On spaces of Triebel—Lizorkin type

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0. Introduction

In this note we study certain spaces of distributions $F_p^{sq}=F_p^{sq}(\mathbf{R}^n)$ where s real, $0<p,\ q\leqq\infty$. They are intimately related to certain spaces studied by Triebel [10] and Lizorkin [5] (cf. also [6]) when $1<p,\ q\leqq\infty$. Our main result is a certain equivalence theorem (see Sec. 3) which says that the spaces do not depend on the special sequence of testfunctions $\{\varphi_\nu\}_{\nu\in\mathbf{Z}}$ entering in their definition. This extends Triebel's corresponding result. But we have to give an entirely new proof, relying on two deep results by Fefferman & Stein: 1° their real variable characterization of the Hardy classes H_p [1], 2° their sequence valued version of the Hardy & Littlewood maximal theorem [2]. (Incidentally it follows from [1] that $F_p^{02}=H_p$ if $0<p<\infty$ while as $F_\infty^{02}=$ B. M. O.!) As an application we prove (see Sec. 5) a multiplier theorem of the Mikhlin type, extending the one by Triebel and Lizorkin. We also give (see Sec. 6) an application to approximation theory related to a theorem of Freud's [3]. Finally we briefly indicate (see Sec. 7) how the result might be extended to the case of a Riemannian manifold.

1. Definitions

By L_p where $0<p\leqq\infty$ we denote the space of measurable functions $f=f(x)$ $(x\in\mathbf{R}^n)$ such that

$$\|f\|_{L_p}=\left(\int|f(x)|^p\,dx\right)^{1/p}<\infty.$$

By l^q where $0<q\leqq\infty$ we denote the space of sequences $\mathbf{t}=\{t_\nu\}_{\nu\in\mathbf{Z}}$ such that

$$\|\mathbf{t}\|_{l^q}=\left(\sum\nolimits_{\nu\in\mathbf{Z}}|t_\nu|^q\right)^{1/q}<\infty.$$

We consider also spaces of sequence valued measurable functions $L_p(l^q)$ and $l^q(L_p)$, defined in the obvious way. If $1\leqq p,\ q\leqq\infty$ these are all Banach spaces, in the general case only quasi-Banach space.

By $\mathscr{S}$ we denote the space of rapidly decreasing functions in $\mathbf{R}^n$ and by $\mathscr{S}'$ the dual space of tempered distributions.

We choose a sequence of testfunctions $\{\varphi_\nu\}_{\nu\in\mathbf{Z}}$, with $\varphi_\nu(x)=2^{\nu n}\varphi(2^\nu x)$, where $\varphi\in\mathscr{S}$ with supp $\hat{\varphi}=\{2^{-1}\leqq|\xi|\leqq 2\}$. For convenience let us also assume that $\{\varphi_\nu\}_{\nu\in\mathbf{Z}}$ is normalized in the sense that

$$\sum\nolimits_{\nu\in\mathbf{Z}}(\hat{\varphi}_\nu(\xi))^2=1 \quad \left(\text{or } \sum\nolimits_{\nu\in\mathbf{Z}}\varphi_\nu*\varphi_\nu=\delta\right).$$

We can now define our principal spaces.

Definition 1.1. Let s real, $0<p,q\leqq\infty$. Then we set (the spaces of Triebel—Lizorkin type)

$$F_p^{sq}=\left\{f\,|\,f\in\mathscr{S}' \ \& \ \{2^{\nu s}\varphi_\nu*f\}_{\nu\in\mathbf{Z}}\in L_p(l^q)\right\}.$$

We equip F_p^{sq} with the quasi-norm

$$\|f\|_{F_p^{sq}}=\|\{2^{\nu s}\varphi_\nu*f\}_{\nu\in\mathbf{Z}}\|_{L_p(l^q)}.$$

Definition 1.2. Let s real, $0<p,q\leqq\infty$, $a\geqq 0$. Then we set (poised spaces of Besov type)

$$B_p^{sq}(a)=\left\{f\,|\,f\in\mathscr{S}' \ \& \ \{2^{\nu s}(1+2^\nu|x|)^a\varphi_\nu*f\}_{\nu\in\mathbf{Z}}\in l^q(L_p)\right\}.$$

We equip $B_p^{sq}(a)$ with the quasi-norm

$$\|f\|_{B_p^{sq}(a)}=\|\{2^{\nu s}(1+2^\nu|x|)^a\varphi_\nu*f\}_{\nu\in\mathbf{Z}}\|_{l^q(L_p)}.$$

If $a=0$ we simply write $B_p^{sq}(0)=B_p^{sq}$ (Besov space).

Remark 1.1. Conformally with the notation of [7] we should perhaps have written $\dot{F}$ and $\dot{B}$, rather than F and B. We also, as is customary in the case of "homogeneous" spaces, have to work modulo polynomials. Thus the above quasi-norms are genuine quasi-norms only after such an identification.

Let us now rapidly state some propertes of these spaces which can be proven in a more or less standard way (cf. [10]).

1. The spaces F_p^{sq} and $B_p^{sq}(a)$ are complete. The embeddings from $\mathscr{S}$ and into $\mathscr{S}'$ are continuous. They are thus quasi-Banach (Banach if $1\leqq p,q\leqq\infty$) spaces of tempered distributions.

2. $\mathscr{S}$ is a dense subspace of F_p^{sq} and $B_p^{sq}(a)$ if $0<p,q<\infty$.

3. We have embedding theorems, e.g. the embedding $B_p^{sq}(a)\to B_{p_1}^{s_1q}(a)$ if $s-n/p=s_1-n/p_1$, $s\geqq s_1$, $q\leqq q_1$.

4. We have duality theorems, e.g. the duality $(F_p^{sq})'\approx F_{p'}^{-sq'}$ if $1\leqq p,q\leqq\infty$.

2. Some lemmata

The following elementary result will do us a great service.

Lemma 2.1. *Let u be any C^1 function in $\mathbf{R}^n$ and let $0<r\leqq\infty$. Then we have the inequality*

$$u^{**} \leqq C\{\delta^{-n/r}(Mu^r)^{1/r}+\delta(\nabla u)^{**}\}, \quad \delta\leqq 1$$

where M denotes the Hardy & Littlewood maximal operator and where we have defined u^{**} by

$$u^{**}(x) = \sup_{y\in\mathbf{R}^n} |u(x-y)|/(1+|y|)^{n/r}$$

and $(\nabla u)^{**}$ in a similar fashion.

Proof. By the mean value theorem we have for any $x, z\in\mathbf{R}^n$

$$|u(x-z)| \leqq C\Big\{\delta^{-n/r}\Big(\int_{|x-z-y|<\delta}|u(y)|^r\,dy\Big)^{1/r}+\delta\sup_{|x-z-y|<\delta}|\nabla u(y)|\Big\}.$$

By definition of M and $(\nabla u)^{**}$ follows

$$|u(x-z)| \leqq C\big\{\delta^{-n/r}(Mu^r(x))^{1/r}+\delta(\nabla u)^{**}(x)\big\}(1+\delta+|z|)^{n/r}.$$

If $\delta\leqq 1$ we clearly get the desired inequality.

We also need a few results connected with M. First we recall the following elementary

Lemma 2.2. *Let f by any measurable function in $\mathbf{R}^n$ and let $b>n$. Then holds*

$$\int |f(y)|/(1+|x-y|)^b\,dy \leqq CMf(x).$$

We need also the following extension of the Hardy & Littlewood maximal theorem.

Lemma 2.3. (Fefferman & Stein [2]) *Let $\mathbf{f}=\{f_\nu\}_{\nu\in\mathbf{Z}}$ be a sequence of measurable functions in $\mathbf{R}^n$ and let $1<p, q\leqq\infty$. Then holds*

$$\|M\mathbf{f}\|_{L_p(l^q)} \leqq C\|\mathbf{f}\|_{L_p(l^q)}$$

where of course $M\mathbf{f}=\{Mf_\nu\}_{\nu\in\mathbf{Z}}$.

3. The equivalence proof

If $f\in F_p^{sq}$ and if $\{\varphi_\nu\}_{\nu\in\mathbf{Z}}$ is the sequence of test functions of Sec. 1 we set

$$\varphi^{**}f(x) = \|\{\varphi_\nu^{**}f(x)\}_{\nu\in\mathbf{Z}}\|_{l^q},$$

$$\varphi_\nu^{**}f(x) = \sup_{y\in\mathbf{R}^n} 2^{\nu s}|\varphi_\nu * f(x-y)|/(1+2^\nu|y|)^a.$$

We also set

$$\varphi^+ f(x) = \|\{\varphi_\nu^+ f(x)\}_{\nu\in \mathbf{Z}}\|_{l^q},$$
$$\varphi_\nu^+ f(x) = 2^{\nu s}\varphi_\nu * f(x).$$

Clearly $\varphi^+ f\in L_p$. Below we show that also $\varphi^{**}f\in L_p$, at least if a is sufficiently large. More generally, let $\{\sigma_\nu\}_{\nu\in\mathbf{Z}}$ be a general sequence of test functions, with $\sigma_\nu(x) = 2^{\nu n}\sigma(2^\nu x)$ (but with no restriction on supp $\hat{\sigma}$) and define $\sigma^{**}f$, $\sigma_\nu^{**}f$, $\sigma_\nu^+ f$ as above. Then we have the following

Theorem 3.1. *Assume that* $\sigma\in B_1^{-sq_1}(a)\cap B_1^{-s+a,q_1}(a)$ *with* $a>n/\min(p,q)$, $q_1=\min(1,q)$. *Then holds:*

$$f\in F_p^{sq} \Rightarrow \sigma^{**}f\in L_p. \tag{3.1}$$

In particular (3.1) *holds with* $\sigma=\varphi$.

Proof. (Cf. Fefferman & Stein [1], pp. 183—187.) Let us start with the identity

$$\sigma_\mu * f = \sum\nolimits_{\nu\in\mathbf{Z}} (\sigma_\mu * \varphi_\nu) * (\varphi_\nu * f).$$

We then get

$$2^{\mu s}|\sigma_\mu * f(x-z)| \leqq \sum 2^{\mu s}\int |(\sigma_\mu * \varphi_\nu)(y)||\varphi_\nu * f(x-z-y)|\,dy \leqq$$
$$\leqq \sum 2^{\mu n}\int 2^{(\mu-\nu)s}|\sigma * \varphi_{\nu-\mu}(2^\mu y)|(1+2^\nu|y|)^a\,dy\,\varphi_\nu^{**}f(x)(1+2^\nu|z|)^a \leqq$$
$$\leqq \sum 2^{(\mu-\nu)s}\int |(\sigma * \varphi_{\nu-\mu})(y)|(1+2^{\nu-\mu}|y|)^a\,dy\,\varphi_\nu^{**}f(x)(1+2^{\nu-\mu})^a(1+2^\mu|z|)^a$$

where we have used the elementary inequality:

$$\max(1+u+v,\, 1+uv) \leqq (1+u)(1+v), \quad u\geqq 0, \quad v\geqq 0.$$

In other words we have

$$\sigma_\mu^{**}f(x) \leqq \sum t_{\nu-\mu}\varphi_\nu^{**}f(x) \tag{3.2}$$

with $t_\nu = \sum 2^{-\nu s}(1+2^\nu)^a\int (1+2^\nu|y|)^a|\sigma * \varphi_\nu(y)|\,dy$. Here by hypothesis

$$\left(\sum |t_\nu|^{q_1}\right)^{1/q_1} \leqq C.$$

Therefore follows

$$\sigma^{**}f \leqq C\varphi^{**}f. \tag{3.3}$$

Thus we have reduced ourselves to proving (3.1) with $\sigma=\varphi$. To this end we first note that (3.3) in particular entails

$$(\nabla\varphi)^{**}f \leqq C\varphi^{**}f.$$

On the other hand lemma 2.1 implies (with $r=n/a$)

$$\varphi_\nu^{**}f \leqq C\{\delta^{-n/r}(M(\varphi_\nu^+ f)^r)^{1/r} + \delta(\nabla\varphi)_\nu^{**}f\}, \quad \delta\leqq 1.$$

Thus we get

$$\|\varphi^{**}f\|_{L_p} \leqq C\{\delta^{-n/r}\|(M(\varphi_v^+ f)^r)^{1/r}\|_{L_p(l^q)} + \delta\|\varphi^{**}f\|_{L_p}\}.$$

By lemma 2.3 we have (since $r<\min(p, q)$)

$$\|(M(\varphi_v^+ f)^r)^{1/r}\|_{L_p(l^q)} = \|M(\varphi_v^+ f)^r\|_{L_{p/r}(l^{q/r})}^{1/r} \leqq$$

$$\leqq C\|(\varphi_v^+ f)^r\|_{L_{p/r}(l^{q/r})}^{1/r} = C\|\varphi_v^+ f\|_{L_p(l^q)} = C\|f\|_{F_p^{sq}}.$$

Thus we have

$$\|\varphi^{**}f\|_{L_p} \leqq C\{\delta^{-n/r}\|f\|_{F_p^{sq}} + \delta\|\varphi^{**}f\|_{L_p}\}, \quad \delta \leqq 1.$$

If we knew already that $\|\varphi^{**}f\|_{L_p}<\infty$ we could, taking δ sufficiently small, conclude that

$$\|\varphi^{**}f\|_{L_p} \leqq C\|f\|_{F_p^{sq}} \tag{3.4}$$

and we were through. But if $\|\varphi^{**}f\|_{L_p}=\infty$ this argument does not apply. To circumvent this difficulty we use an approximation argument. The above proof at least shows that (3.4) is valid if $f\in\mathcal{S}$. For a general $f\in F_p^{sq}$ we find a sequence $\{f_i\}_{i=1}^{\infty}$ in $\mathcal{S}$ such that $f_i\to f$ in $\mathcal{S}'$ as $i\to\infty$, with $\sup_i \|f_i\|_{F_p^{sq}}<\infty$. It is easily seen that

$$\|\varphi^{**}f\|_{L_p} \leqq \varlimsup_{i\to\infty} \|\varphi^{**}f_i\|_{L_p}$$

so an application of (3.3) to f_i effectively yields $\|\varphi^{**}f\|_{L_p}<\infty$. The proof is complete.

Corollary 3.1. *The space F_p^{sq} is independent of the particular sequence of test functions $\{\varphi_v\}_{v\in Z}$ chosen.*

Proof. Obvious.

4. Some variants of the above result

We begin with the following simple variant of th. 3.1.

Theorem 4.1. *Assume that $\sigma\in B_1^{-sq_1}(a)$ with $a>n/\min(p, q)$, $q_1=\min(1, q)$. Then holds:*

$$f\in F_p^{sq} \Rightarrow \sigma^+ f\in L_p \tag{4.1}$$

Proof. The proof of th. 3.1 clearly also gives in place of (3.2)

$$\sigma_\mu^+ f(x) \leqq \sum t'_{v-\mu}\varphi_v^{**}f(x)$$

with $t'_v=2^{-vs}\int(1+2^v|y|)^a|\sigma*\varphi_v(y)|\,dy$. This gives in place of (3.3):

$$\sigma^+ f \leqq C\varphi^{**}f.$$

Since we know already that $\varphi^{**}f\in L_p$ it follows that $\sigma^+ f\in L_p$.

Next we want to relax the condition on σ in th. 4.1. In this direction we can prove:

Theorem 4.2. *Assume that* $\sigma \in B_{\infty}^{-s-n, q_1}(a)$ *where* $a > n/\min(1, p, q)$, $q_1 = \min(1, q)$. *Then holds again* (4.1).

Proof. From lemma 2.2 and lemma 2.3 follows readily that

$$f \in F_p^{sq} \Rightarrow \left\{2^{\nu s}\left(2^{\nu n} \int |\varphi_\nu * f(x-y)|^r/(1+2^\nu|y|)^b\, dy\right)^{1/r}\right\} \in L_p(l^q)$$

where $r < \min(p, q)$, $b > n$. From this follows again readily

$$f \in F_p^{sq} \Rightarrow \left\{2^{\nu(s+n)} \int |\varphi_\nu * f(x-y)|/(1+2^\nu|y|)^a\, dy\right\} \in L_p(l^q)$$

with a as in the hypothesis of the theorem. The proof of th. 3.1 now yields

$$\sigma_\mu^+ f(x) \leqq \sum t''_{\nu-\mu} 2^{\nu(s+n)} \int |\varphi_\nu * f(x-y)|/(1+2^\nu|y|)^a\, dv$$

with $t''_\nu = 2^{-\nu(s+n)} \int (1+2^\nu|y|)^a |\sigma * \varphi_\nu(y)|\, dy$. The rest of the proof is the same.

5. A multiplier theorem

We have the following

Theorem 5.1. *Assume that* $m \in B_1^{0\infty}(a)$ *where* $a > n/\min(p, q)$. *Then* $f \in F_p^{sq} \Rightarrow$ $\Rightarrow m * f \in F_p^{sq}$.

Proof. (Cf. Stein [9], pp. 96—99.) Let us set $g = m * f$. We want to estimate $\varphi^+ g$. Choose σ in such a way that th. 3.1. is applicable and that in addition $\hat{\sigma}_\nu(\xi) = 1$ in supp $\hat{\varphi}_\nu$. Then we have

$$\varphi_\nu * g = (\varphi_\nu * m) * (\sigma_\nu * f)$$

and we get

$$2^{\nu s}|\varphi_\nu * g(x)| \leqq \int |\varphi_\nu * m(y)|(1+2^\nu|y|)^a\, dy\, \sigma_\nu^{**} f(x) \leqq C\sigma_\nu^{**} f(x)$$

or

$$\varphi^+ g \leqq C\sigma^{**} f.$$

Since $\sigma^{**} f \in L_p$ we get $\varphi^+ g \in L_p$ and $g \in F_p^{sq}$.

In order to get a true multiplier theorem we have to express the condition on m in terms of m.

Corollary 5.1. *The conclusion of th. 5.1. is valid in particular if* $|D^\alpha \hat{m}(\xi)| \leqq C|\xi|^{-|\alpha|}$ *for all multi-indices* α *with* $|\alpha| \leqq T$ *where* T *is an integer* $> n/2 + a$.

Proof. Use Bernstein's theorem on absolutely convergent Fourier integrals.

Remark 5.1. Using the results of Sec. 4 it is possible ot relax the assumptions on m (and $\hat{m}$). In particular we can as a special case obtain Hörmander's version of Mikhlin's multiplier theorem [4].

6. An application to approximation theory

We start by recalling the following known result (in the periodic case with $n=1$):

Theorem 6.1. (Freud [3]) Let *f belong to the closure of* $\mathcal{P}$ *in* $B_\infty^{1\infty}(\mathbf{T}^1)$. *Then* $f'(x)$ *exists at a point* $x \in \mathbf{T}$ *iff* $\Phi_n f'(x)$ *tends to a limit as* $n \to \infty$. *Here* $\Phi_n f$ *denote the Fejer sums of* f.

We can now prove the following analogue of th. 6.1, which for $1 < p \leqq \infty$ was given in [8].

Theorem 6.2. *Let* f *be in the closure of* $\mathcal{S}$ *in* $F_p^{0\infty} = F_p^{0\infty}(\mathbf{R}^n)$ *where* $0 < p \leqq \infty$. *Assume that, for some* σ, $\sigma_\nu * f(x)$ *converges as* $\nu \to \infty$ *a.e. for* x *in set of positive measure. Then the same is true for any other kernel such that the difference with the first one belongs to* $B_\infty^{-n1}(a)$ *where* $a > n/\min(1, p)$.

Proof. It suffices of course to prove that $\sigma_\nu * f$ tends to 0 a.e. throughout $\mathbf{R}^n$, for every $\sigma \in B_\infty^{-n1}(a)$. Since $\hat{\sigma}(0) = 0$ this certainly is true if $f \in \mathcal{S}$. On the other hand by th. 4.2. sup $|\sigma_\nu * f(x)| < \infty$ a.e. for a general f. Thus it suffices to apply the usual density argument.

Example 6.1. Th. 6.2 is applicable notably in the case of Riesz means, i.e.

$$\hat{\sigma}(\xi) = \begin{cases} (1-|\xi|)^\lambda & \text{if} \quad |\xi| < 1 \\ 0 & \text{elsewhere} \end{cases}$$

provided $\lambda > a - \frac{1}{2}$.

7. Concluding remarks

In retrospect we notice that in the preceding treatment only very little of the structure of the underlying space $\mathbf{R}^n$ has been utilized. This indicates that there exist generalizations. In the place of $\mathbf{R}^n$ we may indeed consider any (complete) Riemannian manifold Ω. The spaces $F_p^{sq} = F_p^{sq}(\Omega)$ are then defined by a condition of the type

$$\{2^{\nu s}\varphi(\sqrt{-\Delta}/2^{\nu})f\}_{\nu\in Z}\in L_p(l^q)$$

where Δ is the Laplace—Beltrami operator on Ω. (In particular we can thus define Hardy-classes $H_p=H_p(\Omega)$.) We plan to return to this topic in a forthcoming publication.

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