On the convergence almost everywhere of double Fourier series

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Let

$$S_{mn}f(x) = \sum_{k=-m}^{m} \sum_{l=-n}^{n} c_{kl} e^{i(kx_1+lx_2)}$$

denote the partial sums of the Fourier series of a function $f \in L^1(\mathbf{T}^2)$ where $\mathbf{T}=[0, 2\pi]$. It was proved by C. Fefferman [4], P. Sjölin [7] and N. R. Tevzadze [8] that if $p>1$ and $f \in L^p(\mathbf{T}^2)$, then $\lim_{n\to\infty} S_{nn}f(x)$ exists almost everywhere. The method of Fefferman and Tevzadze also shows that if $(m_k)_{k=1}^{\infty}$ and $(n_k)_{k=1}^{\infty}$ are non-decreasing sequences of integers which tend to infinity and $f \in L^2(\mathbf{T}^2)$, then $\lim_{k\to\infty} S_{m_k n_k} f(x)$ exists almost everywhere.

Fefferman [5] also constructed a counterexample which shows that there exists a continuous function f with period 2π in each variable such that $\lim_{m,n\to\infty} S_{mn}f(x)$ exists nowhere. In [7] Sjölin proved that if

$$\sum_{m,n} |c_{mn}|^2 (\log(\min(|m|, |n|)+2))^2 < \infty, \tag{1}$$

then $\lim_{m,n\to\infty} S_{mn}f(x, y)$ exists almost everywhere. From (1) convergence conditions involving the modulus of continuity of f can be obtained. For continuous functions f with period 2π in each variable we set

$$\omega(f; \delta) = \sup_{|x-y| \leqq \delta} |f(x)-f(y)|.$$

It is then known that if

$$\omega(f; \delta) = O\big((\log \delta^{-1})^{-1-\varepsilon}\big), \quad \delta \to 0, \tag{2}$$

for some $\varepsilon>0$, then (1) holds (see Bahbuh [1]). On the other hand it can be proved by use of Fefferman's counterexample that there exists an f with $\omega(f; \delta)=$

$=O((\log 1/\delta)^{-1})$, such that $\lim_{m,n\to\infty} S_{mn}f(x)$ does not exist almost everywhere (see Bahbuh and Nikishin [2]).

The purpose of this paper is to investigate the convergence of $S_{mn}f$ for functions satisfying conditions of the type $\omega(f;\delta)=O((\log 1/\delta)^{-\alpha})$, where $0<\alpha<1$.

We need the following notation. If $f\in L^2(\mathbf{T}^2)$ we extend f to a function on $\mathbf{R}^2$ with period 2π in each variable and set

$$\Delta f(x,t) = f(x_1+t_1, x_2+t_2)-f(x_1, x_2+t_2)-f(x_1+t_1, x_2)+f(x_1, x_2),$$

$$x\in\mathbf{T}^2, \quad |t|\leqq 1,$$

$$\omega'(f;\delta) = \sup_{|t|\leqq\delta}\|\Delta f(\cdot,t)\|_{L^\infty(\mathbf{T}^2)}$$

and

$$\omega_2'(f;\delta) = \sup_{|t|\leqq\delta}\|\Delta f(\cdot,t)\|_{L^2(\mathbf{T}^2)}.$$

We shall prove the following theorem.

Theorem 1. *Assume* $0<\alpha<1$ *and let* $(m_k)_1^\infty$ *and* $(n_k)_1^\infty$ *be non-decreasing sequences of positive integers with* $\lim_{k\to\infty} m_k = \lim_{k\to\infty} n_k = \infty$.

Set

$$\Gamma_k = \{(m,n)\in\mathbf{Z}^2;\ \max(|m-m_k|, |n-n_k|) \leqq e^{(\log\min(m_k,n_k))^\alpha}\} \tag{3}$$

and $\Gamma=\bigcup_{k=1}^\infty \Gamma_k$. *Then the following holds.*

(i) *If* $f\in L^2(\mathbf{T}^2)$ *and*

$$\int_0^1 \omega_2'(f;\delta)^2\delta^{-1}(\log\delta^{-1})^{2\alpha-1}d\delta<\infty, \tag{4}$$

then $\lim_{m,n\to\infty,\,(m,n)\in\Gamma} S_{mn}f(x)$ *exists almost everywhere.*

(ii) *There exists an* $f\in C(\mathbf{T}^2)$ *with period* 2π *in each variable and*

$$\omega'(f;\delta) = O((\log\delta^{-1})^{-\alpha}), \quad \delta\to 0, \tag{5}$$

such that $\lim_{m,n\to\infty,\,(m,n)\in\Gamma} S_{mn}f(x)$ *does not exist almost everywhere.*

The result in (ii) shows that the exponent $2\alpha-1$ in (4) cannot be replaced by a smaller number.

We first prove the following lemma.

Lemma. *If* $0<\alpha\leqq 1$ *and* $f\in L^2(\mathbf{T}^2)$, *then*

$$\sum_{m,n}|c_{mn}|^2(\log(\min(|m|,|n|)+2))^{2\alpha}<\infty \tag{6}$$

if and only if

$$\int_{|t|\leqq 1}\Big(\int_{T^2}|\Delta f(x,t)|^2\,dx\Big)|t|^{-2}(\log|t|^{-1})^{2\alpha-1}dt<\infty. \tag{7}$$

Proof. It follows from Parseval's relation that the inner integral in (7) equals $4\pi\sum|c_{mn}|^2|e^{imt_1}-1|^2|e^{int_2}-1|^2$ and to prove the lemma it is sufficient to prove that

$$C_1(\log m)^{2\alpha}\leqq\int_{|t|\leqq 1}|e^{imt_1}-1|^2|e^{int_2}-1|^2|t|^{-2}(\log|t|^{-1})^{2\alpha-1}dt\leqq C_2(\log m)^{2\alpha} \tag{8}$$

for $3\leqq m\leqq n$, where C_1 and C_2 are positive constants. The integral in (8) is larger than

$$c\int_{1/m}^{1/2}\Big(\int_{t_1}^{1/2}|t|^{-2}(\log|t|^{-1})^{2\alpha-1}dt_2\Big)\,dt_1\geqq c\int_{1/m}^{1/2}\Big(\int_{t_1}^{1/2}t_2^{-2}(\log t_2^{-1})^{2\alpha-1}dt_2\Big)\,dt_1\geqq$$

$$\geqq c\int_{1/m}^{1/2}t_1^{-1}(\log t_1^{-1})^{2\alpha-1}dt_1\geqq c\,(\log m)^{2\alpha},$$

where c denotes positive constants. Thus the left inequality in (8) is proved. To prove the remaining inequality we observe that the integral in (8) is majorized by

$$C\int_{1/m}^{1/2}\int_0^{1/2}|t|^{-2}(\log|t|^{-1})^{2\alpha-1}dt_1\,dt_2+Cm^2\int_0^{1/m}\int_{1/n}^{1/2}t_1^2|t|^{-2}(\log|t|^{-1})^{2\alpha-1}dt_1dt_2+$$

$$+Cm^2n^2\int_0^{1/m}\int_0^{1/n}t_1^2t_2^2|t|^{-2}(\log|t|^{-1})^{2\alpha-1}dt_1\,dt_2=I_1+I_2+I_3.$$

We have

$$I_1\leqq C\int_{1/m\leqq|t|\leqq 1}|t|^{-2}(\log|t|^{-1})^{2\alpha-1}dt=C\int_{1/m}^{1}\delta^{-1}(\log\delta^{-1})^{2\alpha-1}d\delta\leqq C(\log m)^{2\alpha}.$$

For $1/2<\alpha\leqq 1$ we have

$$I_2\leqq Cm^2\int_0^{1/m}t_1^2(\log t_1^{-1})^{2\alpha-1}\Big(\int_{1/n}^{1/2}|t|^{-2}\,dt_2\Big)\,dt_1\leqq Cm^2\int_0^{1/m}t_1(\log t_1^{-1})^{2\alpha-1}dt_1\leqq$$

$$\leqq C(\log m)^{2\alpha-1},$$

and for $0<\alpha\leqq 1/2$

$$I_2\leqq Cm^2\int_0^{1/m}t_1^2\Big(\int_{1/n}^{1/2}|t|^{-2}\,dt_2\Big)\,dt_1\leqq Cm^2\int_0^{1/m}t_1\,dt_1=C.$$

Finally

$$I_3\leqq Cm^2n^2\int_0^{1/m}\int_0^{1/n}t_1t_2(\log|t|^{-1})^{2\alpha-1}dt_1\,dt_2,$$

and for $\alpha>1/2$ we obtain

$$I_3\leqq C\Big(m^2\int_0^{1/m}t_1(\log t_1^{-1})^{2\alpha-1}dt_1\Big)\Big(n^2\int_0^{1/n}t_2\,dt_2\Big)\leqq C(\log m)^{2\alpha-1}.$$

For $\alpha \leqq 1/2$ we get

$$I_3 \leqq C\left(m^2 \int_0^{1/m} t_1\,dt_1\right)\left(n^2 \int_0^{1/n} t_2\,dt_2\right) = C,$$

which completes the proof of the lemma.

Proof of Theorem 1.

(i) If $D_n(u)$ denotes the Dirichlet kernel then $|D_k(u)-D_m(u)| \leqq 2\pi/|u|$ and also $|D_k(u)-D_m(u)| \leqq \pi |k-m|$. It follows that

$$|S_k g(x) - S_m g(x)| \leqq C \log(|k-m|+2)\, g^*(x), \quad x \in \mathbf{T}, \tag{9}$$

where $S_k g$ and $S_m g$ are partial sums of the Fourier series of a function $g \in L^1(\mathbf{T})$ and g^* denotes the Hardy—Littlewood maximal function of g. For $f \in L^1(\mathbf{T}^2)$ we define

$$M_1 f(x) = \sup_{x_1 \in \omega} \frac{1}{|\omega|} \int_\omega |f(t_1, x_2)|\,dt_1, \quad x \in \mathbf{T}^2,$$

where ω denotes subintervals of $\mathbf{T}$,

$$S_1^* f(x) = \sup_n \left|\int_{\mathbf{T}} D_n(x_1 - t_1) f(t_1, x_2)\,dt_1\right|, \quad x \in \mathbf{T}^2,$$

and M_2 and S_2^* in the same way with the variables interchanged.

If $(m, n) \in \Gamma_k$ we write

$$S_{mn} f(x) - S_{m_k n_k} f(x) = S_{mn} f(x) - S_{m n_k} f(x) + S_{m n_k} f(x) - S_{m_k n_k} f(x),$$

and invoking (9) we obtain

$$|S_{mn} f(x) - S_{m_k n_k} f(x)| \leqq C \log(|n-n_k|+2)\, M_2 S_1^* f(x) + C \log(|m-m_k|+2)\, M_1 S_2^* f(x).$$

From the definition of Γ_k it follows that the right hand side in the above inequality is majorized by

$$C\big(\log(\min(m_k, n_k)+2)\big)^\alpha \big(M_2 S_1^* f(x) + M_1 S_2^* f(x)\big).$$

We therefore have

$$|S_{mn} f(x)| \leqq |S_{m_k n_k} f(x)| + C\big(\log(\min(m, n)+2)\big)^\alpha \big(M_2 S_1^* f(x) + M_1 S_2^* f(x)\big).$$

Defining

$$T_\Gamma f(x) = \sup_{(m,n) \in \Gamma} \frac{|S_{mn} f(x)|}{(\log(\min(m, n)+2))^\alpha}$$

we obtain

$$T_\Gamma f(x) \leqq \sup_k |S_{m_k n_k} f(x)| + C\big(M_2 S_1^* f(x) + M_1 S_2^* f(x)\big).$$

It is proved in Fefferman [4] and Tevzadze [8] that the L^2 norm of the first term on the right hand side is majorized by $C\|f\|_2$ and it follows from the L^2 inequality for the Hardy—Littlewood maximal function in one variable that M_1 and M_2 are bounded on $L^2(\mathbf{T}^2)$. Also S_1^* and S_2^* are bounded on $L^2(\mathbf{T}^2)$ since the maximal partial sum operator in one variable is bounded on $L^2(\mathbf{T})$ according to the results of L. Carleson [3] and R. A. Hunt [6]. Hence T_Γ is bounded on $L^2(\mathbf{T}^2)$.

Now let $f\in L^2(\mathbf{T}^2)$ have Fourier coefficients c_{mn} and assume that (6) holds. We set

$$S_\Gamma f(x) = \sup_{(m,n)\in\Gamma} |S_{mn}f(x)|$$

and let g denote the function in $L^2(\mathbf{T}^2)$ which has Fourier coefficients $c_{mn}(\log(\min(|m|,|n|)+2))^\alpha$. Performing a partial summation as in the proof of Theorem 7.2 in Sjölin [7], pp. 85—86, we obtain

$$S_\Gamma f(x) \leqq C(Pg(x)+T_\Gamma g(x)),$$

where P is a bounded operator on $L^2(\mathbf{T}^2)$. Hence

$$\|S_\Gamma f\|_2 \leqq C\|g\|_2 = C(\sum |c_{mn}|^2(\log(\min(|m|,|n|)+2))^{2\alpha})^{1/2}.$$

It follows that $\lim_{m,n\to\infty;\,(m,n)\in\Gamma} S_{mn}f(x)$ exists almost everywhere for each f with Fourier coefficients satisfying (6) and hence by the lemma for each f satisfying (7). To complete the proof of (i) we observe that (7) holds if $\omega_2'(f;\delta)$ satisfies (4).

(ii) Choose $\varphi\in C^\infty(\mathbf{R})$ so that $\varphi(t)=1$ for $1/20\leqq t\leqq 2\pi-1/20$, and $\varphi(t)=0$ for t close to 0 and 2π, and set $h_\lambda(x)=e^{i\lambda x_1x_2}\varphi(x_1)\varphi(x_2)$ for $x\in\mathbf{T}^2$ and $\lambda\geqq 10$. Set $Q=\{x\in\mathbf{T}^2;\ 1/10\leqq x_1, x_2\leqq 2\pi-1/10\}$. Fefferman [5] has proved that

$$|S_{[\lambda x_2],[\lambda x_1]}h_\lambda(x)| \geqq c\log\lambda, \quad x\in Q, \tag{10}$$

where c is a positive constant. The function h_λ can be used to construct the counter-examples mentioned in the introduction. To prove (ii) we shall use a function obtained by multiplying h_λ with a character. We set $\mu_k=(m_k,n_k)$,

$$\lambda_k = \frac{1}{10}e^{(\log\min(m_k,n_k))^\alpha}$$

and

$$g_k(x) = e^{i\mu_k\cdot x}h_{\lambda_k}(x), \quad x\in\mathbf{T}^2,$$

$k=1,2,3,\ldots$. Also set $\mu_k^{(1)}=(m_k,-n_k)$, $\mu_k^{(2)}=(-m_k,n_k)$ and $\mu_k^{(3)}=(-m_k,-n_k)$. We have

$$\sum\nolimits_{|l_1-\mu_1|\leqq m,\ |l_2-\mu_2|\leqq n} \hat{g}_k(l)e^{il\cdot x} = e^{i\mu\cdot x}S_{mn}(e^{-i\mu\cdot x}g_k)(x), \quad \mu\in\mathbf{Z}^2, \tag{11}$$

where $\hat{g}_k(l)$ denotes the Fourier coefficients of g_k. We now take $x\in Q$, $m=[\lambda_k x_2]$, $n=[\lambda_k x_1]$ and $\mu=\mu_k$, $\mu_k^{(1)}$, $\mu_k^{(2)}$ and $\mu_k^{(3)}$ in (11) and add the corresponding four equalities. We then obtain

$$S_{m_k+m,n_k+n}g_k(x)+S_{m_k-m-1,n_k-n-1}g_k(x)-S_{m_k+m,n_k-n-1}g_k(x)-S_{m_k-m-1,n_k+n}g_k(x)=$$
$$= e^{i\mu_k\cdot x}S_{mn}h_{\lambda_k}(x)+\sum_{j=1}^{3}e^{i\mu_k^{(j)}\cdot x}S_{mn}(e^{i(\mu_k-\mu_k^{(j)})\cdot x}h_{\lambda_k})(x). \tag{12}$$

We have

$$|\mu_k-\mu_k^{(j)}|\geqq \min(m_k,n_k)=e^{(\log 10\lambda_k)^{1/\alpha}},\quad j=1,2,3,$$

and it follows from a partial integration in the integral defining Fourier coefficients that

$$|(e^{i(\mu_k-\mu_k^{(j)})\cdot x}h_{\lambda_k})^{\wedge}(l)|\leqq C\lambda_k e^{-(\log 10\lambda_k)^{1/\alpha}},\quad |l_1|\leqq m,\quad |l_2|\leqq n.$$

Hence

$$|S_{mn}(e^{i(\mu_k-\mu_k^{(j)})\cdot x}h_{\lambda_k})(x)|\leqq C\lambda_k^3 e^{-(\log 10\lambda_k)^{1/\alpha}}\leqq C,\quad j=1,2,3.$$

From this estimate and (10) it follows that for $k>k_0$ the right hand side of (12) has absolute value larger than $c\log\lambda_k$ and hence at least one of the terms on the left hand side has absolute value larger than $c\log\lambda_k$, where c denotes positive constants. We have chosen m and n so that the indices of the partial sums on the left hand side of (12) belong to Γ_k and hence we have proved that for $x\in Q$ and $k>k_0$ there exists $\varrho_k=\varrho_k(x)\in\Gamma_k$ such that $|S_{\varrho_k}g_k(x)|\geqq c\log\lambda_k$, where $c>0$.

We now choose an increasing sequence of integers $(k_j)_{j=1}^{\infty}$ so that $k_1>k_0$ and

$$\|S_{mn}g_{k_j}-g_{k_j}\|_\infty\leqq 2^{-i},\quad j=1,2,\dots,i-1,\quad (m,n)\in\Gamma_{k_i}$$

and

$$\min(m_{k_i},n_{k_i})\geqq e^{(\log\max(m_{k_{i-1}},n_{k_{i-1}}))^{3/\alpha}}, \tag{13}$$

for $i=2,3,4,\dots$. This can be done since $S_{mn}g_k$ tends to g_k uniformly for each k. We set $f=\sum_{j=1}^{\infty}c_j g_{k_j}$, where $c_j=(\log\lambda_{k_j})^{-1}$, and shall prove that f has the desired properties.

It is clear that $\omega'(g_k;\delta)\leqq C\min(m_k,n_k)\delta$ and choosing i as the least integer such that

$$e^{(\log 10\lambda_{k_i})^{1/\alpha}}\geqq 1/\delta,$$

we obtain

$$\omega'(f;\delta)\leqq\sum_{j=1}^{\infty}c_j\omega'(g_{k_j};\delta)\leqq C\sum_{j=1}^{i-1}c_j\min(m_{k_j},n_{k_j})\delta+C\sum_{j=i}^{\infty}c_j\leqq$$
$$\leqq C\delta\sum_{j=1}^{i-1}(\log\lambda_{k_j})^{-1}e^{(\log 10\lambda_{k_j})^{1/\alpha}}+Cc_i\leqq$$
$$\leqq C\delta(\log\lambda_{k_{i-1}})^{-1}e^{(\log 10\lambda_{k_{i-1}})^{1/\alpha}}+C(\log\lambda_{k_i})^{-1}.$$

From the choice of i it follows that the last term on the right hand side is less than $C(\log 1/\delta)^{-\alpha}$ and it also follows that

$$e^{(\log 10\lambda_{k_i-1})^{1/\alpha}} \leqq 1/\delta.$$

Using this inequality it is easy to prove that the first term can be majorized in the same way and hence $\omega'(f;\delta) \leqq C(\log 1/\delta)^{-\alpha}$.

We let $x \in Q$ and $\varrho_{k_i} = \varrho_{k_i}(x)$ and write

$$S_{\varrho_{k_i}} f(x) - f(x) = c_i\big(S_{\varrho_{k_i}} g_{k_i}(x) - g_{k_i}(x)\big) + \sum_{j=1}^{i-1} c_j\big(S_{\varrho_{k_i}} g_{k_j}(x) - g_{k_j}(x)\big) +$$

$$+ \sum_{j=i+1}^{\infty} c_j\big(S_{\varrho_{k_i}} g_{k_j}(x) - g_{k_j}(x)\big) = A_1 + A_2 + A_3.$$

From the above estimates it follows that

$$|A_1| \geqq c c_i \log \lambda_{k_i} = c, \quad c > 0,$$

$$|A_2| \leqq (i-1)2^{-i},$$

and we also have

$$|A_3| \leqq \sum_{j=i+1}^{\infty} c_j \|S_{\varrho_{k_i}} g_{k_j} - g_{k_j}\|_\infty \leqq C(\log \max(m_{k_i}, n_{k_i}))^2 \sum_{j=i+1}^{\infty} c_j \|g_{k_j}\|_\infty \leqq$$

$$\leqq C(\log \max(m_{k_i}, n_{k_i}))^2 c_{i+1}.$$

(13) yields

$$\log 10\lambda_{k_{i+1}} \geqq (\log \max(m_{k_i}, n_{k_i}))^3,$$

and hence A_3 tends to zero as i tends to infinity. Also A_2 tends to zero and we conclude that

$$|S_{\varrho_{k_i}} f(x) - f(x)| \geqq c > 0$$

for $x \in Q$ and i large. Hence there exists a set of positive measure on which $\lim_{m,n\to\infty,\,(m,n)\in\Gamma} S_{mn} f(x)$ does not exist. The proof is complete.

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