One-sided minima of indefinite binary quadratic forms and one-sided diophantine approximations

Åke Lindgren

1. Introduction

Let

(1) $$f(x, y) = ax^2 + bxy + cy^2$$

be an indefinite binary quadratic form with real coefficients. We shall be concerned with the set of possible values of

(2) $$m_+(f) = \inf \frac{f(x, y)}{\sqrt{d}},$$

the infimum being taken over integers x, y for which $f(x, y) > 0$. Here $d = b^2 - 4ac$ is the discriminant of f. This is an analogue of the ordinary Markov spectrum, which is the set of possible values of $1/m(f)$, where $m(f) = \inf |f(x, y)|/\sqrt{d}$.

It is sometimes convenient to consider

(3) $$\lambda_+(f) = m_+(f)^{-2}.$$

We always have $\lambda_+(f) \geqq 1$ (see Cassels [1], Ch. II). Dumir [4] proved that we have no λ_+ in the open interval (96/25, 4). This is a special case of the following theorem:

Theorem 1. *There is no λ_+ in the open interval*

$$\left(\left(\frac{k+1}{k}\right)^2 - \frac{4}{k^2(k^2+2k+2)^2}, \left(\frac{k+1}{k}\right)^2\right), \qquad k = 1, 2, 3, \dots .$$

The next theorem implies that in a certain sense Theorem 1 cannot be improved.

Theorem 2. *Let*

$$u_k(x, y) = x^2 + \frac{k^2+k+2}{k^2+2k+2}\, xy - \frac{k^2+k+2}{k(k^2+2k+2)}\, y^2$$

and

$$v_k(x, y) = x^2 + \frac{k-1}{k}\, xy - \frac{1}{k}\, y^2.$$

Then

$$\lambda_+(u_k) = \left(\frac{k+1}{k}\right)^2 - \frac{4}{k^2(k^2+2k+2)^2}$$

and

$$\lambda_+(v_k) = \left(\frac{k+1}{k}\right)^2.$$

Remarks. In this work we have studied the one-sided Markov spectrum. It is defined as the set of possible values of $1/m_+(f)$ and will be denoted by B. It is not difficult to prove that B is a closed subset of $[1, \infty]$ and in [6] it is proved that B contains all real numbers x satisfying

$$x \geqq \sqrt{2}+\frac{\sqrt{5}+1}{2}.$$

Hence B is completely described by the complementary open set

$$\Omega = \left[1, \sqrt{2}+\frac{\sqrt{5}+1}{2}\right] \setminus B$$

which is a union of pairwise disjoint open intervals $\{I_1, I_2, I_3, \ldots\}$. Each interval I_j is called a gap in the one-sided Markov spectrum.

Theorem 1 and 2 prove together that there exist infinitely many gaps. However, the gaps listed in Theorem 1 are not all. For example, the interval $(24\sqrt{2}/23, 31/21)$ constitutes a gap not listed in Theorem 1.

The problem to give a complete list of gaps seems to be very difficult. In Theorem 3 below we give a general characterization of the one-sided Markov spectrum in terms of sequences of positive integers.

2. Basic lemmata

Hightower [5] dealt with the similar symmetric problem of $m(f)$, and I shall need a few lemmata, which he used. Cf. [5] for the proofs.

Lemma 1. *Let* $f(x, y)=ax^2+bxy+cy^2$ *be an indefinite form with* $m_+(f)>0$. *Then there is a form* $g(x, y)=x^2+pxy-qy^2$ *such that*

a) $\lambda_+(f) = \lambda_+(g) = p^2+4q$,

b) $0 \leqq p \leqq 1$,

c) $\inf_{g>0} g(x, y) = 1.$

Lemma 2. *If for every* (b, c) *in* $\alpha < b^2+4c < \beta$, $0 \leqq b \leqq 1$, *we have*

$$\inf_{x^2+bxy-cy^2>0} (x^2+bxy-cy^2) < 1 \quad (x \text{ and } y \text{ are integers}),$$

then there is no λ_+ *in the open interval* (α, β).

Lemma 2 follows immediately from Lemma 1.

Definition. We denote by $T(x, y)$ the points in the (b, c)-plane for which $0 < x^2 + bxy - cy^2 < 1$. (Equivalently expressed

$$\frac{x^2-1}{y^2} + \frac{x}{y} b < c < \frac{x^2}{y^2} + \frac{x}{y} b.)$$

With this notation we get immediately from Lemma 2:

Lemma 3. *There is no* λ_+ *in* (α, β) *if and only if the 2-dimensional region* $\alpha < b^2 + 4c < \beta$, $0 \leqq b \leqq 1$ *in the* (b, c)*-plane can be covered by finitely or infinitely many strips* $T(x, y)$.

Definition. $D = D(b, c) = b^2 + 4c$.

3. A diagram

The diagram of Figure 1 with a few strips $T(x, y)$ will be of great help in understanding the proof of Theorem 1.

By looking at the diagram and performing a few simple calculations we note the following facts:

a) $(b, c) = (1 - 1/k, 1/k)$ is on the border of $T(-1, 1)$, $T(1, k-1)$ and $T(1, k)$ if $k = 2, 3, 4, \ldots$.

b) $D(1-1/k, 1/k) = (1+1/k)^2$.

c) The region $96/25 < D < 4$ is covered by $T(1, 1)$, $T(-3, 2)$ and (not necessary) $T(1, 2)$. That part which lies between $T(1, 1)$ and $T(1, 2)$ is in $T(-3, 2)$.

d) The region $56/25 < D < 9/4$ is covered by $T(-1, 1)$, $T(1, 2)$, $T(-7, 6)$, $T(1, 3)$ and $T(-5, 4)$. That part which lies between $T(1, 2)$ and $T(1, 3)$ is covered by $T(-7, 6)$ and the part between $T(1, 3)$ and $T(1, 4)$ by $T(-5, 4)$.

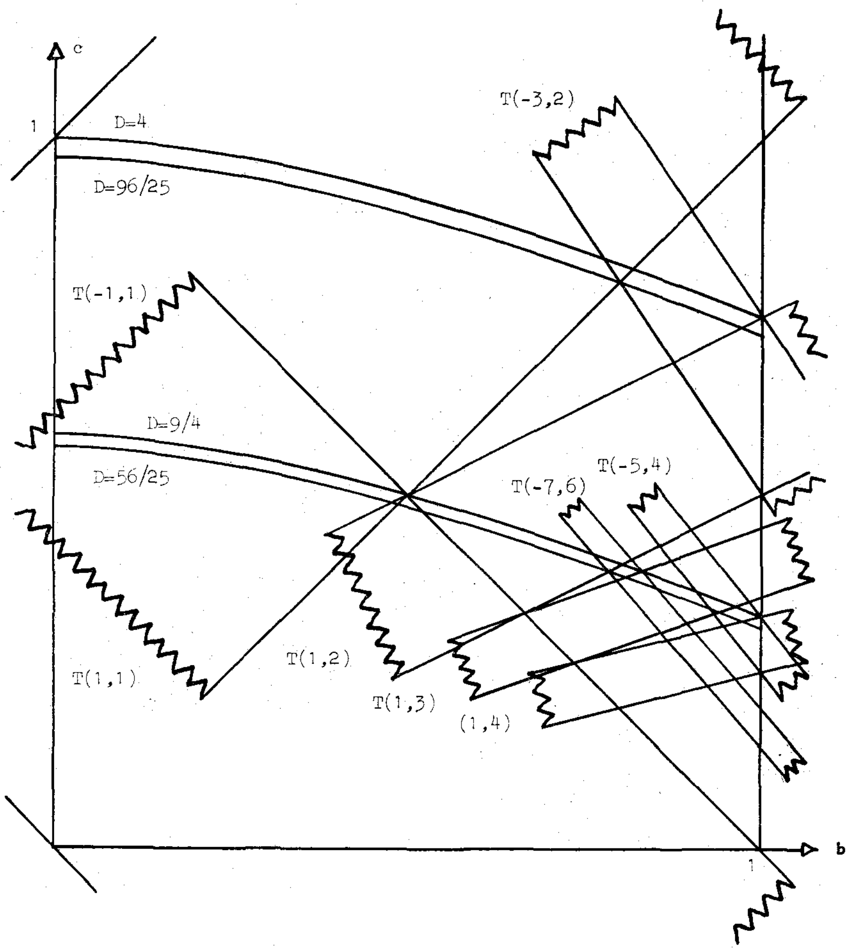

Figure 1

4. Main lemma

What we have expressed in c) and d) above can be generalized. In fact we have the following lemma, which is the essential part of the proof of Theorem 1.

Lemma 4. *That part of the region*

$$\left(\frac{k+1}{k}\right)^2 - \frac{4}{k^2(k^2+2k+2)^2} < D = b^2+4c < \left(\frac{k+1}{k}\right)^2, \quad 0 \leqq b \leqq 1,$$

which lies between $T(1, m)$ *and* $T(1, m+1)$ (*i.e. below* $T(1, m)$ *and above* $T(1, m+1)$)

is covered by $T(-(l+1), 1)$, *where*

$$l = \left[\frac{k(m+1)}{m+1-k}\right], \quad k \geqq 1, \quad k \leqq m \leqq 2k-1. \tag{4}$$

([ ] denotes the integral part.)

Proof of Lemma 4. Let the points A and B have coordinates (b_A, c_A) and $b_B, c_B)$ respectively as defined in Figure 2. It is easy to see that the proof will be complete if we can show that

$$D(A) = D(b_A, c_A) = b_A^2 + 4c_A \geqq \left(\frac{k+1}{k}\right)^2 \tag{5}$$

and

$$D(B) = D(b_B, c_B) = b_B^2 + 4c_B \leqq \left(\frac{k+1}{k}\right)^2 - \frac{4}{k^2(k^2+2k+2)^2}. \tag{6}$$

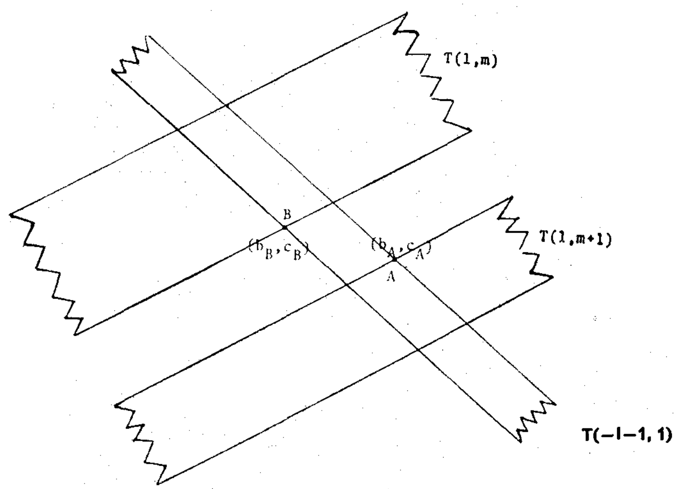

Figure 2

By the definition of $T(x, y)$ we easily get

$$c_A = \left(\frac{1}{m+1}\right)^2 + \frac{1}{m+1} b_A = \left(\frac{l+1}{l}\right)^2 - \frac{l+1}{l} b_A. \tag{7}$$

Thus we have

$$\left(\frac{l+1}{l}+\frac{1}{m+1}\right)b_A = \left(\frac{l+1}{l}\right)^2-\left(\frac{1}{m+1}\right)^2,$$

which gives

(8) $$b_A = \frac{l+1}{l}-\frac{1}{m+1}.$$

And so by (7)

(9) $$c_A = \frac{l+1}{l(m+1)}.$$

By (4)

(10) $$l \leqq \frac{k(m+1)}{m+1-k}.$$

Hence by (8), (9) and (10)

$$D(A) = \left(\frac{l+1}{l}-\frac{1}{m+1}\right)^2+4\frac{l+1}{l(m+1)} = \left(\frac{l+1}{l}+\frac{1}{m+1}\right)^2=$$
$$= \left(1+\frac{1}{l}+\frac{1}{m+1}\right)^2 \geqq \left(1+\frac{m+1-k}{k(m+1)}+\frac{1}{m+1}\right)^2 = \left(\frac{k+1}{k}\right)^2.$$

Thus we have (5).

In the same way

(11) $$c_B = \frac{1}{m}b_B = \frac{(l+1)^2-1}{l^2}-\frac{l+1}{l}b_B,$$

and so $lb_B=m(l+2)-m(l+1)b_B$, which gives

(12) $$b_B = \frac{m(l+2)}{m(l+1)+l} = \frac{m(l+2)}{l(m+1)+m}.$$

By (4)

(13) $$l \geqq \frac{k(m+1)-m+k}{m+1-k}.$$

Using this we get from (12)

(14) $$b_B = \frac{m}{m+1}\cdot\frac{(m+1)(l+2)}{l(m+1)+m} = \frac{m}{m+1}\left(1+\frac{m+2}{l(m+1)+m}\right) \leqq$$
$$\leqq \frac{m}{m+1}\left(1+\frac{m+2}{\frac{k(m+1)-m+k}{m+1-k}(m+1)+m}\right) = \frac{km^2+m^2+2m}{km^2+2km+2k}.$$

Hence by (11)

(15) $$c_B = \frac{1}{m}b_B \leqq \frac{km+m+2}{km^2+2km+2k}.$$

By (14) and (15)

$$D(B) = b_B^2 + 4c_B \leqq \frac{(km^2+m^2+2m)^2+4(km+m+2)(km^2+2km+2k)}{(km^2+2km+2k)^2}.$$

Thus

$$\left(\frac{k+1}{k}\right)^2 - D(B) \geqq \left(\frac{k+1}{k}\right)^2 - \frac{(km^2+m^2+2m)^2+4(km+m+2)(km^2+2km+2k)}{(km^2+2km+2k)^2} =$$

(16)

$$= \left(\frac{2(m+1-k)}{k(m^2+2m+2)}\right)^2.$$

Consider the function $h(x)=(x+1-k)/(x^2+2x+2)$ in the closed interval $[k, 2k-1]$, where $k>0$. We easily get

$$h'(x) = \frac{(x^2+2x+2)-(x+1-k)(2x+2)}{(x^2+2x+2)^2} = \frac{2k(x+1)-(x+1)^2+1}{(x^2+2x+2)^2} > 0.$$

Thus $h(x) \geqq h(k) = 1/(k^2+2k+2)$ if $k \leqq x \leqq 2k-1$.

Hence from (16)

$$\left(\frac{k+1}{k}\right)^2 - D(B) \geqq \left(\frac{2(m+1-k)}{k(m^2+2m+2)}\right)^2 \geqq \frac{4}{k^2(k^2+2k+2)^2}, \tag{17}$$

so we get (6) as well and the proof is complete.

5. An admissible point

As we could see from the proof there is equality in the second step of (17) if and only if $m=k$. In this case there is equality in (13), (14), (15) and (16) too, i.e.

$$(b_B, c_B) = \left(\frac{k^2+k+2}{k^2+2k+2}, \frac{k^2+k+2}{k(k^2+2k+2)}\right).$$

By Theorem 2 this is an admissible point, i.e. it is not contained in any strip $T(x, y)$.

6. A remark

It is easy to see that we need consider m only between k and $2k-1$ in Lemma 4. If for instance (b, c) is situated below $T(1, 2k)$, then we have

$$D(b, c) = b^2+4c \leqq 1^2+4\cdot\frac{1}{2k} = 1+\frac{2}{k} < \left(\frac{k+1}{k}\right)^2 - \frac{4}{k^2(k^2+2k+2)^2}.$$

If on the other hand, (b, c) lies above $T(1, k)$ and $T(-1, 1)$ we have

$$D(b, c) \geqq \left(1-\frac{1}{k}\right)^2+4\cdot\frac{1}{k} = \left(\frac{k+1}{k}\right)^2.$$

With this remark we might regard Theorem 1 as having been fully proved.

7. Proof of the second part of Theorem 2

We have

$$v_k(x, y) = x^2+\frac{k-1}{k}xy-\frac{1}{k}y^2 = (x+y)\left(x-\frac{1}{k}y\right).$$

We may suppose that $x\geqq 0$.

a) $x=0$. Then we have

$$v_k(0, y) = -\frac{1}{k}y^2 \leqq 0.$$

b) $x>0$. Then $v_k(x, y)$ is a function of y of the second degree, that vanishes for $y=-x$ and $y=kx$. v_k is positive if and only if $-x<y<kx$. Thus v_k takes its smallest positive value for integral y when $y=-x+1$ and $y=kx-1$. Hence if $v_k(x, y)>0$ we also have

$$v_k(x, y) \geqq v_k(x, -x+1) = v_k(x, kx-1) = x+\frac{x-1}{k} \geqq x \geqq 1.$$

Thus we see that

$$\inf_{v_k>0} v_k(x, y) = 1.$$

So by (2) $m_+(v_k) = k/(k+1)$ and by (3) $\lambda_+(v_k) = (1+1/k)^2$.

8. A lattice corresponding to f(x, y)

To be able to prove the first part of Theorem 2, we need to point out a few things. The indefinite binary quadratic form $f(x, y)=ax^2+bxy+cy^2$ is equivalent to a so called reduced form

$$f_0(x, y) = a_0(x-\varphi_0 y)(x-\theta_0 y), \tag{18}$$

where $\varphi_0>0$ and $-1<\theta_0<0$. We suppose that φ_0 and θ_0 are irrational, and have the regular continued fraction expansions $\varphi_0=[g_0, g_1, g_2, \ldots]$ and $-\theta_0=[0, g_{-1}, g_{-2}, \ldots]$.

Every reduced form equivalent to f is of the form

$$f_i(x, y) = a_i(x-\varphi_i y)(x-\theta_i y),$$

where

(19) $$\varphi_i = [g_i, g_{i+1}, g_{i+2}, \dots]$$

and

(20) $$-\theta_i = [0, g_{i-1}, g_{i-2}, \dots].$$

We may say that to f corresponds a unique doubly infinite sequence

$$\{\dots, g_{-2}, g_{-1}, g_0, g_1, g_2, \dots\}$$

of positive integers. f and f_i, being equivalent, have the same discriminant. Thus

$$d = b^2-4ac = a_i^2(\varphi_i-\theta_i)^2$$

or

(21) $$|a_i| = \frac{\sqrt{d}}{\varphi_i-\theta_i} = \frac{\sqrt{d}}{\gamma_i},$$

where $\gamma_i = [g_i, g_{i+1}, \dots]+[0, g_{i-1}, g_{i-2}, \dots]$.

These are all well-known facts, see e.g. Dickson [3], Ch. 7.

Further, the absolute infimum of the numbers representable by f is equal to $\inf_i |a_i|$. But we can say more than this, namely

(22) $$\begin{cases} \inf\limits_{f>0} f(x, y) = \inf\limits_{a_i>0} a_i \\ \sup\limits_{f<0} f(x, y) = \sup\limits_{a_i<0} a_i. \end{cases}$$

The numbers a_i are alternately positive and negative. Thus if we suppose $a_0>0$, (22) is equivalent to

(23) $$\begin{cases} \inf\limits_{f>0} f(x, y) = \inf\limits_{i} a_{2i} \\ \sup\limits_{f<0} f(x, y) = \sup\limits_{i} a_{2i+1}. \end{cases}$$

We can see this by considering the fact that to every binary quadratic form corresponds a 2-dimensional lattice, see e.g. Delone—Faddeev [2], Suppl. 1. To the form $f_i(x, y)=a_i(x-\varphi_i y)(x-\theta_i y)$ corresponds the lattice Λ, with lattice points (ξ, η), where

(24) $$\begin{cases} \xi = \alpha(x-\theta_i y) \\ \eta = \beta(x-\varphi_i y), \end{cases}$$

$\alpha\beta=a_i$ and x and y take all integral values. The pair of vectors, (α, β) and $(-\alpha\theta_i, -\beta\varphi_i)$, constitutes a basis of the lattice. Performing the transformation

(25) $$\begin{cases} x = g_i x'+y' \\ y = x', \end{cases}$$

we get by using (19) and (20)

$$f_i(x, y) = f_{i+1}(x', y').$$

Since (by ordinary matrix representation)

$$\begin{pmatrix}\alpha & -\alpha\theta_i\\ \beta & -\beta\varphi_i\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}\alpha & -\alpha\theta_i\\ \beta & -\beta\varphi_i\end{pmatrix}\begin{pmatrix}g_i & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}x'\\ y'\end{pmatrix} = \begin{pmatrix}\alpha(g_i-\theta_i) & \alpha\\ \beta(g_i-\varphi_i) & \beta\end{pmatrix}\begin{pmatrix}x'\\ y'\end{pmatrix} = \begin{pmatrix}-\dfrac{\alpha}{\theta_{i+1}} & \alpha\\ -\dfrac{\beta}{\varphi_{i+1}} & \beta\end{pmatrix}\begin{pmatrix}x'\\ y'\end{pmatrix},$$

the transformation (25) means passing over to the new basis $(-(\alpha/\theta_{i+1}), -(\beta/\varphi_{i+1}))$ and (α, β). According to the general theory ([3], Ch. 7), the lattice points $(-\alpha\theta_i, -\beta\varphi_i)$, (α, β) and $(-(\alpha/\theta_{i+1}), -(\beta/\varphi_{i+1}))$ lie on the hyperbolas $\xi\eta=a_{i-1}$, $\xi\eta=a_i$ and $\xi\eta=a_{i+1}$ respectively. This process can be continued in both directions to yield a doubly infinite sequence of so called relative minima ...A, B, C, ... (see Figure 3 and cf. [2], Suppl. 1). As there is no lattice point inside the triangle OAC, there is no such point in the infinite sector between the lines OA and OC, that yields a better (smaller) value of $\xi\eta$, than the points A and C. An infinity of such sectors fill the first quadrant.

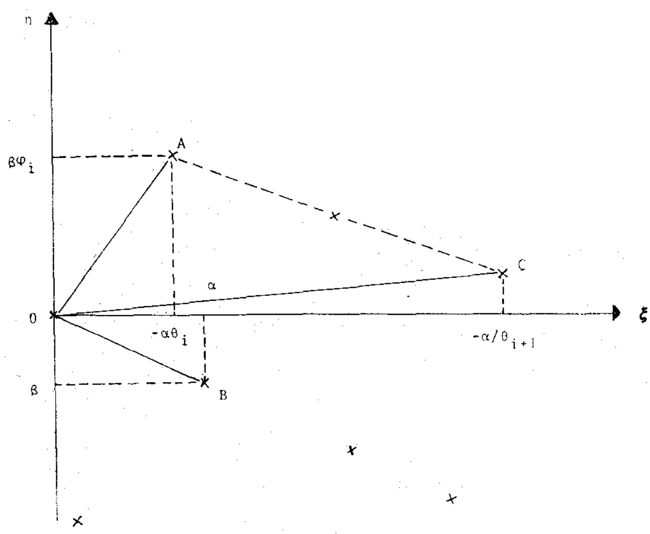

Figure 3

So in order to find $\inf_{\xi\eta>0, (\xi,\eta)\in\Lambda} \xi\eta$ it is sufficient to consider only the positive a_i. By (24) $f_i(x, y)=\xi\eta$ and so we have the first part of (23). The second part is completely similar. Finally (21) and (23) give

$$(26)\qquad \begin{cases} \inf\limits_{f>0} f(x,y) = \sqrt{d}/\sup\limits_i \gamma_{2i} \\ \sup\limits_{f<0} f(x,y) = -\sqrt{d}/\sup\limits_i \gamma_{2i+1}. \end{cases}$$

(This is almost equivalent to what Robinson [8], §3 has shown.)

9. A lemma

If $f=ax^2+bxy+cy^2$ has rational coefficients, we can formulate the above in a lemma, which is of some interest in itself.

Lemma 5. *Let ε be an irrational root of the equation $ax^2+bx+c=0$, where a, b and c are rational. Let $\varepsilon=[a_1, \dots, a_m, \overline{b_1, \dots, b_{2n}}]$ be the periodic regular continued fraction expansion of ε.* (The length of the period can always be made even by taking the primitive period twice if necessary.) *Then there are two integers i and j, $1 \leqq i,\ j \leqq 2n$, such that if*

$$\frac{s_k}{t_k} = [a_1, \dots, a_m, b_1, \dots, b_k]$$

are the convergents of ε, then

$$\inf_{f>0} f(x,y) = f(s_i, t_i)$$

and

$$\sup_{f<0} f(x,y) = f(s_j, t_j).$$

Furthermore,

$$\gamma_{i+1} = \max(\dots, \gamma_{i-1}, \gamma_{i+1}, \gamma_{i+3}, \dots) \quad \textit{if} \ \ 1 \leqq i < 2n$$

and

$$\gamma_1 = \max(\gamma_1, \gamma_3, \dots, \gamma_{2n-1}) \quad \textit{if} \quad i = 2n,$$

where

$$\gamma_k = [\overline{b_k, \dots, b_{2n}, b_1, \dots, b_{k-1}}] + [0, \overline{b_{k-1}, \dots, b_1, b_{2n}, \dots, b_k}]$$

if $1<k \leqq 2n$ and similarly for γ_1. Analogously $\gamma_{j+1}=\max(\dots, \gamma_{j-1}, \gamma_{j+1}, \dots)$ etc.; i and j are not both even or both odd.

We only have to note that there is an equivalent reduced form, whose root (i.e. φ_0 in (18)) has a purely periodic continued fraction expansion. The corresponding doubly infinite sequence is also purely periodic (cf. Perron [7], § 23).

Hightower [5] used the symmetric version of this lemma, however accidently erroneously stated.

10. Concluding the proof of Theorem 2

We are now able to prove the first part of Theorem 2, by proving that $\inf_{u_k>0} u_k(x, y)=1$. $u_k(\varepsilon, 1)=0$ has the root

$$\varepsilon = \frac{\sqrt{(k^3+3k^2+4k+2)^2-4}-(k^3+k^2+2k)}{2k(k^2+2k+2)} = [0, \overline{k, 1, k(k+1), 1}].$$

By Lemma 5 $\inf_{u_k>0} u_k(x, y)=u_k(s, t)$, where $s/t=[0, k]=1/k$. (We understand of course that $(s, t)=1$.) Thus

$$\inf_{u_k>0} u_k(x, y) = u_k(1, k) = 1$$

and the proof of Theorem 2 is complete.

11. Lattice points on the axises

From (2), (3) and (26) we get

$$\sqrt{\lambda_+(f)} = \sup_i \gamma_{2i}$$

and

$$\sqrt{\lambda_+(-f)} = \sup_i \gamma_{2i+1}.$$

This was the case when φ_0 and θ_0 in (18) were irrational. What if we have

$$f(x, y) = a(x-\varphi y)(x-\theta y), \tag{27}$$

where φ or θ or both are rational?

Let us suppose that $\varphi=r/s$ is rational, where $(r, s)=1$. If $a=0$ in (1) there is always an equivalent form g with $a\neq 0$, so we can assume that we have f in the form (27). There are integers t and u such that

$$rt+su = 1. \tag{28}$$

From (27) and (28) we get

$$f(rx'+uy', sx'-ty') = \frac{a}{s}(srx'+suy'-rsx'+rty')(rx'+uy'-s\theta x'+t\theta y') =$$

$$= \frac{a(r-s\theta)}{s} y' \left(x' - \frac{t\theta+u}{s\theta-r} y'\right) = a'y'(x'-\theta' y').$$

$r-s\theta\neq 0$ because $\theta\neq\varphi$ by $d=a^2(\varphi-\theta)^2>0$, since f is indefinite. Hence if we write

$$h(x, y) = y(x-\theta' y) \tag{29}$$

we have $\lambda_+(f)=\lambda_+(h)$ or $\lambda_+(f)=\lambda_+(-h)$ according to a' being positive or negative.

Let the regular continued fraction expansion of θ' be

$$\theta' = [g_0, g_1, g_2, \dots]. \tag{30}$$

The lattice corresponding to (29) has the basis (0, 1), $(1, -\theta')$ (cf. (24) and Figure 4).

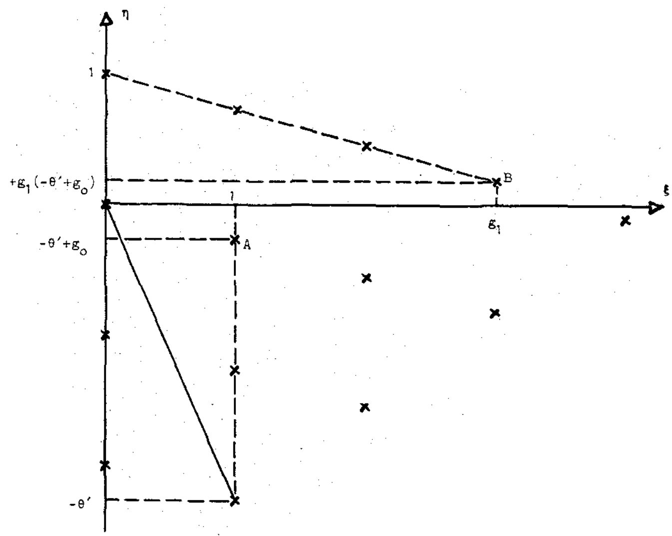

Figure 4

We have

$$h(g_0, 1) = 1\cdot(g_0-\theta') = -[0, g_1, g_2, \dots] = -\frac{1}{[g_1, g_2, \dots]} = \frac{-1}{\gamma_1} \tag{31}$$

and

$$h(g_0g_1+1, g_1) = g_1(g_0g_1+1-\theta' g_1) = \frac{1}{[g_2, g_3, \dots]+\dfrac{1}{g_1}} = \frac{1}{\gamma_2} \tag{32}$$

(see e.g. Perron [7], § 14).

(31) and (32) correspond to the points A and B of Figure 4. Let the coordinates be (ξ_A, η_A) and (ξ_B, η_B) respectively. From the construction of A and B we have $\xi_A=1$, $-1<\eta_A<0$, $\xi_B=g_1$ and $0<\eta_B<-\eta_A$ (if g_1 is not the last partial quotient of θ'). We easily see (by elementary geometry) that there are no points of the lattice that give smaller positive values of $\xi\eta$ than B, or larger negative values than A, with smaller $\xi=y$.

If θ' is irrational we can form an infinite chain of relative minima, in this case as in § 8. Thus in the same way we get

$$(33)\qquad \begin{cases} \inf\limits_{h>0} h(x, y) = \dfrac{1}{\sup\limits_i \gamma_{2i}} \\ \\ \sup\limits_{h<0} h(x, y) = -\dfrac{1}{\sup\limits_i \gamma_{2i+1}} \end{cases}$$

where $\gamma_k = [g_k, g_{k+1}, \dots] + [0, g_{k-1}, \dots, g_1]$. (Note that h has discriminant $d=1$.)
(We also see that $\underline{\lim}_{x, y\to\infty, h>0}\, h(x, y) = 1/\overline{\lim}_{i\to\infty}\, \gamma_{2i}$ etc., cf. [8], § 3.)

The relations (33) are true also if θ' is rational but not integral. Then the lattice corresponding to (29) has points on the ξ-axis. (30) now takes the form

$$\theta' = [g_0, g_1, \dots, g_{k-1}, g_k].$$

We can always achieve $g_k = 1$.

The chain of successive relative minima runs as above up to the last but one, i.e. we get a basis A, B of the lattice such that $A + g_k B = A + B = J$ is on the ξ-axis. Since A and B are relative minima, the rectangles $CAGH$ and $CDBH$ (see Figure 5) are void of lattice points (cf. [2], Suppl. 1). By the symmetry and periodicity of the lattice, $CEFH$ has not any lattice points in its interior either.

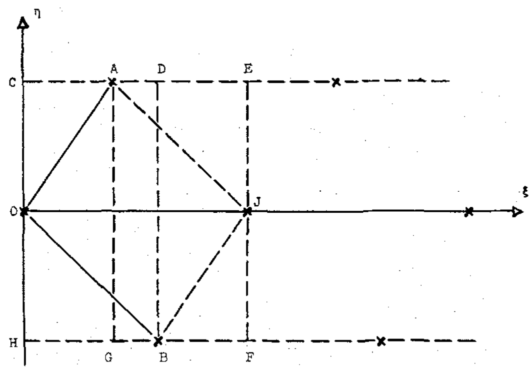

Figure 5

So there are no lattice points between the lines AG and EF, or between the lines DB and EF, that give smaller absolute values of h than A or B respectively. There are no such points to the right of EF either, since if we extend the lines CE and HF to the right there are only points on the ξ-axis between these lines. So we have (33) in this case too.

Finally if θ' is integral we easily see that $\inf_{h>0} h(x, y) = 1$ and $\sup_{h<0} h(x, y) = -1$, so we have in this case $\lambda_+(h) = \lambda_+(-h) = 1$.

We make a summary in the following theorem.

Theorem 3. *The set B of possible values of $\sqrt{\lambda_+(f)}$ is the set of possible values of* $\sup_i \gamma_{2i}$, *where* $\gamma_k=[g_k, g_{k+1}, \ldots]+[0, g_{k-1}, \ldots]$. *The sequence*

$$G = \{\ldots, g_0, g_{-1}, g_1, g_2, \ldots\}$$

of positive integers may be finite, singly infinite or doubly infinite.

We easily see that to every sequence G, there corresponds a quadratic form h, such that (33) holds in the same way as above.

Examples. a) $G=\{k, 1\}$. Then $\gamma_1=k+1$ and $\gamma_2=1+1/k$. This gives $\lambda_+= =(1+1/k)^2$ as a possible value. So we have a different proof of the second part of Theorem 2.

b) $G=\{\ldots, 1, k, 1, k(k+1), 1, k, 1, \ldots\}$. Suitable placing of indices yields

$$\sup_i \gamma_{2i} = [\overline{1, k, 1, k(k+1)}]+[0, \overline{k(k+1), 1, k, 1}] = \sqrt{\left(\frac{k+1}{k}\right)^2-\frac{4}{k^2(k^2+2k+2)^2}}.$$

12. One-sided diophantine approximation

Let A be the set of possible values of

$$\max\left(\sup_i \gamma_{2i},\ a \sup_i \gamma_{2i+1}\right)$$

and A' the set of possible values of

$$\max\left(\overline{\lim_{i\to\infty}}\ \gamma_{2i},\ a \overline{\lim_{i\to\infty}}\ \gamma_{2i+1}\right)$$

where a is an arbitrary positive number and γ_k is defined as before from the doubly infinite sequence $G=\{\ldots, g_{-1}, g_0, g_1, g_2, \ldots\}$ of positive integers.

Tornheim [9] has shown that $A \supseteqq A'$.

Let B be the set of Theorem 3 and B' the set of values of $\overline{\lim}_{i\to\infty} \gamma_{2i}$.

Theorem 4. $B=B'$.

Proof. We use the notations $G=\{\ldots, g_{-1}, g_0, g_1, g_2, \ldots\}$,

$$G' = \{\ldots, g'_{-1}, g'_0, g'_1, g'_2, \ldots\}, \quad \gamma_k = [g_k, g_{k+1}, \ldots]+[0, g_{k-1}, \ldots]$$

and

$$\gamma'_k = [g'_k, g'_{k+1}, \ldots]+[0, g'_{k-1}, \ldots].$$

a) $B \subseteqq B'$.

Suppose $\sup_i \gamma_{2i}=\gamma_k$ for some k. If G is doubly infinite we choose

$$G'=\{\ldots, g_k, 1, g_{k-2}, g_{k-1}, g_k, g_{k+1}, g_{k+2}, 2, g_{k-4}, \ldots, g_{k+4}, 3, g_{k-6}, \ldots, g_{k+6}, 4, \ldots\}.$$

If G is finite, i.e. $\gamma_k=[g_k, \ldots, g_{k+n}]+[0, g_{k-1}, \ldots, g_{k-n}]$, we can always make n and m even since $[\ldots, h, 1]=[\ldots, h+1]$. Then we choose

$$G' = \{\ldots, g_{k-m}, \ldots, g_{k+n}, 1, g_{k-m}, \ldots, g_{k+n}, 2, g_{k-m}, \ldots, g_{k+n}, 3, \ldots\}.$$

If G is singly infinite we mix these two procedures in the obvious way. It is easy to see that in all three cases

$$\overline{\lim_{i\to\infty}}\, \gamma'_{2i} = \gamma_k.$$

If on the other hand $\sup_i \gamma_{2i}=\overline{\lim}_{i\to\infty} \gamma_{2i}$, then there is nothing to prove.

b) $B'\subseteqq B$.

Let $b=\overline{\lim}_{i\to\infty} \gamma_{2i}$. Thus there is a subsequence $\gamma_{2i_0}\to b$. Then there is a subsequence such that all g_{2i_1} are equal, say to g'_0. If $g'_0\neq b$ we cannot have $g_{2i_1-1}\to\infty$ and $g_{2i_1+1}\to\infty$. Thus there is a subsequence, such that, say all g_{2i_2+1} are equal. Denote this number by g'_1. Then there is also a subsequence such that all g_{2i_3+2} are equal, say to g'_2. If this process can be continued in both directions, we will have $\sup_i \gamma'_{2i}=b$. If at some place in the process of finding subsequences we have $g_{2i_n+2m}=g'_{2m}$ and $g_{2i_n+2m+1}\to\infty$, then we choose $G'=\{\ldots, g'_{-1}, g'_0, \ldots, g'_{2m}\}$. With the same situation at the left tail we choose for G' the corresponding finite sequence. In all cases we get $\sup_i \gamma'_{2i}=b$.

The proof will now be complete by noting that we can have $\overline{\lim}_{i\to\infty} \gamma_{2i}=\infty$, and hence also $\sup_i \gamma_{2i}=\infty$.

References

1. Cassels, J. W. S., *An introduction to the geometry of numbers*, Berlin, 1959.
2. Delone, B. N. and Faddeev, D. K., *The theory of irrationalities of the third degree*, Providence, 1964.
3. Dickson, L. E., *Introduction to the theory of numbers*, Chicago, 1929.
4. Dumir, V. C., On positive values of indefinite binary quadratic forms, *Math. Student*, **38** (1970), 177—182.
5. Hightower, C. J., The minima of indefinite binary quadratic forms, *J. Number Theory*, **2** (1970), 364—378.
6. Lindgren Å., Asymmetric and one-sided minima of indefinite binary quadratic forms, *Uppsala University, Dep. of Math., Report*, **6** (1975).
7. Perron, O., *Die Lehre von den Kettenbrüchen I*, Stuttgart, 1954.
8. Robinson, R. M., Unsymmetrical approximation of irrational numbers, *Bull. Amer. Math. Soc.*, **53** (1947), 351—361.
9. Tornheim, L., Asymmetric minima of quadratic forms and asymmetric Diophantine approximation, *Duke Math. J.*, **22** (1955), 287—294.

Received June 10, 1974

Åke Lindgren
Department of Mathematics
Sysslomansgatan 8
Uppsala, Sweden