U_0\,V_0|=1$. $\omega_{10}\omega_{20}\omega_{30}=$

$=0$. $\omega_{10}-\omega_{20}\equiv 0 \pmod{3}$. In particular if $\omega=1$, then $(d=f\pm 2)$ or $(d=4f\pm 1)$ is necessary and sufficient for (19) to have a non-trivial solution.

If furthermore $H\in U$, then in the case M_0 there is a solution if and only if $(\omega=4.\ f=3.\ d=13)$ and in the case N_0 if and only if $(\omega=2.\ f=7.\ d=11)$ or $(\omega=2.\ f=2.\ d=5)$.

N_{01}: If δ is odd and $\sqrt{-d/f^{\delta}}$ is a rational integer then (19) has a non-trivial solution, otherwise there is a solution if and only if $f=2.\ d=4.\ (\omega=1)$. This solution is solitary.

N_{02}: If δ is even and $\sqrt{-d/f^{\delta}}$ is a rational integer then (19) has a non-trivial solution. Otherwise there is a solution if and only if $f=2.\ d=2.\ (\omega=2)$. This solution is solitary.

M_{31}: If $d\equiv 3 \pmod{9}$, then (19) has a non-trivial solution if and only if $\sqrt{-d/3}$ is a rational integer.

If $d\equiv -3 \pmod{9}$, then it is problematic to decide whether (19) has a solution or not. For example, the equation $x^3+y^3+3z^3=33xyz$ has the solution $(x=1.\ y=5.\ z=7)$ but no solution with $|xy|=1$, even though $H\in T$.

M_{32}: There are only trivial solutions.

N_3: If $(3\|d.\ H\equiv\pm 1 \pmod{9})$ or $(3^2|d.\ f\not\equiv\pm 1 \pmod{9})$ then the solubility of (19) can be decided in the same way as for N_0. If $H\in U$, then there is a non-trivial solution if and only if $(\omega=2.\ f=5.\ d=9)$ or $(\omega=4.\ f=2.\ d=9)$.

The extra conditions are necessary as $x^3+y^3+(D^3-3D-1)z^3=3Dxyz$ has the solution $(x=D+1.\ Y=D.\ z=1)$ and would give counterexamples when putting $d=3D=15$ in the first case and $d=9$ in the second case.

N_{31}: If $3\|d.\ H\equiv\pm 1 \pmod{9}$, then there are only trivial solutions.

If $3^2|d.\ f\not\equiv\pm 1 \pmod{9}$, then there is a non-trivial solution if and only if δ is odd and $\sqrt{-d/f^{\delta}}$ is a rational integer.

N_{32}: As in N_{31}, but δ should be even in this case.

M_4: There are only trivial solutions.

N_4: Decidable in the same way as N_0. If $H\in U$ then there are only trivial solutions.

N_{41}: If δ is odd and $\sqrt{-d/f^{\delta}}$ is a rational integer then there exists a non-trivial solution, otherwise none.

N_{42}: As in N_{41}, but δ should be even in this case.

M_5, N_5, N_{51}, N_{52}: Problematic cases, 'fortunately' it is rather unusual that $3^5|d^3-3^3f^\omega$.

If ($d=3D$. $c=D^3-9D+9$. $a=b=1$), then $d^3-27c=3^5(D-1)$ and $x^3+y^3+ +cz^3=dxyz$ has the solution ($x=D-1$. $y=D-2$. $z=1$). Several d-values give: ($H\in T$. $c=f$. f a prime) but still the equation has no solution with $|xy|=1$.

*

In the cases N_3, N_{31} and N_{32} problematic subcases occur; these can be summed up with the conditions:

$f\equiv\pm 1$ (mod9). $d\equiv 0$ (mod3); or

$f^\omega\equiv\pm 4$ (mod9). $d\equiv\mp 3$ (mod9), where upper/lower signs belong together.

Put in this way, these conditions also include all cases where $3^5|G$ (and $f\neq 3$).

Using the same conditions as Mordell [5] did (see above) it is possible to make a very strong statement, namely that (16) lacks solutions in all cases, except if $c=2$. $d=4$, when there is a solitary solution.

*

In order to prove Theorem 3 I give the crucial

Lemma 5. *Consider*

$$ax^3+by^3+cz^3=dxyz \tag{20}$$

If (20) *has a solution* x, y, z *and* ($abc\neq 0$. $(d, abc)=1$. $d\not\equiv 0$ (mod3). $a{:}b$ *a rational cube* $H\in T$), *then it is possible to find an equation*

$$a'\xi^3+b'\eta^3+c'\zeta^3 = d\xi\eta\zeta;$$

where $a'b'c'=abc$, *which has a solution* ξ, η, ζ *satisfying:* $\xi\eta\zeta|z$.

Note. The conditions stated above are valid only for N_0, the most general case. Sylvester [6] states without proof a similar lemma, but he does not distinguish the cases where $3|d$, cases that need additional assumptions as seen from Theorem 3.

Proof. Extend both sides of (20) by $ab(\neq 0)$ to get

$$a^2bx^3+ab^2y^3+abcz^3 = abdxyz; \tag{21}$$

$a{:}b$ a rational cube→(implies) ($a^2b=S^3$. $ab^2=T^3$. $ab=ST$).

(21) rewritten becomes:

$$(3Sx)^3+(3Ty)^3+(dz)^3-3\cdot 3Sx\cdot 3Ty\cdot dz = (d^3-27abc)z^3. \tag{22}$$

Now apply the transformation:

$$\left.\begin{aligned} \Delta\alpha &= 3Sx+3Ty+dz; \\ \Delta\beta &= -6Sx+3Ty+dz; \\ \Delta\gamma &= 3Sx-6Ty+dz; \end{aligned}\right\} \tag{23}$$

where Δ is the g.c.d. of the right hand expressions of (23) and $(\alpha, \beta, \gamma)=1$. The determinant$=3^4 \cdot d \cdot S \cdot T \neq 0$, as $d \neq 0$. (22) transformed becomes:

$$3^2 d^3 \alpha(\beta^2+\beta\gamma+\gamma^2) = (d^3-27abc)(\alpha+\beta+\gamma)^3; \tag{24}$$

$\alpha \neq 0$ if $d^3-27abc \neq 0$ as otherwise $\alpha=0 \to \alpha+\beta+\gamma=0 \to z=0$.

α and $\beta^2+\beta\gamma+\gamma^2$ cannot have a factor in common, which is also a factor in $\alpha+\beta+\gamma$, because if this factor existed, it would divide both $\beta+\gamma$ and $\beta^2+\beta\gamma+\gamma^2= =(\beta+\gamma)^2-\beta\gamma$. i.e. also $\beta\gamma$ and consequently both β and γ and the contradiction $(\alpha, \beta, \gamma) \neq 1$ would occur.

Thus

$$\left.\begin{aligned} \alpha &= k_1 A^3 \\ \beta^2+\beta\gamma+\gamma^2 &= k_2 M^3 \end{aligned}\right\} \text{ where } \quad k_1 k_2 = 3(d^3-27abc). \quad (A, M) = 1.$$

Write $\varepsilon=(\beta, \gamma)$. Then $3 \nmid \varepsilon$ as $3|\varepsilon \to 3|M \to 3|\alpha+\beta+\gamma \to 3|\alpha \to (\alpha, \beta, \gamma) \neq 1$: a contradiction. Let g be an arbitrary prime factor of ε. Suppose $g|M$, this would again give $(\alpha, \beta, \gamma) \neq 1$. Thus $\varepsilon^2|k_2$ i.e. $k_2=K_2\varepsilon^2$ for some K_2. Take a factor $(\neq 3)$ in K_2; it divides both β and γ as this factor$\equiv 2$ (mod3) by assumption. Thus $K_2=1, 3$ as otherwise $\varepsilon \neq (\beta, \gamma)$, contrary to definition of ε.

$$\therefore \beta = \varepsilon\beta_1. \quad \gamma = \varepsilon\gamma_1. \quad \beta_1^2+\beta_1\gamma_1+\gamma_1^2 = \begin{Bmatrix}1\\3\end{Bmatrix} M^3. \quad (\beta_1, \gamma_1) = 1.$$

Case A: $\beta_1^2+\beta_1\gamma_1+\gamma_1^2=M^3$.

$(\beta_1-\varrho\gamma_1, \beta_1-\bar{\varrho}\gamma_1)=1, \varrho-\bar{\varrho}$; but the latter case$\to 3|M \to 3^3|\beta_1^2+\beta_1\gamma_1+\gamma_1^2 \to \to 3|\beta_1$. $3|\gamma_1 \to (\beta_1, \gamma_1) \neq 1$: clearly a contradiction. Thus we have

$$\beta_1-\varrho\gamma_1 = \varrho^j(u-\varrho v)^3, \quad \text{where} \quad M = u^2+uv+v^2. \quad j = 0, \pm 1.$$

It is easily shown: $j=-1$, otherwise the 'usual' contradiction $(\alpha, \beta, \gamma) \neq 1$ would occur.

$$(24) \to \varepsilon^2 dA(u^2+uv+v^2) = (d^3-27abc)A^3-(u^2v+uv^2)\varepsilon^3; \tag{25}$$

(25) is transformed by:

$$\left.\begin{aligned} \Delta_1 P &= dA+\varepsilon u; \\ \Delta_1 Q &= dA+\varepsilon v; \\ \Delta_1 R &= dA-\varepsilon(u+v); \end{aligned}\right\} \tag{26}$$

where Δ_1 is the g.c.d. of the right hand expressions, into: $abc(P+Q+R)^3=d^3PQR$, where $(P, Q, R)=1$ and $(d, abc)=1$.

Lemma 6 gives: $P=a'\xi^3$, $Q=b'\eta^3$, $R=c'\zeta^3$, where $a'b'c'=abc$ and

$$a'\xi^3+b'\eta^3+c'\zeta^3 = d\xi\eta\zeta. \tag{27}$$

It is easily deduced that

$$\xi\eta\zeta = \frac{3z}{\Delta\Delta_1 M};$$

but Case A occurs only if $z\equiv 0\ (\mathrm{mod}3)$ (seen from (23)). i.e.

$$3|\Delta \quad \text{and} \quad \xi\eta\zeta = \frac{z}{\Delta_1\Delta_2 M}, \quad \text{where} \quad \Delta_2 = \Delta/3.$$

Case B: $\beta_1^2+\beta_1\gamma_1+\gamma_1^2=3M^3$.

In this case obviously: $(\beta_1-\varrho\gamma_1, \beta_1-\bar{\varrho}\gamma_1)=\varrho-\bar{\varrho}=2\varrho+1$, i.e. $\beta_1-\varrho\gamma_1=\varrho^j(2\varrho+1)(u-\varrho v)^3$, where $M=u^2+uv+v^2$ and $j=0, \pm 1$. It is a little bit trickier now to dispose of $j=0, 1$ so I give details.

Assume $j=0\rightarrow\beta_1=u^3-v^3+6u^2v+3uv^2$. $\gamma_1=-2(u^3-v^3)-3u^2v+3uv^2$; $\rightarrow$ (via (24)):

$$3\varepsilon^2 dA(u^2+uv+v^2) = (d^3-27abc)A^3+\varepsilon^3(-u^3+v^3+3u^2v+6uv^2). \tag{28}$$

Plainly: $-u^3+v^3+3u^2v+6uv^2\equiv-(u-v)^3\equiv\pm 1\ (\mathrm{mod}9)$ as $u-v\equiv 0\ (\mathrm{mod}3)\rightarrow\rightarrow(\beta_1, \gamma_1)\neq 1$. If $3|A$ one gets $3|\alpha$ but $3|\alpha+\beta+\gamma\rightarrow 3|\beta+\gamma$; as $(\beta_1-\gamma_1)^2+3\beta_1\gamma_1=3M^3$ also: $3|\beta_1-\gamma_1$, but this would again give $(\alpha, \beta, \gamma)\neq 1$.

The contradiction implies: $3\nmid A$.

$$3\nmid A \rightarrow (dA)^3+\varepsilon^3(-u^3+v^3+3u^2v+6uv^2) \equiv \pm 1 \pm 1 \quad (\mathrm{mod}9).$$

But (28) implies that this last expression is divisible by 3. i.e. it is also divisible by 9. (28) then also implies $3|u^2+uv+v^2\rightarrow 3|u-v$ but this would give $(\beta_1, \gamma_1)\neq 1$: a contradiction showing that $j=0$ is not valid.

$j=1$ is excluded by the same arguments.

$$j=-1 \rightarrow \beta_1 = u^3-v^3-3u^2v-6uv^2. \quad \gamma_1 = u^3-v^3+6u^2v+3uv^2;$$

$$(24) \rightarrow 3\varepsilon dA(u^2+uv+v^2) = (d^3-27abc)A^3+\varepsilon^3(2u^3-2v^3+3u^2v-3uv^2). \tag{29}$$

Now use the transformation (where Δ_1 has the same meaning as before):

$$\left.\begin{aligned} \Delta_1 P &= dA-\varepsilon(u+2v); \\ \Delta_1 Q &= dA-\varepsilon(u-v); \\ \Delta_1 R &= dA+\varepsilon(2u+v); \end{aligned}\right\} \tag{30}$$

$$(28),\ (30) \rightarrow \left.\begin{aligned} \Delta_1(P+Q+R) &= 3dA; \\ \Delta_1^3 PQR &= 27abcA^3; \end{aligned}\right\} \quad \text{where} \quad (P, Q, R) = 1$$

As $\Delta_1 P \equiv \Delta_1 Q \equiv \Delta_1 R$ (mod3) and $\Delta_1^3 PQR \equiv 0$ (mod3) then $\Delta_1 = 3\Delta_2$.

As before: $abc(P+Q+R)^3 = d^3 PQR \to P = a'\xi^3.\ Q = b'\eta^3.\ R = c'\zeta^3.$

$$a'b'c' = abc \to a'\xi^3 + b'\eta^3 + c'\zeta^3 = d\xi\eta\zeta.$$

Here

$$\xi\eta\zeta = \frac{z}{\Delta\Delta_2 M}.$$

QED

A lemma used in Lemmas 5,7 and 8 is the

Lemma 6. *If* $abc(P+Q+R)^3 = d^3 PQR$, *where* $(d, abc) = 1$ *and* $(P, Q, R) = 1$, then

$$P = a'\xi^3.\quad Q = b'\eta^3.\quad R = c'\zeta^3.\quad a'b'c' = abc$$

and

$$a'\xi^3 + b'\eta^3 + c'\zeta^3 = d\xi\eta\zeta.$$

Proof. Write

$$\left.\begin{aligned} g_1 &= (Q, R) \\ g_2 &= (R, P) \\ g_2 &= (P, Q) \end{aligned}\right\}$$

where

$$(g_1, g_2) = (g_2, g_3) = (g_3, g_1) = 1;$$

$$\to P = g_2 g_3 P'.\quad Q = g_3 g_1 Q'.\quad R = g_1 g_2 R';$$

where $(P', Q') = (Q', R') = (R', P') = 1$ and $(P', g_1) = (Q', g_2) = (R', g_3) = 1$;

The original equation becomes:

$$abc(g_2 g_3 P' + g_3 g_1 Q' + g_1 g_2 R')^3 = d^3 (g_1 g_2 g_3)^2 P' Q' R';$$

Congruencies (mod g_i) ($i = 1, 2, 3$) implies

$$abc = T(g_1 g_2 g_3)^2 \to \left.\begin{aligned} P' &= t_1 \xi^3; \\ Q' &= t_2 \eta^3; \\ R' &= t_3 \zeta^3; \end{aligned}\right\}$$

where $t_1 t_2 t_3 = T$;

$$\to g_2 g_3 t_1 \xi^3 + g_3 g_1 t_2 \eta^3 + g_1 g_2 t_3 \zeta^3 = d\xi\eta\zeta;$$

Write

$$\left.\begin{aligned} a' &= g_2 g_3 t_1; \\ b' &= g_3 g_1 t_2; \\ c' &= g_1 g_2 t_3; \end{aligned}\right\}$$

and the lemma is proved. QED

The lemmas 1—6 and a descent now give Theorem 3 in the case N_0, but it is readily seen that also M_0 is proved.

The condition $\omega_1\omega_2\omega_3=0$ needs some explanation. Assume that the descent has led to an equation

$$f^{\omega_{10}}U_0^3+f^{\omega_{20}}V_0^3+f^{\omega_{30}}W_0^3 = dU_0V_0W_0; \tag{31}$$

where $\omega_{10}\omega_{20}\omega_{30}\neq 0$, but the other conditions in Theorem 3 are fulfilled, i.e. also $|U_0V_0|=1$. Then $f|W_0$ as $f\nmid d$ and $|U_0V_0|=1$. But this gives $f^k|W_0$, where $k=\min(\omega_{10}, \omega_{20})$ and (31) is converted into an equation where $\omega_{10}'\omega_{20}'\omega_{30}'=(\omega_{10}-k)(\omega_{20}-k)(\omega_{30}+2k)=0$.

I now prove the statement in Theorem 3 when $H\in U$
$\forall_i:\alpha_i\equiv 1 \pmod 2 \rightarrow \varepsilon=1$ (see Lemma 5).
Lemma 5, Case A, equation (25) →

$$dA = (d^3-27f^{\omega})A^3;$$

as it is necessary to have $M=u^2+uv+v^2=1$ to make the descent stop.

$A\neq 0 \rightarrow d=(d^3-27f^{\omega})A^2$; but $(d, d^3-27f^{\omega})=1 \rightarrow$

$(A^2=\pm d.\ d^3-27f^{\omega}=\pm 1)$. Now $d^3-27f^{\omega}=\pm 1 \rightarrow$

$(\omega=1.\ f=19{,}37)\rightarrow d=8{,}10$ which is a contradiction to $A^2=\pm d$.
Lemma 5, Case B, equation (29) →
$3dA=(d^3-27f^{\omega})A^3+2$; with solutions valid for M_0, N_0:

Case	A	ω	f	d		U_0	V_0	W_0
N_0	$-1.$	2	2	5	$2U^3+2V^3+2W^3=5UVW$	1	1	1
N_0	2	2	7	11	$7U^3+7V^3+W^3=11UVW$	1	1	2
M_0	-2	4	3	13	$3^2U^3+3^2V+W^3=13UVW$	1	1	2

Lemma 5 can in fact also be used to prove the Theorem 3 in the cases N_{01} and N_{02}. $H\in T$ in these cases implies: $f\equiv 2 \pmod 3$.

Lemma 4 gives $\omega=1, 2$ in these cases. i.e. $abc=f, f^2$. (19) can be written:

$$x^3+y^3+cz^3=f^{\delta}Dxyz, \quad \text{where} \quad c=f, f^2\cdot\delta\geqq 1$$

This is because $fx^3+fy^3+z^3=f^{\delta}Dxyz$ in the case N_{02} immediately implies $x^3+y^3+f^2z_1^3=f^{\delta}Dxyz_1$, where $z_1=z/f$.

N_{01}: The condition for a non-trivial solution to exist is the existence of U_0, V_0 and W_0, such that

$$U_0^3+V_0^3+fW_0^3=f^{\delta}DU_0V_0W_0, \quad \text{where} \quad |U_0V_0|=1.$$

$U_0V_0=-1\rightarrow$the solution $U_0=1$, $V_0=-1$, $W_0=\pm\sqrt{D}\cdot f\exp((\delta-1/2))$ provided δ is odd and $D=-D_0^2$.

$$U_0V_0=1\rightarrow f=2.\ d=4.\ (\omega=1).$$

The corresponding equation: $x^3+y^3+2z^3=4xyz$ has a solution $x=y=z=1$ and $H=2\cdot 5\in U$. This solution is solitary: Equation (30) in Case B, Lemma 5 gives ($\varDelta_1=3\varDelta_2$ and $P, Q, R=1, 1, 2$)

$$\left.\begin{aligned} 3\varDelta_2 &= 4A-\varepsilon(u+2v);\\ 3\varDelta_2 &= 4A-\varepsilon(u-v);\\ 6\varDelta_2 &= 4A+\varepsilon(2u+v): \end{aligned}\right\}\rightarrow v=0.\quad \varDelta_2=\varepsilon u\rightarrow A=\varepsilon u$$

but $(A, u)=1$ if $v=0$, otherwise $(\beta_1, \gamma_1)\neq 1$. Also $(A, \varepsilon)=1\rightarrow A=\pm 1$. $\varepsilon u=\pm 1$ and this implies that an ascent cannot succeed as the same solution recurs.

N_{02}: $U_0^3+V_0^3+f^2W_0^3=f^\delta DU_0V_0W_0$, where $|U_0V_0|=1$ implies as in case N_{01} when $U_0V_0=-1$:

If δ is even and $D=-D_0^2$ then there is a non-trivial solution $U_0=1$, $V_0=-1$, $W_0=\pm\sqrt{D}\cdot f\exp((\delta/2-1))$.

$$U_0V_0=1\rightarrow f=2.\ D=1.$$

$x^3+y^3+4z^3=2xyz$ thus has the solution $x, y, z=1, 1, -1$ and $H=-2^2\cdot 5^2\in T$. Using the same arguments as in N_{01}, this is a solitary solution.

*

The correspondence to Lemma 5 when $3|d$ and $3^3\|F$ is now shown, omitting a lot of details.

Lemma 7. *Consider* $ax^3+by^3+cz^3=dxyz$; (20) *again. If* (20) *has a solution* x, y, z *and* $(abc\neq 0.\ (d, abc)=1.\ 3^3\|F.\ a{:}b$ *a rational cube.* $H\in T)$ *and also* (*when* $abc=f^\omega.\ \omega\not\equiv 0\ (\text{mod}3))3\|d.\ H\equiv\pm 1\ (\text{mod}9)$ *or* $3^2|d.\ f\not\equiv\pm 1\ (\text{mod}9)$, *then there is an equation* $a'\xi^3+b'\eta^3+c'\zeta^3=d\xi\eta\zeta$; *where* $a'b'c'=abc$, *having a solution* ξ, η, ζ *satisfying:* $\xi\eta\zeta|z$.

Proof. Write $d=3D$. Apply the transformation (see Lemma 5)

$$\left.\begin{aligned} \varDelta\alpha &= Sx+Ty+Dz;\\ \varDelta\beta &= -2Sx+Ty+Dz;\\ \varDelta\gamma &= Sx-2Ty+Dz; \end{aligned}\right\},\quad (\alpha, \beta, \gamma)=1, \tag{33}$$

on (20), resulting in

$$3^2D^3\alpha(\beta^2+\beta\gamma+\gamma^2)=(D^3-f^\omega)(\alpha+\beta+\gamma)^3.$$

As in Lemma 5:

$$\left.\begin{aligned} \alpha &= k_1A^3;\\ \beta^2+\beta\gamma+\gamma^2 &= k_2M^3; \end{aligned}\right\}\quad \text{where}\quad k_1k=3(D^3-f^\omega).\quad (A, M)=1.\quad D^3-f^\omega\in T$$

and $$\beta = \varepsilon\beta_1. \quad \gamma = \varepsilon\gamma_1. \quad (\beta_1, \gamma_1) = 1. \quad \beta_1^2+\beta_1\gamma_1+\gamma_1^2 = \begin{Bmatrix} 1 \\ 3 \end{Bmatrix} M^3.$$

Case A: $\beta_1^2+\beta_1\gamma_1+\gamma_1^2=M^3$.

As in Lemma 5 (same notation & arguments)

$$\varepsilon^2 D \cdot A(u^2+uv+v^2) = (D^3-f^{\omega})A^3-\varepsilon^3(u^2v+uv^2); \tag{34}$$

$$\left.\begin{aligned} \Delta_1 P &= DA+\varepsilon u; \\ \Delta_1 Q &= DA+\varepsilon u; \\ \Delta_1 R &= DA-\varepsilon(u+v); \end{aligned}\right\} \quad \text{where} \quad (P, Q, R) = 1.$$

Finally $P=a'\xi^3$. $Q=b'\eta^3$. $R=c'\zeta^3$; and $a'\xi^3+b'\eta^3+c'\zeta^3=3D\xi\eta\zeta=d\xi\eta\zeta$; where

$$\xi\eta\zeta = \frac{z}{\Delta\Delta_1 M};$$

Case B: $\beta_1^2+\beta_1\gamma_1+\gamma_1^2=3M^3$.

This is where difficulties occur and it is easily shown that the extra conditions are necessary.

As in Lemma 5 $j=0$ implies

$$3\varepsilon \cdot D \cdot A(u^2+uv+v^2) = (D^3-f^{\omega})A^3+\varepsilon^3(-u^3+v^3+3u^2v+6uv^2); \tag{35}$$

Also here: $3 \nmid \varepsilon$. $3 \nmid A$. $3 \nmid u^2+uv+v^2$.

Suppose: $3 \nmid D$. $D^3-f^{\omega} \equiv \pm 1$ (mod9);

$$(35) \to \pm 3 \equiv \pm(D^3-f^{\omega})\pm 1 \quad (\text{mod}9) \to$$
$$D^3-f^{\omega} \equiv \pm 2, \ \pm 4 \quad (\text{mod}9).$$

A contradiction!

Suppose instead: $3|D. f \not\equiv \pm 1$ (mod9)

$$(35) \to 0 \pm f^{\omega} \pm 1 \quad (\text{mod}\,9) \to f^{\omega} = \pm 1 \quad (\text{mod}9) \to$$
$$\to f \equiv \pm 1 \quad (\text{mod}9) \quad (\text{as } 3 \nmid \omega).$$

A contradiction again.

The case $j=1$ is excluded with the same conditions and arguments.

$j=-1 \to \beta_1 = u^3-v^3-3u^2v-6uv^2$. $\gamma_1 = u^3-v^3+6u^2v+3uv^2$; $\to$

$$3\varepsilon^2 DA(u^2+uv+v^2) = (D^3-f^{\omega})A^3+\varepsilon^3(2u^3-2v^3+3u^2v-3uv^2); \tag{36}$$

$$\left.\begin{aligned} \Delta_1 P &= DA-\varepsilon(u+2v); \\ \Delta_1 Q &= DA-\varepsilon(u-v); \\ \Delta_1 R &= DA+\varepsilon(2u+v); \end{aligned}\right\} \to$$

$$P = a'\xi^3. \quad Q = b'\eta^3. \quad R = c'\zeta^3 \quad \text{and}$$

$$a'\xi^3+b'\eta^3+c'\zeta^3 = 3D\xi\eta\zeta, \quad \text{where} \quad a'b'c' = abc.$$

Also
$$\xi\eta\zeta = \frac{z}{\Delta\Delta_1 M};$$
QED

Lemma 7 and a descent now proves Theorem 3 in the case N_3. There remains to investigate the subcase $H \in U$. Equation (34) in Case A, Lemma 7→($\varepsilon = M = 1$)

$$DA = (D^3 - f^{\omega})A^3; \quad \to D = (D^3 - f^{\omega})A^2; \quad \text{as} \quad A \neq 0.$$

But $f \nmid D \to D = \pm A^2$. $D^3 - f^{\omega} = \pm 1$, two equations having no consistent solutions.

Equation (35) in Case B, Lemma 7→

$3DA = (D^3 - f^{\omega})A^3 + 2$; with solutions valid in the case N_3 (and $H \in U$):

A	f	ω	d	H		U_0	V_0	W_0
-1	2	4	9	11	$4U^3+4V^3+W^3 = 9UVW$	1	1	1
2	5	2	9	2	$5U^3+5V^3+W^3 = 9UVW$	1	1	2

M_{31} and M_{32}: Lemma 7 can be used, just watch the extra conditions due to $f=3$. Put $d=3D$.

In the case M_{31}: $H = D^3 - 3 \equiv \pm 1 - 3 \equiv -2, -4 \pmod{9}$.

As $3 \| d$, the case is undecidable (at least with above theorems) unless Case B in Lemma 7 can be excluded.

This is in fact possible, if $d \equiv 3 \pmod{9}$.

Proof. $x^3 + y^3 + 3z^3 = 3Dxyz$; where $D \equiv 1 \pmod{3}$.

$$3 | x^3 + y^3 \to 3 | x + y \to 3^2 | x^3 + y^3 \to 3^2 | 3z(Dxy - z^2) \to 3 | z(Dxy - z^2).$$

Assume: $3 \nmid z$; Then $3 | Dxy - z^2 \to xy \equiv z^2 \pmod{3}$
$\to xy \equiv 1 \pmod{3} \to x \equiv y \equiv \pm 1 \pmod{3} \to x + y \equiv \mp 1 \pmod{3}$.
Clearly a contradiction! Thus $3 | z$.
The first equation also gives:

$$(x+y+z)\big((x+y+z)^2 - 3(xy+yz+zx)\big) = z\big(3(D-1)xy - 2z^2\big) \equiv 0 \pmod{3^3};$$

$$x+y+z \equiv 0 \pmod{3} \to 3 \| (x+y+z)^2 - 3(xy+yz+zx)$$

(because $3^2 | (x+y+z)^2$. $3^3 | 3z(x+y)$. $3 \| 3xy$)
$\to x+y+z \equiv 0 \pmod{3^2}$;
$\Delta\alpha = x+y+Dz = x+y+z+(D-1)z \equiv 0 \pmod{3^2}$
$\Delta\beta = -2x+y+Dz = \Delta\alpha - 3x \not\equiv 0 \pmod{3^2}$
$\Delta\gamma = x-2y+Dz = \Delta\alpha - 3y \not\equiv 0 \pmod{3^2}$
The conclusion is: $3 | \Delta$ but $3^2 \nmid \Delta \to 3 | \alpha$ and only the Case A occurs. QED

In the case M_{32}: $H=D^3-3^2\equiv\pm1$ (mod9). i.e. the 'correct' condition when $3\|d$. Equation (19) has a non-trivial solution if and only if $U^3+V^3+9W^3=3DUVW$ has a non-trivial solution U_0, V_0, W_0 where $|U_0V_0|=1$.

$U_0V_0=-1\to3|D$, a contradiction.

$U_0V_0=1\to2+9W_0^3=3DW_0$, clearly impossible.

N_{31}, N_{32}: Lemma 7 can be used also in these cases, with minor modifications. Write $d=3f^\delta D$, where $\delta\geqq1$. $f\neq3$.

The extra conditions in Lemma 7 are:

$3|D$. $f\not\equiv\pm1$ (mod9) or $3\nmid D$. $(Df^\delta)^3-f^\omega\equiv\pm1$ (mod9).

$$\omega=1\to U_0^3+V_0^3+fW_0^3=3f^\delta DU_0V_0W_0,\quad\text{where}\quad |U_0V_0|=1.$$

$U_0V_0=-1$ gives the statement in Theorem 3.

$U_0V_0=1$ implies $f=2$. $D=0$, which is already excluded.

$$\omega=2\to U_0^3+V_0^3+f^2W_0^3=3f^\delta DU_0V_0W_0,\quad\text{where}\quad |U_0V_0|=1.$$

$U_0V_0=-1$ gives the statement in Theorem 3.

$$U_0V_0=1\to W_0=1.\quad f=2.\quad \delta=1.\quad D=1,\quad\text{but}\quad H=4\not\equiv\pm1\quad(\text{mod}\,9)$$

thus the conditions for Lemma 7 are not fulfilled in this specific case.

To prove Theorem 3 in the cases where $3^4\|G$, it is necessary to have a lemma, corresponding to Lemma 5 and 7, namely

Lemma 8. *Can be stated like Lemma* 5, *except for* $d\not\equiv0$ (mod3), *which is replaced by* $G=3^4$. *Also suppose:* ($abc=f^\omega$. $\omega\neq0$ (mod3)), *even though* $a{:}b$ *a rational cube is a sufficient condition.*

As in Lemma 7:

$$3^2D^3\alpha(\beta^2+\beta\gamma+\gamma^2)=(D^3-f^\omega)(\alpha+\beta+\gamma)^3;$$

where now: $3\|D^3-f^\omega$, giving

$$3D^3\alpha(\beta^2+\beta\gamma+\gamma^2)=\frac{D^3-f^\omega}{3}(\alpha+\beta+\gamma)^3. \tag{37}$$

As in Lemma 5:

$$\left.\begin{aligned}\alpha&=k_1A^3\\ \beta^2+\beta\gamma+\gamma^2&=k_2M^2\end{aligned}\right\}\quad\text{where}\quad k_1k_2=3(D^3-f^\omega).\quad (A,M)=1.$$

Now $H=(D^3-f^\omega)/3$ and three subcases occur:

$3\nmid k_1$. $3^2|k_2$

Here: $3^2|\beta^2+\beta\gamma+\gamma^2 \to 3^2|(\beta-\gamma)^2+3\beta\gamma \to 3|\beta-\gamma \to 3|\beta\gamma \to (3|\beta.\ 3|\gamma)$, but together with $3|\alpha+\beta+\gamma$ we get $(\alpha, \beta, \gamma) \neq 1$, a contradiction.

$3|k_1.\ 3|k_2$

$$3|\beta^2+\beta\gamma+\gamma^2 \to 3|\beta-\gamma, \quad \text{but} \quad 3|\alpha+\beta+\gamma \to 3|\beta+\gamma$$

thus $3|\beta.\ 3|\gamma$ and again $(\alpha, \beta, \gamma) \neq 1$.

$3^2|k_1.\ 3\nmid k_2$.

As in earlier lemmas ($j=-1$ here too)

$$\varepsilon^2 DA(u^2+uv+v^2) = (D^3-f^\omega)A^3-\varepsilon^3(u^2v+uv^2).$$

Apply transformation (26) again, finally

$$a'\xi^3+b'\eta^3+c'\zeta^3 = d\xi\eta\zeta;$$

and

$$\xi\eta\zeta = \frac{z}{\Delta\Delta_1 M}. \qquad \text{QED}$$

The case N_4 of Theorem 3 is now readily shown. If $H \in U$ then the proof is similar to that for N_3.

The cases M_4, N_{41} and N_{42} are shown after a slight modification of Lemma 8.

The difficulty to decide the solubility of (19) when $3^5|G$, lies therein that it is not possible by means of elementary congruences to exclude asymmetrical (in u and v) equations of the type:

$$\varepsilon^2 DA(u^2+uv+v^2) = \frac{D^3-f^\omega}{3^2} \cdot A^3+\varepsilon^3(u^3-v^3+3u^2v);$$

$$\text{where } 3^2|D^3-f^\omega.$$

*

Theorem 3 now proves to be powerful within its restrictions. In Table 1 I have marked cases of 'non-solubility' with a '$-$' and cases where there exists at least one non-trivial solution of $x^3+y^3+abcz=dxyz$; with a '$+$'. The density of decided cases is good when $abc=f^\omega$. $\omega \not\equiv 0 \pmod 3$ and $|d| \leqq 27$. Note in these cases that a '$+$' marks that *every* equation in the family: $ax^3+by^3+cz^3=dxyz$ has a non-trivial solution.

The cases when $d=0$ have been taken from Selmer [11]. All combinations of a, b, c and d with $|abc| \leqq 27$ and $|d| \leqq 27$ have been tested to give solutions of ax^3+ $+by^3+cz^3=dxyz$ where $a \geqq b \geqq c \geqq 1$ and $|x| \leqq 50$. $|y| \leqq 100$. $|z| \leqq 100$. The solutions with a minimal x are given in Table 4. Note that a solution of $ax^3+by^3+cz^3=dxyz$ generally gives a solution of $x^3+y^3+abcz^3=dxyz$ due to Sylvester's 'Theorem of Derivation', given at the beginning of Part 3.

Table 1

$$x^3+y^3+abcz^3 = dxyz$$

$\boxed{+}$: non-trivial solution exists
$\boxed{-}$: no solution exists
$\boxed{}$: undecided

d																												
27		+	−	−	+		+	+	+		−	+	−			+		+			+		−	+	+		+	
26	+			+		+	−	+	−	+	+		+		+	+	−	+	+	+	+	+	+	+	−		+	
25	−	−	+	−	+	+	−		−		−			+	+	+	−		−		+		+	+			+	
24		−		+	+	+	−	−	−		−		+	+		+	+	+	+	+		+		+	+	+	−	
23	−	−	+	+	−	+	−		+	+	−		+			+	−	+	+	+	+		−	+	+		+	
22	−	−	+	+	+			−	−	+	−	+	+	+	+	−	−	+	−	+	+	+	+	+	+	+	+	
21	+		−	−	+	+	+		−		−		−			+			+				+		−		−	
20	+	−	−	−	+		+	+	−	+	−		−	+	+	+	−	+	−		+	+	+	+	−	+	+	
19	+	−	−	+	+	+	−	+	−	+	+	+	+			+	+		+	+	+		−		−	+	+	
18	+	−	−	−	+		+	+	+	+	−	+		+	+	+	+			+				−	−	+	+	
17	+	+	+	+	+	+	−		−	+	+		+		+	−	−		+	+	+		+		+		+	
16	+	+	+	+	−		+	−	+	+	−	+	+	+		+	+	+	+	+	+		+	−	−	+	+	
15	+	−	+	+	−		−		−		+		+		+	+	+	+	+		+		−	+	−	+	+	
14	+	−	−	−	−		+	−	+		−	+	−	+	+	+	+		+			+	+	−	−	+		
13	+	+	+	+	−		−		−	+	+		+	+	+	−	+	+	+	+		+	−		−		+	
12	−	+	+	−			+	+	−	+	−		−	+	+	−	+	+	+	+				+	+	+	−	
11	−	−	+	−	+		+	+	+	+	−		+	+		+	+	+	+		+		−	+	+			
10	+	+	+	+	−	+	+	+	+		+	+	−	+		−	+	+	+	+			+	+	+	+	+	
9	+	+	−	−	−		+		+		+	+	+		+	+	+		+	+	+	+	−	+	+		+	
8	−	+	−	−	+	+	+	−	−	+	+	+	+		+	+	−	+	−		+	+	+	−	+			
7	−	+	−	−	+		+	+	+		−	+	−			−	−		−	+		+	+		−	+	+	
6	+	−	+	+	−	+	−	+	−		−			+		−		+	+	+	+		−	−	−		−	
5	+	−	+	+	−		+		−	+	+		−	+		+	−	+	+	+		+	+		−		+	
4	−	+	−	−	−	+	−	−	−		−		+	+		−	+		−		+	+	−	+	+	+		
3	+	−	−	−	+		−		−		−	+	−		+	−	+	+			+				−		−	
2	−	−	+	+	+		+	−	−	+	−		+			−	−	+	−	+	+	+	+	+	−	+		
1	−	−	+	−	+		−	+	+	+	+		+		+	−	+		−	+		+	+	+	+	+		
0	−	+	−	−	−	+	+	−	+	−	−	+	+	−	+	+	+	−	+	+	−	+	−	−	−	+	−	
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	abc

It is fairly easy to realize from the proof of Theorem 3 that it is effective even if abc is not of the form f^{ω}, if you just succeed to exclude (e.g. by means of congruences) all combinations of a, b, c except $a'=1$. $b'=1$. $c'=abc$, when given abc. The condition $abc=f^{\omega}$. $\omega\not\equiv 0$ (mod3) is used only to secure that $a{:}b$, $b{:}c$ or $c{:}a$ is a rational cube at every factorization of abc into three factors.

Mordell [5] shows that

$$x^3+by^3+cz^3 = dxyz; \tag{38}$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	*abc*
−1	+	−	−	+	−		+		+		−	+	−			+	−		−		+		−	+	+	+	+	
−2	−	+	−	−	+	+	−	+	−		+	+	−			−	−	+	−	+	+	+	+	−	−	+	+	
−3	−	−	+	+	−		−	−	−		−	+	+	+	+	−	+		+			+	−	+	+		+	
−4	+	−	−	+	+	+	−	−	+	+	+		−		+	+	+	+	+				−	−	+	+		
−5	−	−	+	−	+		+	+	+		+	+	+	+	+	−	+		−	+	+	+	+	+	−	+		
−6	−	−	−	−	+	+	−	−	−	+	+	+	+	+		−	+		+		+	+	+	+	−		−	
−7	−	+	−	−	−	+	+		−		−	+	−	+	+	+	−		−		+		+		−		+	
−8	−	+	−	+	−		−	+	+	+	−		−	+		−	−	+	−	+		+	+	−	−		+	
−9	+	−	−	+	−	+	−		+	+	+		−	+		+	+		+	+	+		−	+	+		−	
−10	+	+	−	+	−		−	−	−	+	+		+		+	−	+	+	−			+		+	+		+	
−11	+	+	−	−	+	+	+		+	+	+		−	+	+	+	−		−	+	+		+	+	−		+	
−12	+	−	+		−		−	−	+	+	−	+	+			−	+	+	+			+	−	−	−	+	+	
−13	−	+	+	−	−	+	−		+		+		+		+	−	−		+			+	−	+	−	+		
−14	−	+	−	+	−		+	−	+	+	−		+	+	+	+	−		−	+	+		+	−	+	+	+	
−15	−	+	−	+	+		−	+	+		−	+	−		+	−	+		+	+	+		+		−	+	−	
−16	+	−	+	+	+	+	−	−	+		−		+	+	+	+	−		−	+		+	−	−	+	+	+	
−17	+	−	−	−	+		+	+	−		−	+	+	+		+	+	+		+	+	+	+		+		+	
−18		+	+	−	+		+	+	+		−		+	+		−	+	+	+	+			−	−	+		−	
−19	−		−	+	+		−	+	+		+	+	+		+	−	−		+	+		+	−		−			
−20	−	+	−	−	+		+	+	−	+	−	+		+	+	+	+		−	+			+	−	−	+		
−21	+	−	+		−		+		−		−				+	−			+	+	+				−		−	
−22	+	−	+	+	−	+	+	+	+		+	+	−		+	+	−				+	+	−	−	+	+		
−23		−	+	−	+	+	−		+		+	+	−	+		+	−	+	−				+	+	+			
−24	+	−	+	+		+	−	+	−			+	+	+	+	−	+	+		+		+		+	−		−	
−25	+		−	+	−		−	+	+	+	+	+	−		+	+	+	+	−		+	+	−		+			
−26	−	+	−	−	+	+	+	−	+		+	+		+	+	+	+	+	−		+			+	−	+		
−27	+	+	+				−		+	+	+	+				−	+		+		+		−	+	+		+	

lacks solutions if $b \equiv 2$ (mod7). $c \equiv 4$ (mod7), $d \equiv -1$ (mod7). So, it might be possible to use Theorem 3 when $abc=8$ or $abc=6$* for example, as the tricky combinations of a, b and c can be excluded in some cases.

However, suppose $abc=p_1p_2$ where p_1 and p_2 are primes and $p_1 \equiv 2$ (mod7). $p_2 \equiv 4$ (mod7). Also suppose $d \equiv -1$ (mod 7), then $d^3-27abc \equiv 0$ (mod7) i.e. $7 | d^3 - -27abc$ and $H \notin T$, which means that Theorem 3 cannot be used.

Suppose $f \equiv \pm 4$ (mod9) and f a prime. Then $x^3+2y^3+fz^3 \equiv 9Dxyz$ has only trivial solutions, which is easily shown by congruence (mod9).

Sylvester [9] probably made use of this fact when stating a theorem on the non-solubility of

$$x^3+y^3+2fz^3 = 9Dxyz. \qquad (39)$$

Here $3^3 \| G$. i.e. to be able to use Theorem 3 we must have $2f \not\equiv \pm 1$ (mod9), but $f \equiv \pm 4$ (mod9) on the contrary gives $2f \equiv \mp 1$ (mod9). As seen from the counterexample

---

* In the case $abc=6$ write (38): $x^3+2y^3+(-3)(-z)^3=(-d)xy(-z)$; where now $d \equiv 1$ (mod 7).

given at the beginning of part 3, Sylvester lacked this condition in other statements too. However, I have not found any counterexample to his statement about (39) for $f=5, 13, 23, 31$ and max $(|x|, |y|, |z|) \leqq 40$.

Sylvester [9] also made an analogous statement on $x^3+y^3+4f\cdot z^3=18Dxyz$ where $f\equiv \pm 2 \pmod 9$ and a prime. Also here there is no immediately found counterexample.

Some further examples of the same kind:

$x^3+fy^3+9z^3=9Dxyz$; has no solution if $f\equiv \pm 2, \pm 4 \pmod 9$, but here $3^5\|G$ and the equation

$$x^3+y^3+9f\cdot z^3 = 9Dxyz;$$

is problematic.

Also $x^3+f\cdot y^3+7^\alpha z^3=7Dxyz$; where $f\equiv \pm 2, \pm 4 \pmod 7$. $\alpha=1, 2$ has no solution, a congruence (mod7) shows this. But $H \notin T$ as $7|H$ and it is not possible to use Theorem 3.

Using Selmer's [11] results on cubic residues, further examples are easily constructed.

*

To find parametric solutions of

$$x^3+y^3+cz^3 = dxyz; \tag{40}$$

where c and d are unrestricted, proceed as follows: Plainly shown by insertion is that:

If $x^3+y^3+c_0z^3=d_0xyz$ has a solution x_0, y_0, z_0, then the equation where

$$\left.\begin{aligned} c &= c_0+\delta x_0y_0 \\ d &= d+\delta z_0^2 \end{aligned}\right\}, \quad \delta \text{ a rational integer}$$

has the same solution.

$x^3+y^3+\gamma^3z^3=3\gamma xyz$; has (infinitely) many solutions satisfying:

$$x+y+\gamma z = 0;$$

x, y and z can be written parametrically

$$x = -\frac{\gamma u}{2}+\tau. \quad y = -\frac{\gamma u}{2}-\tau. \quad z = u; \tag{41}$$

where τ is chosen so that x, y become integers. i.e. τ is a rational integer or the half an odd such.

Equation (40), where:

$$\left.\begin{aligned} c &= \gamma^3+\delta\left(\frac{\gamma^2u^2}{4}-\tau^2\right); \\ d &= 3\gamma+du^2; \end{aligned}\right\}$$

thus has a solution given by (41). This is easily generalized so that c changes into cm^3, d into dm and z into mu, but x and y in (41) remain unchanged.

These forms for c and d are also necessary as shown by the surprisingly easy proved:

Theorem 4. *If* $x^3+y^3+cz^3=dxyz$; (40) *where c and d are unrestricted, has a solution x, y, z satisfying $xyz\neq 0$ and $x+y\neq 0$, then*

$$\left.\begin{aligned} cm^3 &= \gamma^3+\delta\left(\frac{\gamma^2u^2}{4}-\tau^2\right); \\ dm &= 3\gamma+\delta u^2; \end{aligned}\right\} \tag{42}$$

for some values of the rational integers m, γ, δ, u and the (half-) integer τ.

If furthermore $z=3^\alpha t$, where $t\in T$ and $\alpha\geqq 0$, then $m=1, 3$ and u is of the same form as z and given c and d it is possible to determine all values (if they exist) of m, γ, δ, u and τ satisfying (42). Thus it is always possible to decide whether (40) has any solution satisfying $z=3^\alpha t$, and the number of such z-values is always finite.

Proof. Suppose x, y, z is a solution of (40). For simplicity also suppose $(x, y)=1$. Write $\gamma/m=-(x+y)/z$, where $(\gamma, m)=1$, and $u=z/m$.

Then $x+y=-\gamma u$ and for some (half-) integer τ

$$\left.\begin{aligned} x &= -\frac{\gamma u}{2}+\tau; \\ y &= -\frac{\gamma u}{2}-\tau; \\ z &= mu. \end{aligned}\right\}$$

Inserting these expressions into (40), it is easily obtained:

$$u^2(cm^3-\gamma^3) = xy(dm-3\gamma); \quad \text{but} \quad (u^2, xy) = 1 \text{ gives (42).}$$

Now (42) implies:

$$m^3(d^3-27c) = \delta\left(\left(\frac{9\gamma u}{2}+\delta u^3\right)^2+27\tau^2\right); \tag{43}$$

As $(\gamma, m)=1$ we must have $(\delta, m)=1$ by (42) and consequently $\delta|d^3-27c$.

From (43) it is concluded that if the m-value is known (and c, d given), then there is only a finite number of possibilities for δ, γ, u, τ and $z(=mu)$.

Suppose first $\alpha=0$ i.e. $z\in T$.

$$z = \Pi_i p_i^{\alpha_i}. \quad \forall_i : p_i \equiv 2 \pmod 3 \rightarrow \forall_i : x^3+y^3 \equiv 0 \pmod{p_i^{\alpha_i}}$$

but then $\forall_i : p_i^{\alpha_i}|x+y$ as $p_i|x^2-xy+y^2$ would give $x\equiv y\equiv 0 \pmod{p_i}$ because $p_i\equiv 2$ (mod3).

Thus $z|x+y$ or $x+y+\gamma z=0$ for some γ and $m=1$.

Then suppose $3|z$ i.e. $3^{\alpha}\|z$ and $\alpha\geqq 1$.

If $3\nmid d$, then $3^{\alpha-1}\|x+y\to 3(x+y)+\gamma z=0$, where $3\nmid\gamma$ and $m=3$.

If $3|d$, then $3^{\alpha+1}|x^3+y^3\to 3^{\alpha}|x+y\to x+y+\gamma z=0$ for some γ and $m=1$. QED

Note. If $x+y=0$ in (40) then $x=-y=1$ and $\sqrt{-d/c}$ must be an integer for (40) to have non-trivial solutions. If $(x, y)=r>1$ then γ and τ in (42) and (43) should be replaced by $r\gamma$ and $r\tau$.

Theorem 4 gives an efficient way to find solutions of $x^3+y^3+cz^3=dxyz$. If there exists a solution $z=3^{\alpha}t$, where $t\in T$ (62 of the natural numbers $\leqq 100$ satisfy this) this can be found by putting $m=1, 3$ in (43) and try finite number of possible values for δ, u and τ. If no solution is found, then continue to try with $m=7, 13, 19, 21, 31, 37, 39, 43, 49, \dots$ i.e. the numbers containing only primes $\equiv 1$ (mod6) or three times such a number if $d\not\equiv 0$ (mod3). The theorem is used below (at the end of part 4) to give parametric solutions of $x^3+y^3+z^3=nxyz$. i.e. for special n-forms it is always possible to find a non-trivial solution.

4. On $f^3x^3+g^3y^3+h^3z^3=nxyz$

Consider

$$f^3x^3+g^3y^3+h^3z^3=nxyz \tag{44}$$

where $fgh\neq 0$.

The case $fgh=1$. $n=0$ is 'Fermats last theorem' for cubes and there are no solutions even in $K(\sqrt{-3})$.

Mordell [10] proved that (44) has essentially one (solitary) solution when $fgh=1$. $n=-1, 5$. This was long an outstanding question from Hurwitz' paper [12].

The case $fgh=1$. $n=3$ is trivial since (44) then can be written: $(x+y+z)\big((x-y)^2+ +(y-z)^2+(z-x)^2\big)=0$.

Rather famous became the case, where $fgh=1$. $n=1$ as it remained undecided for several years. However Cassels [15] and Sansone, Cassels [16] proved that the corresponding equation has no solutions. The method in the last paper is elementary, but cannot be used for other n-values.

The problem to deal with (44) using the method in part 3 is obvious: $F=d^3-27abc=n^3-(3fgh)^3\notin T$, as F contains at least one pair of conjugated primes in $K(\sqrt{-3})$ unless $n=3fgh$. Therefore (44) should be transformed to a form, more suitable for Theorem 3.

Sylvester [6] stated without proof: (44) can be made to 'depend upon'

$$Au^3+Bv^3+Cw^3 = Duvw; \tag{45}$$

where $ABC=(n-6fgh)^3-(n-3fgh)^3$ and $D=3(n+6fgh)$. He concludes from this that $x^3+y^3+z^3=-6xyz$ has no solutions, for example.

A proof is easily obtained from Theorem 1 by changing $x \to fx$. $y \to gx$. $z \to hz$ and putting $a=fgh$. $b=0$. $d=-n$. The result is

$$fgh(n^2+3nfgh+9(fgh)^2)(U+V+W)^3 = (n+6fgh)^3 UVW. \tag{46}$$

But $fgh(n^2+3nfgh+9(fgh^2))=1/3^3((n+6fgh)^3-(n-3fgh)^3)$ and the statement is proved.

The condition for the transformation to be permitted is $(n+6fgh)(n-3fgh) \neq 0$, so Sylvesters conclusion was proved is some other way.

One should use the transformation:

$$\left.\begin{aligned} p &= x+y+z; \\ q &= -2x+y+z; \\ r &= x-2y+z; \end{aligned}\right\}$$

to bring $x^3+y^3+z^3=-6xyz$ into $P^3+Q^3+R^3=0$ (via an intermediate step). Here P, Q and R are all rational integers, cf. the transformation in part 2, where u, v and $w \in K(\sqrt{-3})$.

In (44), $(n, fgh)=1$ can be assumed, as otherwise the common factors certainly could be included into x, y or z. Equation (44) then in general leads to a difficult equation

$$au^3+bv^3+cw^3 = (n+6fgh)uvw; \tag{47}$$

where $abc=fgh(n^2+3nfgh+9(fgh)^2)$ (if $3 \nmid n$). To make any progress, suppose $fgh = 1, 2$. (46) in the first case becomes:

$$(n^2+3n+9)(U+V+W)^3 = (n+6)^3 UVW; \tag{48}$$

where $n \neq -6$ and $n \neq 3$, but these n-values have already been discussed.

Four distinct subcases occur:

Case #	$n(\text{mod} 9)$	$(n^2+3n+9, (n+6)^3)$
1	1, 2, 4, 5, 7, 8	1
2	0	3^2
3	3	3^3
4	6	3^2

Case 1. If $n^2+3n+9=f^\omega$. $\omega \not\equiv 0$ (mod3). f a prime, then Theorem 3 can be used immediately, provided that $d^3-27abc=(n+6)^3-27(n^2+3n+9)=(n-3)^3 \in T$. i.e. $n-3 \in T$. $n^2+3n+9=f^\omega$ has many solutions, especially for small n-values, when $\omega=1$.

When $\omega=2$ it has only the solutions $f=7$. $n=-8,5$.

When $\omega > 3$ there are probably no solutions, but in fact one could safely state only that it has a finite number of solutions for each ω [17].

Necessary and sufficient for $x^3+y^3+z^3=nxyz$; to have at least one non-trivial solution if $n^2+3n+9=f$. $n-3\in T$ is thus that

$$u^3+v^3+(n^2+3n+9)w^3 = (n+6)uvw$$

has a non-trivial solution satisfying $|uv|=1$.

This is possible only when $n=-1$, and the solution is $u=1$. $v=1$. $w=-1$.

If $w=2$, there is only need to investigate if

$$\left.\begin{aligned} u^3+v^3+49w^3 &= 11uvw; \\ \text{or} \quad 7u^3+7v^3+w^3 &= 11uvw; \end{aligned}\right\} \quad n=5$$

or

$$\left.\begin{aligned} u^3+v^3+49w^3 &= -2uvw; \\ \text{or} \quad 7u^3+7v^3+w^3 &= -2uvw; \end{aligned}\right\} \quad n=-8$$

has a solution, satisfying $|uv|=1$.

Only the second equation has such a solution $u, v, w=1, 1, 2$, which gives the (solitary) solution $1, 1, 2$ of $x^3+y^3+z^3=5xyz$. The other equation $x^3+y^3+z^3= =-8xyz$ consequently has no solution.

Case 2. Here $n\equiv 0$ (mod9). Write $n=9N$.

$$(48) \rightarrow (9N^2+3N+1)(U+V+W)^3 = 3(3N+2)^3 UVW; \tag{49}$$

To get forward suppose N=0(n=0). (49) turns into $u^3+v^3+9w^3=6uvw$; Then $d^3-27c=-3^3$ and the case M_{32} of Theorem 3 is applicable, and no solution exists.

Case 3. Here $n\equiv 3$ (mod9). Write $n=3(3N+1)$.

$$(48) \rightarrow (3N^2+3N+1)(U+V+W)^3 = 3^3(N+1)^3 UVW; \tag{50}$$

Suppose $3N^2+3N+1=f^\omega$. $\omega\not\equiv 0$ (mod3). f a prime.

$$(50) \rightarrow au^3+bv^3+cw^3 = 3(N+1)uvw; \tag{51}$$

where

$$abc = f^\omega = 3N^2+3N+1$$

and

$$F = d^3-27abc = 3^3(N+1)^3-3^3(3N^2+3N+1) = 3^3N^3.$$

To use Theorem 3 we have to demand: $N\in T$.

If $3\|d\Leftrightarrow N+1\not\equiv 0$ (mod3), then N_3 of Theorem 3 presupposes $H\equiv\pm 1$ (mod9), and this is fulfilled if $N\equiv 1$ (mod3)$\Leftrightarrow n\equiv 12$ (mod27).

If $3^2|d\Leftrightarrow N+1\equiv 0$ (mod3), then we must have $f\not\equiv\pm 1$ (mod9). However $N\equiv -1$ (mod3)$\rightarrow N\equiv 2, 5, 8$ (mod9)$\rightarrow 3N^2+3N+1\equiv 1$ (mod9) and because $\omega\not\equiv 0$ (mod3) this implies $f\equiv\pm 1$ (mod9). The contradiction shows that no cases when $n\equiv 21$ (mod27) can be decided by means of Theorem 3.

$3N^2+3N+1=f^\omega$ has solutions satisfying: f a prime. $\omega\not\equiv 0$ (mod3); for $\omega=1,2$ there are perhaps infinitely many, but probably none if $\omega>3$.

Equation (51), when $\omega=1, 2$ becomes:

$$u^3+v^3+fw^3 = 3(N+1)uvw; \qquad (\omega = 1)$$

$$\left.\begin{aligned} u^3+v^3+f^2w^3 &= 3(N+1)uvw; \\ fu^3+fv^3+w^3 &= 3(N+1)uvw; \end{aligned}\right\} \quad (\omega = 2)$$

Only if $N=0(f=1)$, then these equations have a non-trivial solution satisfying $|uv|=1$, but $N=0\Leftrightarrow n=3$, which is excluded.

Case 4. Here $n\equiv 6$ (mod9). Write $n=3(3N-1)$.

$$(48) \rightarrow (9N^2-3N+1)(U+V+W)^3 = 3(3N+1)^3 UVW;$$

Only $N=0(n=-3)$ gives a decidable case:

$$u^3+v^3+9w^3 = 3uvw;$$

$$F = d^3-27abc = -3^3\cdot 2^3 \rightarrow H = -2^3 \in U.$$

Thus we have the case M_{32} and no solution exists.

Summary of the results for

$$x^3+y^3+z^3 = nxyz; \tag{52}$$

a) If n^2+3n+9 is a prime and $n-3\in T$, then there is no solution except if $n=-1$, when there is one solitary solution.

b) If $n^2+3n+9=f^2$ (f a prime) and $n-3\in T$ (this occurs only when $n=-8,5$), then there is one solitary solution when $n=5$, but no solution when $n=-8$.

c) If $n\equiv 0$ (mod9), then Theorem 3 can be used only when $n=0$ and (52) has no solution, as well known.

d) If $n\equiv 12$ (mod27), write $N=(n-3)/9$. Then if $N\equiv 1$ (mod3), $N\in T$ and $3N^2+3N+1=f$ or $3N^2+3N+1=f^2$ (f a prime), then (52) has no solution.

e) If $n\equiv 6$ (mod9), then Theorem 3 can be used only when $n=-3$ and no solution exists.

*

These results have been marked in Table 2$(-3^4\leqq n\leqq 3^4-1)$. According to Hurwitz [12] and Mordell [10] there exist an infinite number of solutions of (52) if there exists at least one, except if $n=-1, 5$. The n-values, for which there is a solution have mostly been found by trial. All x, y, z satisfying max $(|x|, |y|, |z|)\leqq$ $\leqq 1200$ have been tested to give n-values and eq. (47) with $fgh=1$ has been tested for solutions when $-3^4\leqq n\leqq 3^4-1$ and $a>b>c\geqq 1$ and $|u|\leqq 75$. $|v|\leqq 150$. $|w|\leqq 150$. When $b=c=1$ then the search limits were $|u|\leqq 75$. $|v|\leqq 100$. $|w|\leqq 100$.

Table 2

$$x^3+y^3+z^3 = nxyz$$

Same notation as in Table 1
⊕ : solitary solution

−81	+		+		+	+			+
−72	+			−	+		+		+
−63			−	+	+		+		+
−54			−			+		+	+
−45	+	+		−	−	+		+	+
−36	+	+	+	+		−		+	+
−27	+	−	+	+		+	+	−	−
−18		+	+	−	−	−	+	+	+
− 9	+	−	−	−	−	+	−	−	⊕
n (mod9)	0	1	2	3	4	5	6	7	8
0	−	−	−	+	−	⊕	+	−	−
9	+	+	−	−	+	+	+	+	+
18	+	+	+	+		−		−	+
27		−	+	+	+	−			+
36	+	−	+	−	+	+		−	+
45			+		−	−	+		+
54	+		−	+	−				+
63	+	−		+	+		+		+
72			+		+	+			

It is notable that there exist non-trivial parametric solutions of (52). Besides the simple $n, x, y, z=-A^2, A, 1, -1$ we have $n, x, y, z=-(A^2+A+4)$, $-(A^2+A+1)$, $-(A-1)$, $A+2$; where $(A-1)(A+2)\neq 0$, which can be deduced from (42) in Theorem 4 by putting $m=1$ and $\delta=-1$.

*

If $fgh=2$ then I have found only one decidable case, namely when $n=-3$ in (46).

Solubility of (46) then implies the solubility of $u^3+v^3+2w^3=3uvw$; which equation however has only trivial solutions by Theorem 3, Case N_3. Cf. the example given in connection with the 'Theorem of Derivation'.

5. Miscellaneous results

In part 2 it was proved that

$$a(x^3+y^3+z^3)+b(x^2y+y^2z+z^2x)+c(xy^2+yz^2+zx^2)+dxyz = 0;$$

can be transformed into

$$X^3+Y^3+CZ^3 = DXYZ;$$

where

$$C = \frac{1}{3^3}((6a-d)^3+(3a+d)^3+(3b)^3+(3c)^3-3(3a+d)\cdot 3b\cdot 3c);$$

and $D=6a-d$;

Now the question is: What conditions should be laid down upon a, b, c, d to make Theorem 3 useful?

Clearly such a, b, c, d exist, for example:

a	b	c	d	C	D	Case
1	1	1	1	5	5	N_0
1	-3	-3	4	13	-1	N_0
2	3	3	0	9	6	M_{32}
2	-2	-2	3	2	9	N_3
2	3	0	3	3	3	M_{31}
2	4	2	3	2	3	N_3

(See Ward [14].) (cf. Part 4. Case 2).

From the form of D^3-27C it is seen that $b=c$ is necessary to make $D^3-27C\in T$. Also necessary is then $3a+6b+d\in T$ and $3a-3b\in T$.

Necessary to make $D^3-27C=t\cdot 3^\alpha$, where $t\in T$. $\alpha=3, 4$ is of course $3|d$, but also one of the conditions $b=c$, $3a+d=3b$ or $3a+d=3c$ should be fulfilled. Even if $\alpha>4$, it is sometimes possible to use Theorem 3, namely if $3^3|C$ and $3|D$ when an inclusion of the factor 3 into z can be done.

A minor problem, which has not been solved above, is: Has the equation $x^3+y^3+cz^3=dxyz$ a non-trivial solution when $c=0$? i.e. given d, does x, y, z exist so that:

$$x^3+y^3 = dxyz; \text{ where } (x, y, z)=1 \text{ and } xyz\neq 0?$$

The answer is affirmative and the general solution given by Hans Riesel is:

$$\left.\begin{aligned} x &= ep^2q; \\ y &= epq^2; \\ z &= (p^3+q^3)/f; \end{aligned}\right\} \quad \text{where} \quad d=ef \quad \text{and} \quad (p,q)=1.$$

Here p^3+q^3 should be divisible by f and $(e, (p^3+q^3)/f)=1$. It is always possible to find such p, q, e and f as seen from the example: $p=d-v$, $q=2d+v$, $e=1, f=d$ and $v=\pm 1$ is chosen so that $3\nmid d-v$. QED

6. Summary

In part 2 of this paper the connection between

$$a(x^3+y^3+z^3)+b(x^2y+y^2z+z^2x)+c(xy^2+yz^2+zx^2)+dxyz = 0$$

and

$$AX^3+BY^3+CZ^3 = DXYZ$$

is discussed. A theorem stated by Sylvester and valid when $b=c$ is generalized to the case $b\neq c$.

The equation $ax^3+by^3+cz^3=dxyz$, when $abc=f^\omega$ (f a prime and $\omega\not\equiv 0 \pmod 3$) is discussed in part 3. A lemma used for reduction of 'the degree' ω is given, as well as a theorem for simultaneous solutions of all equations belonging to the set with a given $abc=f^\omega$. Necessary conditions for the method given here are:

I) $d^3-3^3abc\not\equiv 0 \pmod{3^5}$

II) d^3-3^3abc contains no rational prime factor $\equiv 1 \pmod 3$.

A classification of $ax^3+by^3+cz^3=dxyz$ into 16 classes is made, out of which 12 satisfy the first condition. The conditions are sharpened in the different classes and the problem to decide if there any integer solutions (apart from those with $xyz=0$) is reduced to a simple test. The criteria prove to be powerful (within the restricitons) and it was relatively easy to find solutions in most of the remaining cases when $|d|\leqq 27$.

Some types of equations, where progress can be made by means of congruences are given. However, the conditions imposed upon the coefficients make the main theorem non-applicable in these cases.

Another theorem in part 3 gives the most general forms for c and d (in x^3+y^3+ $+cz^3=dxyz$) in five parameters. These forms can be used to give specific and parametric solutions, as shown.

In part 4, $x^3+y^3+z^3=nxyz$ is transformed so that the earlier results of this paper become applicable. Simple criteria for non-existence of solutions are given when $3\nmid n$ and $n\equiv 12$ (mod27). It is also shown that some quadratic n-forms always permit non-trivial solutions of $x^3+y^3+z^3=nxyz$.

Finally in part 5 some necessary conditions on a, b, c, d of

$$a(x^3+y^3+z^3)+b(x^2y+y^2z+z^2x)+c(xy^2+yz^2+zx^2)+dxyz = 0$$

are given, so as to be able to use the results in part 3.

*

Table 3

Solutions of $x^3+y^3+z^3 = nxyz$

$-3^4 \leqq n \leqq 3^4-1$

n	x, y, z
3	(1, 1, 1) or $x+y+z = 0$
5	(1, 1, 2) (solitary)
6	(1, 2, 3)
9	(2, 3, 7)
10	(5, 7, 18)
13	(9, 13, 38)
14	(2, 7, 13)
15	(−7, 1, 3)
16	(−70, 9, 31)
17	(5, 18, 37)
18	(13, 42, 95)
19	(1, 5, 9)
20	(−61, 13, 14)
21	(2, 13, 21)
26	(9, 38, 91)
29	(27, 43, 182)
30	(2, 21, 31)
31	(−37, 1, 27)
35	(−97, 14, 19)
36	(−151, 7, 78)
38	(70, 151, 629)
40	(−9, 1, 2)
41	(1, 2, 9), (1, 5, 14)
44	(−819, 19, 554)
47	(−845, 38, 367)
51	(9, 13, 77)
53	(2, 7, 27)
54	(2, 43, 57)
57	(19, 91, 310)
62	(−13559153, 1513300, 1950953)
63	(−3775, 247, 903)
66	(1, 3, 14)
67	(1133, 7525, 23517)
69	(2, 57, 73), (42, 95, 523)
71	(−67, 7, 9)
74	(133, 2502, 4607)
76	(−45, 2, 13)
77	(−52, 5, 7)

$-n$	x, y, z
1	(−1, 1, 1) (solitary)
4	(−1, 1, 2)
9	(−1, 1, 3)
10	(−4, 1, 7)
11	(−4, 9, 19)
12	(−3, 14, 19)
16	(−1, 1, 4), (−5, 2, 13)
17	(−1, 7, 9)
21	(−37, 7, 78)
22	(−1, 4, 9)
24	(−2, 1, 7)
25	(−1, 1, 5), (−1, 2, 7)
27	(−28, 109, 279)
28	(−325, 362, 1813)
29	(−9, 74, 127)
33	(−3, 13, 35)
34	(−7, 4, 31)
35	(−1333, 14220, 23233)
36	(−1, 1, 6)
37	(−52, 19, 193)
38	(−1581475, 28251, 1934524)
40	(−217, 2692, 4345)
44	(−19, 67, 234)
45	(−52, 21, 223)
46	(−8, 5, 43)
47	(−9, 196, 221)
49	(−1, 1, 7), (−1, 7, 18)
55	(−7, 76, 163)
57	(−1, 3, 13)
59	(−6244, 817, 17739)
60	(−3, 2, 19), (−5, 117, 158)
64	(−1, 1, 8)
66	(−127, 3423, 4432)
68	(−35, 914, 1251)
72	(−9, 1, 26), (−19, 6, 91)
73	(−715, 13483, 24577)
76	(−10, 7, 73)
77	(−2394, 853, 12581)
79	(−823, 43, 1764)
81	(−1, 1, 9)

Note. For some n-values, there are two solutions given, as in these cases it has not been possible to generate all found solutions with one basic solution.

Table 4
Solutions of $ax^3+by^3+cz^3 = dxyz$

$A, B, C = 2, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
−27	14	1	−11
−26	13	−1	−23
−20	2	11	−3
−18	3	1	−1
−15	6	−11	−31
−14	2	5	−1
−13	18	11	−5
−11	2	−1	−5
−10	1	−3	−5
−8	2	1	−1
−7	2	3	−1
−2	1	1	−1
0	1	−1	−1
4	1	1	1
7	2	−1	−1
8	1	1	−3
9	2	1	1
10	1	3	1
12	3	5	1
13	4	3	1
16	7	−1	−5
17	38	75	11
27	4	23	5

$A, B, C = 3, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
−27	3	−40	−41
−24	3	−4	−17
−23	9	2	−5
−22	27	−1	−44
−21	49	17	−20
−18	3	−1	−8
−16	18	29	−5
−13	1	−2	−5
−12	1	5	−2
−5	1	2	−1
−3	1	−1	−2
1	1	−1	−1
2	1	1	−2
5	1	1	1
6	1	2	1
10	2	1	−5
11	2	−1	−1
12	3	−1	−2
13	2	1	1
15	1	1	−4
16	1	8	5
17	1	4	1
22	5	4	1
23	1	10	−17
25	7	17	2

$A, B, C = 4, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
−25	5	2	−2
−24	21	71	−11
−22	1	11	−5
−19	4	5	−1
−16	2	1	−1
−15	3	−2	−10
−14	7	5	−3
−10	1	3	−1
−9	3	2	−2
−8	1	−1	−3
−4	1	1	−1
−3	1	−2	−2
−1	1	2	−2
2	1	−1	−1
5	1	2	2
6	1	1	1
10	2	3	1
13	1	6	10
15	2	−1	−1
16	9	−1	11
17	1	2	−6
19	1	6	2
22	7	47	15
23	19	−6	−10
24	1	1	−5
26	1	5	1

$A, B, C = 2, 2, 1$

$D=$	$X=$	$Y=$	$Z=$
−25	1	−1	−5
−24	11	−71	−42
−22	5	−11	−2
−19	1	−5	−8
−16	1	−1	−4
−15	1	5	−3
−14	3	−5	−14
−10	1	−3	−2
−9	1	−1	−3
−8	1	3	−2
−4	1	−1	−2
−3	1	1	−1
−1	1	−3	4
2	1	1	−2
5	1	1	1
6	1	1	2
10	1	3	4
13	3	−5	1
15	1	1	−4
16	1	11	−18
17	1	−3	1
19	1	3	−8
22	15	47	14
23	3	5	−19
24	1	−5	2
26	1	5	2

$A, B, C = 5, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
−26	1	−8	−13
−23	4	−1	−11
−20	2	1	−1
−19	19	−6	−53
−18	21	2	−29
−17	1	4	−1
−16	7	−26	−51
−15	1	−1	−4
−11	2	3	−1
−8	1	2	−1

$A, B, C = 5, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
-5	1	-2	-3
-4	1	3	-2
-2	1	-1	-2
1	1	1	-2
2	2	-1	-3
3	1	-1	-1
7	1	1	-3
8	1	4	3
11	1	3	1
17	9	13	-46
18	3	2	-11
19	2	-1	-1
20	1	3	-8
21	1	11	8
22	5	3	2
24	3	2	1
25	9	13	2
27	3	8	1

$A, B, C = 6, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
-26	1	5	-1
-24	1	-1	-5
-13	4	1	-5
-6	1	1	-1
-4	13	-23	-35
-2	19	-7	-37
0	21	-17	-37
4	1	-1	-1
6	3	7	5
8	1	1	1
19	2	7	-17
21	2	1	-7
23	2	-1	-1
25	2	1	1
26	4	7	1

$A, B, C = 3, 2, 1$

$D=$	$X=$	$Y=$	$Z=$
-23	31	-4	-65
-22	3	5	-1
-16	3	2	-1
-11	3	-2	1
-9	1	2	-1
-7	1	-2	-1
-4	1	1	-1
-2	1	-1	1
0	1	-4	5
6	1	-2	1
10	1	2	1
17	11	-4	-5
24	1	1	-5
26	1	1	5

$A, B, C = 7, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
-26	31	3	-46
-22	5	1	-6
-21	3	4	-1
-20	7	-33	-64
-18	2	5	-1
-17	1	11	-6
-14	13	-19	-60
-11	1	3	-1
-7	1	-3	-4
-5	2	1	-3
-1	1	-1	-2
0	1	1	-2
2	1	2	-3
5	1	-1	-1
7	1	3	2
8	1	2	1
9	1	1	1
10	4	5	3
11	1	2	-5
12	3	11	4
14	1	1	-4
16	27	80	19
18	1	4	1
20	7	-1	-10
21	3	5	1
27	2	-1	-1

$A, B, C = 8, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
-25	2	27	-13
-24	3	-28	-38
-22	2	-9	-19
-20	1	14	-8
-18	1	-2	-6
-15	1	-6	-8
-8	1	-2	-4
-5	13	56	-38
-2	1	-2	-2
1	13	56	-62
6	1	-1	-1
7	4	7	5
10	1	1	1
12	1	3	1
18	1	7	3
19	1	10	8
20	5	36	14
26	9	76	26
27	2	3	-13

$A, B, C = 4, 2, 1$

$D=$	$X=$	$Y=$	$Z=$
-19	1	-2	-6
-17	1	-3	-1
-5	1	-3	5
-3	1	-1	-2
1	1	-5	6
7	1	-3	2
11	1	1	3

$A, B, C = 9, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
-27	7	-83	-100
-26	7	6	-3
-25	5	-61	-64
-23	7	3	-6
-22	7	-15	-48
-19	7	12	-3
-18	1	4	-1
-16	4	3	-3
-15	3	1	-4
-14	1	-1	-4
-13	1	-3	-6
-12	3	5	-2
-11	1	6	-3
-9	1	-4	-5
-8	1	2	-1
-5	1	-3	-3
-4	2	3	-3
-1	1	1	-2

$A, B, C = 9, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
0	1	−1	−2
1	2	−3	−3
7	1	−1	−1
9	1	2	1
10	1	3	−6
11	1	1	1
14	1	6	3
16	5	12	3
18	7	22	5
23	1	1	−5
27	1	5	1

$A, B, C = 3, 3, 1$

$D=$	$X=$	$Y=$	$Z=$
−26	1	−2	−7
−25	1	−1	−5
−23	1	−2	7
−22	5	16	−7
−19	1	−4	−7
−18	1	−4	−3
−16	1	−1	−4
−15	1	−4	9
−14	1	4	−3
−13	1	2	−1
−12	2	−5	−9
−11	1	−2	−1
−9	1	−1	−3
−8	1	−2	−3
−5	1	1	−1
−4	1	−1	−2
−1	1	−2	3
0	1	2	−3
1	1	1	−2
7	1	1	−3
9	1	−16	21
10	1	−2	1
11	1	1	3
14	1	2	1
16	1	4	5
18	5	22	21
23	1	−5	3
27	1	5	3

$A, B, C = 10, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
−27	6	−1	−17
−12	1	3	−1
−10	1	1	−1
−9	2	1	−3
−6	1	−1	−3
−4	37	11	−73
2	43	−37	−79
8	1	−1	−1
12	1	1	1
18	2	3	1

$A, B, C = 5, 2, 1$

$D=$	$X=$	$Y=$	$Z=$
−25	3	−16	−7
−20	3	−2	1
−14	1	−3	7
−11	3	−2	7
−10	1	−13	19
−8	1	−1	−3
−6	1	−17	23
−4	1	−1	1
1	1	2	−3
2	1	−6	7
5	1	−2	1
8	1	−4	3
11	1	2	1
13	5	4	3
16	1	−3	1
17	1	4	7
19	3	−26	19
20	1	3	1
22	3	4	1
23	1	2	−7
26	1	2	7

$A, B, C = 11, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
−27	1	5	−1
−26	7	−5	−32
−25	43	96	−17
−23	1	−1	−5
−22	1	−3	−8
−19	2	3	−1
−13	14	−25	−69
−11	1	−5	−6
−10	5	−6	−19
−9	1	2	−1
−6	3	2	−5
−5	1	3	−2
−4	1	−2	−3
−2	1	1	−2
1	1	−1	−2
5	1	1	−3
8	2	5	3
9	1	−1	−1
10	1	2	1
13	1	1	−4
15	3	23	16
17	9	16	5
19	1	4	1
26	13	37	6

$A, B, C = 12, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
−27	3	2	−2
−22	2	5	−1
−12	1	1	−1
−6	21	−59	−79
−5	49	26	−94
−3	1	2	−2
−2	37	−1	−85
−1	1	−2	−2
0	39	−19	−89
7	1	2	2
8	7	25	19
9	21	46	26
10	1	−1	−1
14	1	1	1

$A, B, C = 6, 2, 1$

$D=$	$X=$	$Y=$	$Z=$
−26	3	−17	−32
−25	3	−7	−1
−24	1	−7	−10
−23	1	−7	−5

$A, B, C = 6, 2, 1$

$D=$	$X=$	$Y=$	$Z=$
−20	3	7	−2
−19	3	−1	5
−17	1	−1	4
−15	1	−1	−4
−7	1	1	−1
−6	1	−1	2
−5	1	−1	1
−2	1	−1	−2
0	1	1	−2
3	1	−7	8
8	1	1	2
9	1	−5	4
14	1	1	−4
16	1	5	4
18	1	1	4
19	3	1	4
22	1	−7	4
27	1	7	4

$A, B, C = 4, 3, 1$

$D=$	$X=$	$Y=$	$Z=$
−26	1	−3	−1
−25	1	−6	14
−24	5	−1	7
−23	5	−2	14
−20	1	−3	−7
−19	5	6	−2
−17	2	1	−1
−15	2	−1	1
−7	1	−2	−2
−6	1	1	−1
−5	1	2	−2
−2	1	−1	1
0	1	−1	−1
3	1	−2	2
8	1	1	1
9	1	2	2
14	1	3	5
16	2	1	5
18	2	1	1
19	2	3	1
22	2	1	−7
27	2	1	7

$A, B, C = 3, 2, 2$

$D=$	$X=$	$Y=$	$Z=$
−27	3	−1	1
−22	4	−1	5
−12	2	−1	1
−7	37	−55	−13
−6	42	−79	−59
−5	49	−47	13
−3	1	−1	1
−1	1	−1	−1
7	1	1	1
8	14	19	25
9	21	13	23
10	2	−1	−1
14	2	1	1

$A, B, C = 13, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
−24	3	1	−4
−19	1	−7	−10
−18	6	−7	−29
−17	1	−3	−7
−16	5	−1	−14
−14	9	7	−10
−13	1	−6	−7
−12	1	−2	−5
−10	1	2	−1
−6	3	−10	−11
−5	1	−1	−3
−3	1	1	−2
0	3	−2	−7
1	1	2	−3
2	1	−1	−2
4	13	−1	−30
8	1	3	2
9	3	5	4
11	1	−1	−1
13	7	11	6
15	1	1	1
16	19	60	17
17	2	1	−7
19	5	7	3
22	1	15	−22
23	1	2	−7
24	4	25	7
26	1	7	2

$A, B, C = 14, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
−23	2	3	−1
−17	2	−5	−13
−14	1	1	−1
−7	2	5	−3
−6	7	5	−13
4	1	1	−3
10	31	67	37
12	1	−1	−1
14	1	3	1
16	1	1	1
20	4	13	3
22	1	1	−5
25	4	1	−13

$A, B, C = 7, 2, 1$

$D=$	$X=$	$Y=$	$Z=$
−26	1	−1	5
−24	1	−1	−5
−20	1	3	−1
−18	1	−5	−3
−16	1	−3	−1
−11	1	2	−1
−9	1	4	−3
−8	1	1	−1
−6	1	−2	−3
−5	1	−2	−1
−3	1	−2	3
4	1	−8	9
5	3	−4	−1
6	1	1	−3
10	1	−6	5
11	3	2	5
12	1	1	3
13	1	4	5
18	5	2	9
24	1	4	9

$A, B, C = 15, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
−25	1	7	−2
−24	1	−2	−7
−22	5	2	−7
−16	2	−1	−7

$A, B, C = 15, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
−15	1	−7	−8
−14	1	−4	−7
−11	1	2	−1
−7	7	−17	−29
−4	1	1	−2
−3	3	1	−7
1	13	−7	−31
3	1	−1	−2
9	3	11	7
12	1	1	−4
13	1	−1	−1
17	1	1	1
18	3	7	2
20	1	4	1
22	4	−1	−7
26	9	17	4

$A, B, C = 5, 3, 1$

$D=$	$X=$	$Y=$	$Z=$
−26	1	−4	11
−21	2	1	−1
−20	2	3	−1
−19	2	−1	1
−14	1	−3	−2
−13	1	3	−2
−10	1	−2	−1
−7	1	1	−1
−5	1	−1	2
−3	1	−1	−2
0	1	1	−2
1	1	−3	4
8	1	1	2
9	1	−2	1
14	1	1	−4
15	1	2	1
18	1	1	4
22	2	1	1
25	1	−3	1

$A, B, C = 16, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
−26	9	11	−5
−25	5	4	−4
−23	13	12	−10
−22	1	−1	−5
−20	1	−6	−10
−17	2	1	−3
−16	1	1	−1
−14	1	3	−1
−11	1	−4	−6
−9	1	4	−2
−7	1	−4	−4
−4	1	−1	−3
−1	1	4	−4
0	1	−2	−2
5	1	2	−4
8	1	2	2
9	1	4	4
11	1	4	2
14	1	−1	−1
15	3	−2	−4
16	1	2	−6
18	1	1	1
19	5	6	4
20	1	6	2
21	3	4	2
23	1	4	−10
24	3	10	2
25	7	44	12
26	2	3	1
27	1	10	4

$A, B, C = 8, 2, 1$

$D=$	$X=$	$Y=$	$Z=$
−26	5	−36	−22
−25	1	−5	−2
−23	3	13	−5
−22	1	−4	10
−20	3	−2	10
−17	1	8	−6
−16	1	−4	−2
−14	1	−4	−6
−11	1	−1	3
−9	1	−14	20
−7	1	−16	22
−4	1	−4	6
−1	1	−49	62
0	1	−2	2
5	1	−9	10
8	1	7	2
9	1	1	2
11	1	−7	6
14	1	−4	2
15	1	−6	4
16	1	2	−6
18	1	4	2
19	1	5	3
20	1	2	6
21	1	3	1
23	1	1	−5
24	1	6	10
25	3	7	22
26	1	8	6
27	1	1	5

$A, B, C = 4, 4, 1$

$D=$	$X=$	$Y=$	$Z=$
−26	5	−11	−36
−25	1	−1	−5
−23	5	−6	−26
−22	1	5	−4
−20	3	5	−2
−17	1	−3	8
−16	1	−1	−4
−14	1	−3	−4
−11	2	3	−2
−9	1	−2	−2
−7	1	1	−1
−4	1	−1	−2
−1	1	−1	−1
0	1	1	−2
5	1	−2	2
8	1	1	2
9	1	1	1
11	1	2	2
14	1	1	−4
15	1	2	−6
16	1	−3	2
18	1	1	4
19	2	3	10
20	1	3	2
21	1	2	6

$A, B, C = 4, 4, 1$

$D=$	$X=$	$Y=$	$Z=$
23	2	−5	2
24	1	5	6
25	3	11	7
26	1	3	8
27	2	5	2

$A, B, C = 17, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
−27	1	12	−5
−26	2	5	−1
−25	7	−11	−45
−24	1	15	−8
−20	1	4	−1
−18	37	30	−41
−17	1	−8	−9
−15	7	−53	−54
−12	1	−1	−4
−10	4	1	−9
−9	1	−3	−5
−6	1	3	−2
−5	1	1	−2
−4	9	23	−20
−3	1	−2	−3
0	7	1	−18
1	2	−1	−5
3	1	1	−3
4	1	−1	−2
9	1	4	3
10	1	2	−5
11	2	9	5
12	7	10	9
13	1	2	1
14	14	29	−81
15	1	−1	−1
16	5	11	4
18	1	4	−9
19	1	1	1
24	7	73	36

$A, B, C = 18, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
−25	4	15	−3
−18	1	1	−1
−8	2	3	−3
−4	1	−3	−3
−2	1	3	−3
5	2	−3	−3
8	1	3	3
10	19	69	39
11	2	3	3
16	1	−1	−1
20	1	1	1
24	3	−1	−5
27	2	5	1

$A, B, C = 9, 2, 1$

$D=$	$X=$	$Y=$	$Z=$
—26	5	−3	−21
—25	1	−6	− 3
—24	1	−2	7
—23	1	6	−3
—17	5	6	−3
—12	1	2	−1
—10	1	1	−1
—8	1	−15	21
—4	1	−2	−1
2	1	−3	3
3	1	−2	1
6	1	−10	11
10	1	2	−5
12	1	−8	7
13	1	2	1
15	1	2	5
22	1	−6	3
23	1	12	−21
26	1	6	3

$A, B, C = 6, 3, 1$

$D=$	$X=$	$Y=$	$Z=$
−26	1	7	−5
−25	2	1	−1
−24	2	−7	−3
−23	2	−1	1
−17	2	−1	5
−12	2	−1	3
−10	1	−1	3
−8	1	−5	9
−4	1	−7	11
2	1	−1	−1
3	2	−1	−3
6	1	1	−3
10	1	1	1
12	1	1	3
13	2	1	3
15	2	5	3
22	2	−1	−1
23	4	−7	1
26	2	1	1

$A, B, C = 3, 3, 2$

$D=$	$X=$	$Y=$	$Z=$
−25	1	−5	−4
−18	1	−1	−3
−8	1	−1	−2
−4	1	1	−1
−2	1	−5	6
2	13	55	−67
5	1	1	−2
8	1	1	1
10	13	23	19
11	1	1	2
16	1	1	−3
20	1	1	3
24	1	5	−9
27	1	5	6

$A, B, C = 19, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
−27	19	9	−25
−21	1	8	−3
−19	1	−9	−10
−18	19	−24	−97
−15	1	3	−1
−13	1	2	−1
−12	31	75	−34
−9	7	−3	−22
−6	1	−3	−4
−4	1	5	−4
−3	1	−1	−3
0	1	2	−3

$A, B, C = 19, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
5	1	−1	−2
6	13	−10	−27
9	1	3	2
10	5	16	9
11	1	1	−4
12	1	6	5
13	1	5	−9
14	1	2	1
15	13	29	12
16	7	50	27
17	1	−1	−1
19	13	18	11
21	1	1	−5
23	5	34	11
24	49	69	32
26	7	2	−25

$A, B, C = 20, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
−24	3	−1	−11
−20	1	1	−1
−19	1	6	−2
−17	1	−2	−6
−16	1	7	−3
−9	13	14	−22
−5	1	2	−2
0	7	−1	−19
1	1	−2	−2
2	1	1	−3
9	1	2	2
10	13	61	49
13	2	1	−7
16	1	3	1
18	1	−1	−1
22	1	1	1
23	2	11	3

$A, B, C = 10, 2, 1$

$D=$	$X=$	$Y=$	$Z=$
−21	1	3	−1
−18	1	−1	4
−15	1	−3	−1
−14	1	−1	−4
−11	1	1	−1
−9	1	−3	−4
−8	1	−1	2
2	1	1	−2
0	1	−1	−2
5	1	1	−3
6	1	−3	2
7	1	−11	12
10	1	1	2
12	1	3	2
13	1	−9	8
17	3	−1	−4
19	1	1	4
24	1	3	8
26	3	17	8

$A, B, C = 5, 4, 1$

$D=$	$X=$	$Y=$	$Z=$
−21	2	−1	6
−18	1	2	−1
−15	2	−1	−6
−14	1	−2	−1
−11	2	−1	2
−9	1	−2	−3
−8	1	1	−1
−2	1	−1	1
0	1	−2	3
5	2	−3	2
6	1	1	−3
7	1	6	−11
10	1	1	1
12	1	1	3
13	1	−2	1
17	3	−2	−1
19	1	2	1
24	1	4	3
26	3	4	17

$A, B, C = 5, 2, 2$

$D=$	$X=$	$Y=$	$Z=$
−24	6	−11	−1
−20	2	−1	1
−19	1	−1	3
−17	1	−3	−1
−16	2	−3	7
−9	13	−11	7
−5	1	−1	1
0	14	−19	−1
1	1	−1	−1
2	2	−3	1
9	1	1	1
10	26	49	61
13	4	−7	1
16	2	1	3
18	2	−1	−1
22	2	1	1
23	4	3	11

$A, B, C = 21, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
−21	1	−10	−11
−15	3	2	−5
−14	1	2	−1
−11	1	−1	−4
−9	3	−5	−13
−7	1	1	−2
−5	19	17	−43
−2	14	−11	−41
2	1	4	−5
3	21	−13	−53
6	1	−1	−2
11	7	25	13
15	1	2	1
16	1	8	−13
19	1	−1	−1
23	1	1	1
26	7	13	4

$A, B, C = 7, 3, 1$

$D=$	$X=$	$Y=$	$Z=$
−27	2	−1	1
−26	5	−29	−22
−25	1	−3	−1
−22	1	−5	−8
−17	1	−1	4
−15	1	−1	−4
−9	1	−2	−1
−6	1	−1	2
−5	1	−1	1
−2	1	−1	−2
−1	1	1	−2
3	1	−4	5

$A, B, C = 7, 3, 1$

$D=$	$X=$	$Y=$	$Z=$
4	2	−3	1
8	1	−2	1
9	1	1	2
11	1	−3	2
16	1	2	1
17	5	3	4
20	4	1	−11
22	17	−5	−16
23	1	1	−5
25	1	3	8
26	1	6	5
27	1	1	5

$A, B, C = 22, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
−25	2	1	−3
−22	1	1	−1
−16	1	3	−1
−13	4	3	−7
−12	2	−1	−7
−8	1	5	−3
−2	1	−1	−3
5	2	−1	−5
13	2	7	3
14	1	3	−7
20	1	−1	−1
22	14	3	−47
24	1	1	1
26	4	5	3

$A, B, C = 11, 2, 1$

$D=$	$X=$	$Y=$	$Z=$
−24	11	43	−15
−19	5	−6	23
−17	1	−4	9
−13	1	−2	−5
−12	1	−4	−3
−10	1	−1	1
−6	1	−1	−3
−5	3	4	−5
−3	1	−2	−1
0	1	2	−3
1	5	6	−13
2	1	−2	1
4	1	3	−5
7	5	−2	−9
8	1	−12	13
9	1	2	3
14	1	−10	9
22	1	3	1

$A, B, C = 23, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
−23	1	−11	−12
−20	13	20	−11
−17	4	−3	−17
−15	1	2	−1
−14	1	5	−2
−11	1	−2	−5
−8	1	1	−2
−7	1	3	−2
−6	2	1	−5
−5	1	4	−3
−2	1	−2	−3
1	1	1	−3
2	9	16	−31
5	1	3	−5
7	1	−1	−2
8	14	−3	−37
10	1	1	−4
14	9	40	17
16	1	2	1
17	1	3	1
20	14	17	15
21	1	−1	−1
22	1	4	1
25	1	1	1
26	2	−1	−3

$A, B, C = 24, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
−26	1	−4	−10
−24	1	1	−1
−10	1	4	−2
−6	1	−2	−4
−3	3	2	−8
2	1	−2	−2
4	1	2	−4
9	3	10	8
10	1	2	2
11	19	94	64
12	1	4	2
20	1	1	−5
22	1	−1	−1
24	3	−2	−4
26	1	1	1

$A, B, C = 12, 2, 1$

$D=$	$X=$	$Y=$	$Z=$
−27	1	−1	5
−23	1	−1	−5
−13	1	1	−1
−11	1	−1	1
−9	1	−1	2
−5	1	2	−2
−3	1	−5	7
−1	1	−2	2
1	1	−1	−2
9	1	−13	14
11	1	1	2
15	1	−11	10
23	1	−5	2
25	3	−83	85
27	1	5	2

$A, B, C = 8, 3, 1$

$D=$	$X=$	$Y=$	$Z=$
−26	1	−1	5
−24	1	−6	−8
−10	1	−2	−4
−6	1	−6	10
−5	22	−31	59
−3	1	6	−8
−1	20	−43	61
2	1	−2	2
4	1	−1	−1
9	2	3	5
10	1	2	2
11	16	19	47
12	1	1	1
20	1	4	−10
22	1	−4	2
24	1	−6	4
26	1	4	2

$A, B, C = 6, 4, 1$

$D=$	$X=$	$Y=$	$Z=$
−27	2	5	−2
−23	2	−5	−2
−13	2	−1	2
−11	2	1	−2
−9	1	1	−1
−5	1	−1	2
−3	1	−1	−2
−1	1	1	−2
1	1	−1	−1
9	1	1	2
11	1	1	1
15	1	5	−11
23	1	1	−5
25	3	−1	−2
27	1	1	5

$A, B, C = 4, 3, 2$

$D=$	$X=$	$Y=$	$Z=$
−27	1	−2	−5
−23	1	−2	5
−13	1	2	−1
−11	1	−2	−1
−9	1	−5	7
−5	1	1	−1
−3	1	−1	1
−1	1	−6	7
1	1	−2	2
9	1	−2	1
11	1	2	2
15	1	2	1
23	5	−2	−2
25	1	−3	1
27	5	2	2

$A, B, C = 25, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
−27	7	10	−5
−25	1	−12	−13
−23	2	15	−5
−22	1	−5	−10
−18	1	10	−5
−17	1	3	−1
−16	1	2	−1
−14	7	−15	−40
−10	1	−1	−4
−9	1	−5	−5
−4	2	5	−5
−3	3	−5	−10
−1	1	−1	−3
1	1	3	−4
4	7	−10	−15
8	1	−1	−2
9	2	5	5
10	1	3	2
11	1	5	5
12	3	10	5
17	1	2	1
22	18	5	−65
23	1	−1	−1
24	3	5	−20
27	1	1	1

$A, B, C = 5, 5, 1$

$D=$	$X=$	$Y=$	$Z=$
−27	1	−2	−7
−25	1	−1	−5
−23	1	−3	−2
−22	1	−4	−5
−18	1	−2	−1
−17	1	−39	70
−16	1	−2	−5
−14	3	8	−7
−10	1	4	−5
−9	1	−2	5
−4	1	−1	−2
−3	1	2	−3
−1	1	−11	19
1	3	−4	5
4	2	3	−7
8	1	2	−5
9	1	1	2
10	2	3	5
11	1	1	1
12	1	2	3
17	1	−2	1
22	1	−13	18
23	1	1	−5
24	1	−4	3
27	1	1	5

$A, B, C = 26, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
−26	1	−5	−11
−20	1	−1	−5
−15	4	−15	−29
−14	1	9	−5
−13	14	45	−19
−12	7	9	−11
−1	4	−9	−11
0	1	1	−3
12	2	5	3
14	9	−5	−19
18	1	3	1
24	1	−1	−1

$A, B, C = 13, 2, 1$

$D=$	$X=$	$Y=$	$Z=$
−22	1	3	−1
−16	3	−7	−1
−14	1	−3	−1
−12	1	−4	−5
−5	1	−2	−3
−4	1	−2	3
−2	1	−2	−1
1	1	−2	1
2	3	−7	5
4	1	1	−3
7	3	−2	−5
10	1	−14	15
14	1	−5	3
15	1	2	1
16	1	−12	11
19	1	6	5
20	3	11	23
22	1	1	−5

$A, B, C = 27, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
−27	1	−13	−14
−17	1	−6	−9
−16	9	77	−38
−12	1	−3	−6
−11	7	81	−57
−10	1	1	−2
−7	1	7	−5

$A, B, C = 27, 1, 1$

$D=$	$X=$	$Y=$	$Z=$
-3	1	-3	-3
5	7	81	-93
9	1	-1	-2
10	27	91	62
13	5	13	7
15	1	6	3
16	1	5	2
18	1	2	1
22	1	2	-7
23	1	4	1
25	1	-1	-1
27	1	7	2

$A, B, C = 9, 3, 1$

$D=$	$X=$	$Y=$	$Z=$
-16	1	-3	-6
-14	1	-2	-5
-11	1	-3	-3
-8	1	-3	6
-7	1	-2	-3
-2	1	-2	3
-1	1	-1	-2
5	1	-5	6
7	1	-2	1
10	1	-12	15
13	1	-4	3
17	1	2	1
19	1	1	4
20	1	5	6
26	2	15	-33

References

1. Sylvester, J. J. Phil. Mag. XXXI (1847), pp. 467—471 or *Collected Works* (1909), vol 1, p. 118. (Note the misprints!)
2. Selmer, E. S. The exceptional points of a cubic curve which is symmetric in the homogeneous variables. *Math. Scand.* **2** (1954), pp. 227—236.
3. Carmichael, R. D. Analyse Indeterminée. Les Presses Universitaires, Paris (1929), pp. 65—69. Also: *Diophantine Analysis. Mathematical Monographs* NO 13. N.Y. (1913).
4. Carmichael, R. D. On the representation of numbers in the form $x^3+y^3+z^3-3xyz$. *Bull. Amer. Math. Soc.* **22** (1915), pp. 111—117.
5. Mordell, L. J. *Diophantine Equations. Academic Press. London and New York* 1969.

6, 7, 8, 9. Sylvester, J. J. *Collected Works* (1909). Vol 1, pp. 106—109, 110—113, 114—118. Vol 2, pp. 63—64.

10. Mordell, L. J. *The Diophantine equation* $x^3+y^3+z^3+kxyz=0$. *Colloque sur la Théorie des Nombres. Brussels, December,* 1955.
11. Selmer, E. S. The diophantine equation $ax^3+by^3+cz^3=0$. *Acta Math.* **85** (1951), pp. 203—362.
12. Hurwitz, A. Über ternäre diophantische Gleichungen dritten Grades. *Vierteljahrsschrift Naturf. Ges. Zürich.* **62** (1917), pp. 207—229. Also *Math. Werke* (Birkhäuser Cie, Basel) Vol 2, (1933), pp. 446—468.
13. Desboves, A. Résolution en nombres entiers et sous sa forme la plus générale, de l'équation cubique, homogene, à trois inconnues. *Nouv. Ann. de Math. Ser* III, 5, (1886), pp. 545—579.
14. Ward, M. The vanishing of the homogeneous product sum of the roots of a cubic. *Duke Math. J.* **26** (1959), pp. 553—562.
15. Cassels, J. W. S. On a diophantine equation. *Acta Arithm.* **6** (1960—61), pp. 47—52.
16. Sansone, G. and Cassels, J. W. S. Sur le problème de M Werner Mnich. *Acta Arithm.* **7** (1962), pp. 187—190.
17. Thue, A. Über die unlösbarkeit der Gleichung $ax^2+bx+c=dy^n$ in grossen Zahlen x und y. *Arch. Math. Naturv. Kristiania* Nr 16, **34** (1917).

Received October 29, 1973

Erik Dofs
38 Rue de Longchamp
92200 Neuilly
France