Bessel potentials and extension of continuous functions on compact sets

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1. Introduction

Let K be a compact subset of R^m. H. Wallin [18] proved that if K has classical α-capacity zero for a certain α, then every $f_0 \in C(K)$ can be extended to a continuous function $f \in W_l^p(R^m)$, where $1 \leqq p < \infty$, and l is a positive integer. The number α depends on m, p and l. He also proved a converse statement. However, his results give a complete solution to this extension problem only when $p=2$ [18, Theorem 3, Theorem 4]. We are going to give a solution to this problem by considering $L_\alpha^p(R^m)$, $1 < p < \infty$, $\alpha > 0$, α not necessarily an integer. The case studied by H. Wallin is then included since $L_\alpha^p(R^m) = W_\alpha^p(R^m)$, when $1 < p < \infty$ and α is a positive integer.

We state our main result in an even more general form by considering potentials relative to general kernels $k(r)$, of L^p-functions. For notations and statement of the theorem, see section 2. See [9] for classical potential theory.

2. Preliminaries and statement of the theorem

We consider R^m with Euclidean norm. All sets are sets of points in R^m. Compact and open sets are denoted by K and V respectively.

The spaces $C(K)$, $C^\infty(V)$, and $C_0^\infty(V)$ are defined in the usual way.

The Lebesgue measure of a set E is denoted by mE and integration with respect to Lebesgue measure is written $\int_E dx$. The spaces $L^p(E)$, $1 \leqq p < \infty$, with norm $\|\cdot\|_{L^p(E)}$ are defined in the usual way. When $E = R^m$ we write L^p and $\|\cdot\|_p$. The class of positive elements in $L^p(E)$ is denoted by $L_+^p(E)$. As a general rule, a sub index $+$ denotes positive elements. The conjugate of p is $q = p/p-1$.

The class A_1 consists of all sets which are measurable for all non-negative

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Radon measures in R^m. When $E\in A_1$, $\mathscr{L}^1_+(E)$ denotes the class of complete, non-negative Radon measures μ, which are concentrated on E and satisfy

$$\|\mu\|_1 = \text{total variation of } \mu < \infty.$$

For $\alpha>0$ and $1\leqq p<\infty$, $L^p_\alpha(R^m)$ with norm $\|\cdot\|_{\alpha,p}$ is the space of Bessel potentials of order α of L^p-functions. The space $W^p_\alpha(R^m)$, with norm $|\cdot|_{\alpha,p}$, is the usual Sobolev space. See [3] for details.

A kernel $k(r)$ is a non-negative, non-increasing, and lower semi-continuous function $k\colon (0,\infty)\to[0,\infty)$. In order to exclude trivialities we assume that,

(i) $\int_{|y|\leqq 1} k(|y|)\,dy<\infty$,

(ii) $\int_{|y|\geqq 1} (k(|y|))^q\,dy<\infty$,

(iii) $k(r)\not\equiv 0$.

Also in all cases considered by us we have

(iv) $\int_{|y|\leqq 1} (k(|y|))^q\,dy=\infty$.

See [17, Lemma 2].

The k-potential of a non-negative function f or measure μ is defined by convolution and written $k*f$ and $k*\mu$ respectively.

We now define two capacities, cf. N. G. Meyers [11]. Let k be a kernel and let $1<p<\infty$. The $C_{k,p}$-capacity of an arbitrary set E is defined by

$$C_{k,p}(E) = \inf \|f\|_p^p,$$

where the infimum is taken over all $f\in L^p_+$ satisfying $(k*f)(x)=\int k(|x-y|)\cdot f(y)\,dy\geqq 1$ for every $x\in E$. We call such functions f test functions for $C_{k,p}(E)$.

For $E\in A_1$, we define

$$c_{k,p}(E) = \sup \|\mu\|_1,$$

where the supremum is taken over all $\mu\in\mathscr{L}^1_+(E)$ for which $\|k*\mu\|_q\leqq 1$. Such a measure μ is called a test measure for $c_{k,p}(E)$.

The $c_{k,p}$-capacity, which is a kind of dual capacity, satisfies $C_{k,p}(A)=(c_{k,p}(A))^p$ for every analytic set A [11, Theorem 14]. All analytic sets are $C_{k,p}$-capacitable [11, Theorem 8].

Capacities of this type have been studied by many authors [1, 2, 7, 11, 13, 14, 16, 17, 20].

We are now in a position to state our main result.

Theorem 1. *Suppose that $1<p<\infty$ and that k is a kernel satisfying the conditions* (i)—(iv) *of section 2. Then $C_{k,p}(K)=0$ is a necessary and sufficient condition for every function $f_0\in C(K)$ to be the restriction to K of a continuous k-potential*

$f=k*v$, where $v\in L^p$. Furthermore, the above statement remains true if we also require the norm $\|v\|_p$ to be arbitrarily small.

Theorem 1 remains true for $p=1$ if we replace the $C_{k,p}$-capacity by the classical k-capacity [18, p. 56] and if the conditions (i)—(iv) of section 2 are replaced by the conditions (a)—(c) in [18, pp. 56—57]. This was proved by H. Wallin [18, Theorem 1 and Theorem 2]. See also S. Ya. Havinson [8].

The solution of the extension problem described in the introduction is contained in Theorem 1. To prove this, we note that for $1<p<\infty$, $\alpha>0$, and $\alpha\cdot p\leqq m$, the Bessel kernel $G_\alpha(r)$ [3, p. 220] satisfies the conditions (i)—(iv) of section 2 [3, p. 224], and that the space of potentials $G_\alpha*v$, where $v\in L^p$, is precisely the space $L^p_\alpha(R^m)$. When $k(r)=G_\alpha(r)$, $\alpha>0$, the capacities $C_{k,p}$ and $c_{k,p}$ are denoted by $B_{\alpha,p}$ and $b_{\alpha,p}$ and are called Bessel capacities. If α is an integer we have $L^p_\alpha(R^m)=W^p_\alpha(R^m)$ [3, Theorem 11: 1] and Theorem 1 gives a complete solution of the extension problem studied by H. Wallin [18, Theorem 3 and Theorem 4]. The case when $k(r)=G_\alpha(r)$ and $\alpha\cdot p>m$ was excluded from Theorem 1. However, this case is trivial since every function in $L^p_\alpha(R^m)$ satisfies (when redefined on a set of Lebesgue measure zero) a Hölder condition. See [17, Proposition 3].

A relation which holds except for a set of $C_{k,p}$-capacity or $c_{k,p}$-capacity zero is said to hold $C_{k,p}$-a.e. and $c_{k,p}$-a.e. respectively.

Now we define the concepts of capacitary distributions and potentials.

A function $f\in L^p_+$ such that $(k*f)(x)\geqq 1$ $C_{k,p}$-a.e. on E, and $\|f\|_p^p=C_{k,p}(E)$ is called a $C_{k,p}$-capacitary distribution for E. The potential $k*f$ is called a $C_{k,p}$-capacitary potential for E. Let $E\in A_1$, then any test measure μ for $c_{k,p}(E)$ satisfying $\|\mu\|_1=c_{k,p}(E)$, is called a $c_{k,p}$-capacitary distribution for E and $k*\mu$ is called a $c_{k,p}$-capacitary potential for E.

Existence and properties of capacitary distributions and potentials was proved by N. G. Meyers [11]. See also V. G. Maz'ja and V. P. Havin [13, 14] and Yu. G. Rešetnyak [16, Theorem 3.1]. Every compact set K satisfying $C_{k,p}(K)<\infty$ has capacitary distributions. In particular this holds for the Bessel capacity $B_{\alpha,p}$, $\alpha>0$.

3. Some lemmas

We begin with a generalization of the classical boundedness principle for potentials of measures. (See for example [9, p. 72]).

Lemma 1. (D. R. Adams and N. G. Meyers [1]). *Let k_1 and k_2 be two kernels and let μ be a non-negative Radon measure. Let $u(x)=k_1*(k_2*\mu)^{1/p-1}(x)$, for $1<p<\infty$. Then*

$$\sup_{x\in R^m} u(x)\leqq M\cdot \sup_{x\in \operatorname{supp}\mu} u(x)$$

where M depends on m and p only.

See also [13, Theorem 1].

The distance between a point x and a set E is denoted by dist (x, E).

Lemma 2. *Let k be a kernel satisfying the conditions* (i)—(iv) *of section* 2. *Let $1<p<\infty$ and let K be a compact set such that $C_{k,p}(K)=0$. For any positive numbers ε and η, there is a function $v\in C_0^\infty(R^m)$ such that the continuous potential $f=k*v$ satisfies*

$f(x)\geqq 1$, *for all x belonging to some neighbourhood of K,*

$0\leqq f(x)\leqq M$, *for every $x\in R^m$, where M depends on m and p only.*

$f(x)\leqq \varepsilon\cdot M(a)$, *for all points x such that* dist $(x, K)>a$, *for all $a\geqq\eta$,*

$$\|v\|_p<\varepsilon.$$

Here $M(a)$ is a positive number depending on m, p, k and a.

Lemma 2 was proved with $k(r)=G_\alpha(r)$, including differentiability properties, in [17, Lemma 3]. We sketch the proof given there. For any $\delta>0$, K_δ is the set of points x such that dist $(x, K)\leqq\delta$. The set K_δ has capacitary distributions v_δ and μ_δ such that

$$v_\delta(x) = c_{k,p}(K_\delta)\cdot(k*\mu_\delta)^{\frac{1}{p-1}}(x), \quad \text{a.e. [11].}$$

The $C_{k,p}$-capacitary potential $f_\delta=k*v_\delta$ satisfies

$$f_\delta(x)\geqq 1,\ C_{k,p}\text{-a.e. on } K_\delta,$$

$$f_\delta(x)\leqq 1,\ \text{everywhere on } \operatorname{supp}\mu_\delta,$$

$$\|v_\delta\|_p = \big(C_{k,p}(K_\delta)\big)^{1/p} = c_{k,p}(K_\delta) = \|\mu_\delta\|_1.$$

Now Lemma 1 gives that

$$f_\delta(x)\leqq M, \quad \text{for every} \quad x\in R^m,$$

where M depends on m and p only. We finish the proof of Lemma 2 by regularizing f_δ and choosing δ small enough.

A proof of the sufficiency part of Theorem 1 using Lemma 2 and a method used by H. Wallin [18, Theorem 1] was given in [17] and is omitted here.

The following lemma is known in the linear case, J. Deny and J. L. Lions [6, p. 353] (see H. Wallin [18, Lemma 6] and J. Deny [5, Theorem 5] for a proof), and in the non-linear case, V. G. Maz'ja, V. P. Havin [14, Lemma 5.8]. Our proof differs from [14] and uses an idea of H. Wallin [18].

Lemma 3. *Let k be a kernel and let p be a real number, $1<p<\infty$. Let g_i, $i=1, 2$, be functions with the following property: For every $\varepsilon>0$ there is a Borel set E such that the restrictions of g_1 and g_2 to $R^m\setminus E$ are continuous and $C_{k,p}(E)<\varepsilon$. Assume* *that the set of those points where $g_1(x) \neq g_2(x)$ is a Borel set. If furthermore $g_1(x) = g_2(x)$ a.e. then $g_1(x) = g_2(x)$, $C_{k,p}$-a.e.*

Remark. The lemma is trivial if k does not satisfy the conditions (i)—(iv) of section 2.

Proof of Lemma 3. Let E be the Borel set consisting of those points where $g_1(x) \neq g_2(x)$ and assume that $C_{k,p}(E) = a > 0$. We first assume that a is finite. Choose E_1, for $0 < \varepsilon < a$, such that the restrictions of g_1 and g_2 to $R^m \setminus E_1$ are continuous and $C_{k,p}(E_1) < \varepsilon$. By the subadditivity of $C_{k,p}$ we have $C_{k,p}(E \setminus E_1) > a - \varepsilon$. Since $C_{k,p}$ is an inner capacity, there is a compact set K such that $K \subset E \setminus E_1$ and $C_{k,p}(K) > a - \varepsilon$. The set K has $c_{k,p}$-capacitary distribution μ satisfying

$$\|\mu\|_1^p = C_{k,p}(K).$$

We can find a non-negative function $\varphi \in C_0^\infty(R^m)$ which is supported by the unit ball and has L^1-norm equal to one. Put $\varphi_n(x) = n^m \cdot \varphi(nx)$, and $\mu_n = \mu * \varphi_n$, $n = 1, 2, \ldots$. Then μ_n is a test measure for the set

$$\{x \in R^m;\ \operatorname{dist}(x, K) \leqq n^{-1}\} \setminus E,$$

since μ_n is absolutely continuous with respect to the Lebesgue measure and $mE = 0$, $n = 1, 2, \ldots$. This yields, with $K_n = \{x \in R^m;\ \operatorname{dist}(x, K) \leqq n^{-1}\}$,

$$\big(C_{k,p}(K_n \setminus E)\big)^{1/p} = c_{k,p}(K_n \setminus E) \geqq \|\mu_n\|_1 = \|\mu\|_1 = \big(C_{k,p}(K)\big)^{1/p} > (a-\varepsilon)^{1/p}$$

and thus

$$C_{k,p}\big((K_n \setminus E) \setminus E_1\big) > (a-\varepsilon) - \varepsilon > 0,$$

if ε is small enough. Then $(K_n \setminus E) \setminus E_1 \neq \emptyset$, for every $n = 1, 2, \ldots$. Choose points $x_n \in (K_n \setminus E) \setminus E_1$ and $y_n \in K$ satisfying $|x_n - y_n| \leqq n^{-1}$, $n = 1, 2, \ldots$. Since K is compact we may assume that $\lim_{n \to \infty} y_n = y$, $y \in K$. Then $\lim_{n \to \infty} x_n = y$, and

$$|g_1(y) - g_2(y)| \leqq |g_1(y) - g_1(x_n)| + |g_1(x_n) - g_2(x_n)| + |g_2(x_n) - g_2(y)|, \quad n = 1, 2, \ldots. \tag{3.1}$$

The middle term in the right-hand member of (3.1) equals zero. The remaining terms tend to zero when n tends to infinity, by continuity. We conclude that $g_1(y) = g_2(y)$, which is a contradiction.

Some obvious modifications are necessary when $C_{k,p}(E) = \infty$. The lemma is proved.

The following lemma is analogous to [18, Lemma 3].

Lemma 4. *Let p be a real number, $1 < p < \infty$, and k a kernel satisfying*

$$\int k(|y|)^q\, dy = \infty. \tag{3.2}$$

Suppose that K is a compact set having positive $C_{k,p}$-capacity. Further, suppose that

t^* is a non-negative and non-decreasing function defined for $r \geqq 0$ such that $t^*(r)>0$ whenever $r>0$ and $\lim_{r\to 0} t^*(r)=t^*(0)=0$.

Then there exists a function $f_0 \in C(K)$ having the following property: If the positive number ε is chosen small enough there exist, for every Borel set E satisfying $C_{k,p}(E)<\varepsilon$, points x_1 and x_2 in $K \setminus E$ such that $x_1 \neq x_2$ and $|x_1-x_2|$ is arbitrarily small and

$$|f_0(x_1)-f_0(x_2)| \geqq t^*(|x_1-x_2|). \tag{3.3}$$

H. Wallin proved this for the classical α-capacity [18, Lemma 3]. Since (3.2) implies that all finite sets have $C_{k,p}$-capacity zero, Lemma 4 can be proved analogously. The proof is therefore omitted.

4. Proof of Theorem 1

As remarked in section 3 we only prove the necessity of the condition $C_{k,p}(K)=0$ in Theorem 1.

Let k be a kernel satisfying the conditions (i)—(iv) of section 2 and let $1<p<\infty$. Recall that, assuming $C_{k,p}(K)>0$, we must prove that there exists a function $f_0 \in C(K)$ which is not the restriction to K of a continuous potential $f=k*v$, $v \in L^p$.

The idea of the proof is analogous to the proof of [18, Theorem 4].

We are going to prove that there exists a strictly positive kernel k_1 satisfying the conditions (i)—(iv) of section 2, such that

$$C_{k_1,p}(K) > 0, \tag{4.1}$$

and

$$\lim_{r\to 0} k(r)\cdot(k_1(r))^{-1} = 0. \tag{4.2}$$

The $c_{k,p}$-capacitary distribution μ for K satisfies $\|\mu\|_1=(C_{k,p}(K))^{1/p}>0$. N. G. Meyers [11, Lemma 9] proved that there exists a kernel k_1 such that (4.2) holds and $k_1*\mu \in L^q$. It is easy to see that this kernel k_1 can be modified to satisfy the conditions (i)—(iv) of section 2. Then μ is a test measure for $c_{k_1,p}(K)$ since $k_1*\mu \in L^q$ and $\mu \not\equiv 0$, which implies that (4.1) holds. Compare [4, Theorem 2] for the case of classical α-capacity.

Next we find a modulus of continuity for the potentials $k*v$, $v\in L^p$.

This modulus of continuity is independent of v. More precisely: There exists a non-negative function $t(r)$, defined for $r\geqq 0$, satisfying $\lim_{r\to 0} t(r)=t(0)=0$ and a positive number M, such that

$$|(k*v)(x_1)-(k*v)(x_2)| \leqq M\cdot t(|x_1-x_2|), \tag{4.3}$$

for all points x_1 and x_2 with $|x_i|\leqq R$, and $(k_1*|v|)(x_i)\leqq a$, $i=1,2$. The function t depends on m, p, k, and k_1. The number M depends on m, p, a, R and v [17, p. 47].

A basic fact is that

$$C_{k,p}(\{x\in R^m;\ (k*|v|)(x)\geqq a\})\leqq\left(\frac{\|v\|_p}{a}\right)^p, \tag{4.4}$$

for any $v\in L^p$ and $a>0$ [11, Theorem 2].

The proof of the necessity part of Theorem 1 is now easily completed in the following way:

Let K satisfy $C_{k,p}(K)>0$ and choose a kernel k_1 such that (4.1) and (4.2) hold. Then we construct the function t in (4.3) and choose a function t^* satisfying the assumptions of Lemma 4 and

$$\lim_{r\to 0} t^*(r)\cdot(t(r))^{-1}=\infty. \tag{4.5}$$

The function constructed in Lemma 4 is denoted by f_0. Now suppose that there exists a continuous potential $k*v$, $v\in L^p$, such that

$$f_0(x)=(k*v)(x),\quad \text{for every}\quad x\in K.$$

Combining (4.3), (4.4), and (4.5) with (3.3) leads to a contradiction. Thereby Theorem 1 is proved.

5. Further results

We define two capacities introduced by V. G. Maz'ja [12] and studied by many others [2, 10, 16, 17, 19, 20].

Let $B=B(0,R)$ be a fixed open ball. For any positive integer l and a real number p, $1\leqq p<\infty$, we define the $\Gamma_{l,p}$-capacity of $K\subset B$ by

$$\Gamma_{l,p}(K)=\inf|f|_{l,p},$$

where the infimum is taken over all $f\in C_0^\infty(B)$ which satisfy $f(x)\geqq 1$, for every $x\in K$. Similarly we define

$$N_{l,p}(K)=\inf|f|_{l,p},$$

where the infimum is taken over all $f\in C_0^\infty(B)$ such that $0\leqq f(x)\leqq 1$ for every $x\in R^m$ and $f(x)=1$, for every $x\in K$.

The following important result follows from the properties of the capacitary distributions for the Bessel capacity and a method used by W. Littman [10, p. 865]:

Let l be a positive integer and let $1<p<\infty$, and $p\cdot l\leqq m$. Then the capacities $N_{l,p}$, $\Gamma_{l,p}$, and $B_{l,p}$ are equivalent for compact sets.

An even more general result was proved by D. R. Adams, John C. Polking [2, Theorem A].

Remark. H. Wallin proved the equivalence between $\Gamma_{l,p}$ and $B_{l,p}$ in [20, Theorem 1] and studied the connection between $\Gamma_{l,p}$-capacity and classical α-capacity in [19].

Now we consider the case $p=1$ with the extended function belonging to $W_l^1(R^m)$, where l is a positive integer. This case is not completely solved.

A sufficient condition for every $f_0 \in C(K)$ to be extendable to a continuous function f having arbitrarily small norm in $W_l^1(R^m)$ is that $N_{l,1}(K)=0$. A necessary condition is given by $\Gamma_{l,1}(K)=0$ [17, Theorem 8 and Proposition 4]. We know that $N_{l,1}$ and $\Gamma_{l,1}$ are equivalent for compact sets when $l=1$ [10, p. 861]. It is an open question if this holds also when l is an integer greater than one.

There exists a compact set $K \subset R^2$ such that every $f_0 \in C(K)$ has an extension in $W_1^1(R^2) \cap C(R^2)$ but it is in general not possible to make the norm in $W_1^1(R^2)$ of the extended function arbitrarily small [18, p. 58]. This contrasts to Theorem 1 where the two properties:

(a) Every $f_0 \in C(K)$ is the restriction to K of a continuous potential $k * v$, $v \in L^p$,

(b) Property (a) holds and $\|v\|_p$ can be made arbitrarily small, are equivalent.

Finally we state a consequence of the Open Mapping Theorem.

Theorem 2. *Suppose that $\alpha > 0$, $1 \leqq p < \infty$ and that K is such that every $f_0 \in C(K)$ can be extended to a continuous function f belonging to $L_\alpha^p(R^m)$ $(W_\alpha^p(R^m))$. Then there is a positive number M such that every $f_0 \in C(K)$ can be extended to a continuous function f satisfying*

$$\|f\|_{\alpha,p} \leqq M \cdot \sup_{x \in K} |f(x)|,$$

$$\left(|f|_{\alpha,p} \leqq M \cdot \sup_{x \in K} |f(x)|\right).$$

The number M is independent of f.

For a proof see [17, Theorem 9]. When $p > 1$, the $L_\alpha^p(R^m)$-case of Theorem 2 is contained in Theorem 1.

References

1. Adams, D. R., Meyers, N. G., Bessel potentials. Inclusion relations among classes of exceptional sets, *Indiana Univ. Math. J.*, **22** (1973), 873—905.
2. Adams, D. R.; Polking, J. C., The equivalence of two definitions of capacity, *Proc. Amer. Math. Soc.*, **37** (1973), 529—534.
3. Aronszajn, N., Mulla, F., Szeptycki, P., On spaces of potentials connected with L^p-classes, *Ann. Inst. Fourier (Grenoble)*, **13** (1963), fasc. 2, 211—306.
4. Carleson, L., On the connection between Hausdorff measures and capacity, *Ark. Mat.*, 3 (1957), No. 36, 403—406.
5. Deny, J., Théorie de la capacite dans les espaces fonctionelles, *Séminarie de Théorie du Potentiel, Paris, 9éme année,* 1964—65, No. 1.

6. Deny, J., Lions, J. L., Les espaces du type de Beppo—Levi, *Ann. Inst. Fourier (Grenoble)*, **5** (1953/54), 305—370.
7. Fuglede, B., Extremal length and functional completion, *Acta Math.*, **98** (1957), 171—219.
8. Havinson, S. Ya., On the representation and approximation of functions on thin sets, (Russian), *Sovremennye problemy teorii analitičeskih funkcii* (Proc. Conf. Erevan 1965), Nauka, Moscow 1966, 314—318.
9. Kunugui, K., Étude sur la théorie du potentiel généralisé, *Osaka Math. J.*, **2** (1950), 63—103.
10. Littman, W., A connection between α-capacity and $(m-p)$-polarity, *Bull. Amer. Math. Soc.*, **73** (1967), 862—866.
11. Meyers, N. G., A theory of capacities for potentials of functions in the Lebesgue classes, *Math. Scand.*, **26** (1970), 255—292.
12. Maz'ja, V. G., Some p-conductance and theorems of imbedding certain function spaces into the space C, *Dokl. Akad. Nauk. SSSR, Tom* **140** (1961), 299—302. *Soviet Math. Dokl.* **2** (1961), 1200—1203.
13. Maz'ja, V. G., Havin, V. P., A non-linear analogue of the Newton potential and metric properties of the (p, l)-capacity, *Dokl. Akad. Nauk. SSSR, Tom* **194** (1970), 770—773, *Soviet Math. Dokl.*, **11** (1970), No. 5., 1294—1298.
14. Maz'ja, V. G., Havin, V. P., Non-linear potential theory, *Usp. Mat. Nauk.* **27:6** (1972), 67—138. *Russian Math. Surveys* **27:6** (1972), 71—148.
15. Du Plessis, N., Some theorems about the Riesz fractional integral, *Trans. Amer. Math. Soc.*, **80** (1955), 124—134.
16. Rešetnyak, Yu. G., On the concept of capacity in the theory of functions with generalized derivatives, *Sibirskii Matem. Zhur.*, **10** (1969), No. 5., 1109—1138. *Siberian Math. J.*, **10** (1969), 818—842.
17. Sjödin, T., Bessel potentials and extension of continuous functions on compact sets, *Research Report, Univ. of Umeå Sweden, No.* **1** (1973).
18. Wallin, H., Continuous functions and potential theory, *Ark. Mat.*, **5** (1963), 55—84.
19. Wallin, H., A connection between α-capacity and L^p-classes of differentiable functions, *Ark. Mat.*, **5** (1963), 331—341.
20. Wallin, H., Riesz potentials, k, p-capacity and p-modules, *Mich. Math. J.*, **18** (1971), 257—263.

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