On temperate fundamental solutions supported by a convex cone

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Let $P(D)$ be a partial differential operator in $\mathbf{R}^n$ with constant coefficients and Γ a closed convex cone in $\mathbf{R}^n$. Thus we assume that $x, y \in \Gamma$ and $s, t \geqq 0$ implies that $sx+ty \in \Gamma$. The problem discussed here is to decide when $P(D)$ has a temperate fundamental solution with support in Γ.

An arbitrary differential operator with constant coefficients has a temperate fundamental solution. This was first proved by Hörmander [5] and Lojasiewicz [6]. Later Atiyah [1] has given a shorter proof using results of Hironaka [3].

In [7] and [8] Melrose has given necessary and sufficient conditions for the existence of a temperate fundamental solution with support in a half space. In this paper we give necessary and sufficient conditions in the case of an arbitrary closed convex cone Γ. The existence of a temperate fundamental solution does not immediately imply the existence of a temperate solution to the equation $P(D)U=F$, where F is temperate. Therefore, we prefer to discuss this more general problem directly.

The intersection $\Gamma \cap (-\Gamma) = W$ is a linear subspace and $x \in \Gamma$ implies $x+y \in \Gamma$ for every $y \in W$. This shows that Γ is the inverse image in $\mathbf{R}^n$ of the image V of Γ in $\mathbf{R}^n/W$ under the quotient map. It is clear that V is a proper cone. It follows from Theorem 2.11 in [2] that there is no restriction in assuming that Γ (and V) has interior points. Thus we shall assume this later on. We shall use the notations $n'=\dim W$, $n''=n-n'$ and coordinates $x=(x', x'')$ such that W is defined by $x''=0$. We will also need the following norms on $\mathscr{S}(\mathbf{R}^n)$,

$$\|u\|_s = \Big(\int (1+|x|^2)^s \sum_{|\alpha| \leqq s} |D^\alpha u|^2\, dx\Big)^{1/2}, \quad u \in \mathscr{S}(\mathbf{R}^n),$$

$$\|u\|_s^{\Gamma_-} = \inf\{\|\varphi\|_s;\ \mathscr{S} \ni \varphi = u \text{ on } \Gamma_- = -\Gamma\}, \quad u \in \mathscr{S}(\mathbf{R}^n).$$

Theorem 1. *The following conditions on $\Gamma = \mathbf{R}^{n'} \times V$ and the differential operator $P(D)$ are equivalent.*

(i) *$P(D)$ has a temperate fundamental solution with support in Γ.*

(ii) *$P(D)U=F$ has a solution $U\in\mathscr{S}'(\mathbf{R}^n)$ with* supp $U\subset\Gamma$ *for every $F\in\mathscr{S}'(\mathbf{R}^n)$ with* supp $F\subset\Gamma$.

(iii) *For every $\xi'\in\mathbf{R}^{n'}$ either $P(\xi',\zeta'')\neq 0$ if* $\operatorname{Im}\zeta''\in -\operatorname{int} V^*$, *where $V^*=\{\eta''\in\mathbf{R}^{n''};\ \langle\eta'',x''\rangle\geqq 0$ for all $x''\in V\}$, or $P(\xi',\zeta'')=0$ for all $\zeta''\in\mathbf{C}^{n''}$.*

Proof. It is trivial that (i) follows from (ii). We will now use Theorem 3 below to prove that (iii) implies (ii). Since $F\in\mathscr{S}'(\mathbf{R}^n)$ and supp $F\subset\Gamma$, there are constants C and s such that $|\check{F}(u)|=|\check{F}(\varphi)|\leqq C\|\varphi\|_s$ for all $u\in\mathscr{S}(\mathbf{R}^n)$ and all $\varphi\in\mathscr{S}(\mathbf{R}^n)$, $\varphi=u$ on Γ_-. This implies that $|\check{F}(u)|\leqq C\|u\|_s^{\Gamma_-}$ for all $u\in\mathscr{S}(\mathbf{R}^n)$. From Theorem 3 we obtain that there are constants C_0 and s_0 such that

$$|\check{F}(u)|\leqq C_0\|P(D)u\|_{s_0}^{\Gamma_-},\quad \text{for all}\quad u\in\mathscr{S}(\mathbf{R}^n).$$

Thus it follows from the Hahn—Banach theorem that the linear form

$$P(D)\mathscr{S}(\mathbf{R}^n)\ni P(D)u\mapsto \check{F}(u)$$

can be extended to a continuous linear form $\check{U}$ on $\mathscr{S}(\mathbf{R}^n)$ with supp $\check{U}\subset\Gamma_-=-\Gamma$. Thus $U\in\mathscr{S}(\mathbf{R}^n)$, $P(D)U=F$ and supp $U\subset\Gamma$.

Now, assume that (iii) is false. Thus there are $\xi_0'\in\mathbf{R}^{n'}$ and $\zeta_0''\in\mathbf{C}^{n''}$ such that $P(\xi_0',\zeta_0'')=0$, $\operatorname{Im}\zeta_0''\in-\operatorname{int}V^*$ but $P(\xi_0',\zeta'')\neq 0$ for some $\zeta''\in\mathbf{C}^{n''}$. Take N'' such that $P(\xi_0',\zeta_0''+tN'')\neq 0$ for some $t\in\mathbf{C}$ and write $q(\xi',t)=P(\xi',\zeta_0''+tN'')$ as a product of irreducible factors $q(\xi',t)=\Pi q_i(\xi',t)$. Let $b(\xi')$ be the coefficient of the term of highest degree with respect to t of the polynomial q. Denote by $\Delta(\xi')$ the product of b and the discriminants of the factors q_i considered as polynomials of t. Since the zeros of a polynomial depend continuously on the coefficients and $q(\xi_0',0)=0$ we can choose a closed ball $B\subset\mathbf{R}^{n'}$ with positive radius and centre near ξ_0' and a function $B\ni\xi'\mapsto t(\xi')\in\mathbf{C}$, such that $q(\xi',t(\xi'))=0$ and $\operatorname{Im}(\zeta_0''+t(\xi')N'')\in-\operatorname{int}V^*$ if $\xi'\in B$. Moreover, we can assume that $\Delta(\xi')\neq 0$ in B so that $t(\xi')$ can be chosen analytic in B. Thus we have an analytic function $B\ni\xi'\mapsto\zeta''(\xi')=(\zeta_0''+t(\xi')N'')\in\mathbf{C}^{n''}$ such that $P(\xi',\zeta''(\xi'))=0$ and $\operatorname{Im}\zeta''(\xi')\in-\operatorname{int}V^*$ for all $\xi'\in B$. Let $w\in C_0^\infty(B)$, $0\neq w\geqq 0$ and let $\varphi\in C^\infty(\mathbf{R}^{n''})$ be 1 in a neighbourhood of $V_-=-V$, $\varphi(x'')=0$ if $d(x'',V_-)\geqq 1$ and assume that φ has bounded derivatives. Set

$$v(x)=\varphi(x'')\int_B e^{i\langle x',\xi'\rangle+i\langle x'',\zeta''(\xi')\rangle}w(\xi')\,d\xi'.$$

There are constants C and $\delta>0$ such that $|(D^\alpha\varphi)(x'')e^{i\langle x'',\zeta''(\xi')\rangle}|\leqq C_\alpha e^{-\delta|x''|}$ if $\xi'\in\operatorname{supp}w$. From this we obtain by means of partial integration that $D^\beta x^\gamma v(x)$ is bounded so that $v\in\mathscr{S}(\mathbf{R}^n)$. Further, $P(D)v=0$ on $\Gamma_-=\mathbf{R}^{n'}\times V_-$. If $E\in\mathscr{S}'(\mathbf{R}^n)$ is a fundamental solution of $P(D)$ with supp $E\subset\Gamma$, then $v(0)=\check{E}(P(D)v)=0$. However, $v(0)=\int w(\xi')\,d\xi'\neq 0$ and this contradiction proves that (i) implies (iii). #

We will now state a theorem which shows that condition (iii) implies a condition "stronger" than (i).

Theorem 2. *Let P be a polynomial satisfying condition* (iii) *of Theorem* 1. *Then there are constants C and s' and temperate distributions $E=E(t, P)$ such that $P(D)E=\delta_t$ and $|\check{E}(u)|\leqq C\|u\|_{s'}^{\Gamma_-}$ for all $u\in\mathcal{S}(\mathbf{R}^n)$ and all $t\in\Gamma_-$.*

For the proof of this theorem we need some preliminaries, so we postpone it. Instead we will prove now that Theorem 2 implies the required estimate.

Theorem 3. *Let P be a polynomial satisfying condition* (iii) *of Theorem* 1. *Then for every s there are constants C and s_0 such that*

$$\|u\|_s^{\Gamma_-} \leqq C\|P(D)u\|_{s_0}^{\Gamma_-} \quad \textit{for all} \quad u\in\mathcal{S}(\mathbf{R}^n).$$

Proof. First we observe that for some s_1

$$\|\psi\|_s \leqq C_1\|\hat{\psi}\|_s = C_1\Big(\int(1+|\xi|^2)^s \textstyle\sum_{|\alpha|\leqq s}|D^\alpha\hat{\psi}(\xi)|^2\,d\xi\Big)^{1/2}\leqq$$
$$\leqq C_2\sup\big((1+|x^2|)^{s_1/2}\textstyle\sum_{|\alpha|\leqq s_1}|D^\alpha\psi(x)|\big),$$

which implies that

$$\|u\|_s^{\Gamma_-} \leqq C\inf\big\{\sup\big((1+|x|^2)^{s_1/2}\textstyle\sum_{|\alpha|\leqq s_1}|D^\alpha\psi(x)|\big);\ \mathcal{S}\ni\psi=u \text{ on } \Gamma_-\big\}.$$

However, since Γ_- is regular in the sence of Whitney (see e.g., [5]) we obtain from this that there is an integer s_2 such that

(1) $$\|u\|_s^{\Gamma_-} \leqq C\sup_{\Gamma_-}\big((1+|x|^2)^{s_2/2}\textstyle\sum_{|\alpha|\leqq s_2}|D^\alpha u(x)|\big).$$

If β is a multi-index and $P(\xi)\neq 0$ then $D_\xi^\beta\hat{u}(\xi)=D_\xi^\beta\big(P(\xi)\hat{u}(\xi)/P(\xi)\big)=$ $=\big(L(\xi, D_\xi)P(\xi)\hat{u}(\xi)\big)/(P(\xi))^{|\beta|+1}$, which shows that we have an identity of the form $(P(D))^{|\beta|+1}x^\beta u(x)=L(D, x)P(D)u(x)$, where $L(D, x)$ is a differential operator with polynomial coefficients. We observe that $P^{|\beta|+1}$ also satisfies condition (iii) of Theorem 1. Let $t\in\Gamma_-$ and let $E=E(t, P^{|\beta|+1})$ be the distribution we obtain from Theorem 2 applied to $P^{|\beta|+1}$. Then $t^\beta u(t)=\check{E}\big((P(D))^{|\beta|+1}x^\beta u\big)=\check{E}\big(L(D, x)P(D)u\big)$, which implies that

$$\sup_{\Gamma_-}|t^\beta u(t)| \leqq C_1\|L(D, x)P(D)u\|_{s'}^{\Gamma_-} \leqq C_2\|P(D)u\|_{s''}^{\Gamma_-}.$$

If we apply this to $D^\alpha u$ for all α and β with $|\alpha|\leqq s_2$ and $|\beta|\leqq s_2$ then we obtain that there are constants C_0 and s_0 such that

$$\sup_{\Gamma_-}\big((1+|t|^2)^{s_2/2}\textstyle\sum_{|\alpha|\leqq s_2}|D^\alpha u(t)|\big) \leqq C_0\|P(D)u\|_{s_0}^{\Gamma_-},$$

which proves the theorem by means of (1). #

For the hard part of the proof of Theorem 2 we need the following theorem due to Hironaka.

Theorem 4. *Let F be a real analytic function ($\neq 0$), defined in a neighbourhood of $0\in\mathbf{R}^n$. Then there exists an open set $U\ni 0$, a real analytic manifold $\tilde{U}$ and a proper analytic map $\varphi\colon\tilde{U}\to U$ such that*

(i) *$\varphi\colon\tilde{U}\setminus\tilde{A}\to U\setminus A$ is an isomorphism, where $A=F^{-1}(0)$ and $\tilde{A}=\varphi^{-1}(A)=(F\circ\varphi)^{-1}(0)$,*

(ii) *for each $P\in\tilde{U}$ there are local analytic coordinates $(y_1,\dots,y_n)$ centred at P so that, locally near P, we have*

$$F\circ\varphi = \varepsilon(y)\prod_1^n y_i^{k_i},$$

where ε is an invertible analytic function and $k_i\geqq 0$.

Proof. See Atiyah [1]. #

We will now use Theorem 4 to prove the following lemma. The proof is a slight modification of the proof Melrose gave in [7].

Lemma 5. *If Q is a polynomial in k variables and $A=\{\xi\in\mathbf{R}^k;\ Q(\xi)=0\}$, then there is a constant C and an integer s such that, if $\psi\in\mathcal{S}(\mathbf{R}^k)$ and ψ/Q is bounded on $\mathbf{R}^k\setminus A$ then*

$$\sup_{\mathbf{R}^k\setminus A}|\psi(\xi)/Q(\xi)| \leqq C\|\psi\|_s.$$

Proof. If $\operatorname{supp}\psi\subset B=\{\xi\in\mathbf{R}^k;\ |\xi|\leqq 2\}$ and $q(\xi)=\prod_1^k \eta_i^{k_i}$ then it is trivial that $\sup|\psi(\xi)/q(\xi)|\leqq C\sup\sum_{|\alpha|\leqq s}|D^\alpha\psi(\xi)|$ for some constants C and s. Now, let U be a small neighbourhood of a point in $\mathbf{R}^n$, so that Theorem 4 can be applied with $F=Q$ and assume that $\operatorname{supp}\psi\subset U$. If φ is the map we obtain from Theorem 4 then $\psi\circ\varphi$ has compact support. Thus we obtain from condition (ii) of Theorem 4 that we can choose a finite partition of unity on $\tilde{U}$, $1=\sum\chi_i$, so that for suitable coordinates $Q\circ\varphi=\varepsilon(\eta)\prod_1^k\eta_j^{k_j}$ in $\operatorname{supp}\chi_i$. Then the simple case above implies that

$$\sup_\xi|\psi(\xi)/Q(\xi)| = \sup|\psi\circ\varphi/Q\circ\varphi| \leqq \sum\sup_\eta|\chi_i(\eta)(\psi\circ\varphi)(\eta)/Q\circ\varphi(\eta)| \leqq$$

$$\leqq C_0\sup_{i,\eta}\sum_{|\alpha|\leqq s}\left|D_\eta^\alpha(\chi_i(\psi\circ\varphi))(\eta)\right| \leqq C_1\sup_\xi\sum_{|\alpha|\leqq s}|D^\alpha\psi(\xi)| \leqq C\|\psi\|_{s_1}$$

where the last estimate follows from the Sobolev inequality. From this we obtain the lemma for all ψ with $\operatorname{supp}\psi\subset B$ by means of a finite partition of unity. Now, let $\chi\in C_0^\infty(B)$ be 1 in a neighbourhood of $\{\xi\in\mathbf{R}^k;\ |\xi|\leqq 1\}$. If $\psi\in\mathcal{S}(\mathbf{R}^k)$ we set $\psi_1=\chi\psi$ and $\psi_2=\psi-\psi_1$. Then there are constants C_1 and s_1 such that

$$\sup_{\mathbf{R}^k\setminus A}|\psi_1(\xi)/Q(\xi)| \leqq C_0\|\psi_1\|_{s_1} \leqq C_1\|\psi\|_{s_1}.$$

Further, set $\varphi(\eta)=\psi_2(\eta/|\eta|^2)|\eta|^{2m}$ and $q(\eta)=Q(\eta/|\eta|^2)|\eta|^{2m}$, where $m=\deg Q$. Then $\varphi\in C_0^\infty(B)$, q is a polynomial in η and φ/q is bounded. Thus, there are constants C_2 and s_2 such that

$$\sup_{\mathbf{R}^k\setminus A}|\psi_2(\xi)/Q(\xi)| = \sup_{\mathbf{R}^k\setminus q^{-1}(0)}|\varphi(\eta)/q(\eta)| \leqq C_2'\|\varphi\|_{s_2} \leqq C_2''\|\psi_2\|_{s_2+k} \leqq C_2\|\psi\|_{s_2+k}.$$

This proves the lemma with $C=C_1+C_2$ and $s=\max(s_1, s_2+k)$. #

Let P be a polynomial satisfying condition (iii) of Theorem 1 and let $\zeta_0''\in\mathbf{C}^{n''}$ with $\operatorname{Im}\zeta_0''\in-\operatorname{int}V^*$. Set $s=s'+2n'$ where s' is the integer we obtain from Lemma 5 with $Q(\xi')=P(\xi',\zeta_0'')$. If we complete $\mathscr{S}(\mathbf{R}^{n'})$ with respect to the norm $\|\cdot\|_s$, then we obtain a Hilbert space $\mathscr{S}_{(s)}(\mathbf{R}^{n'})$. Let Π denote the orthogonal projection of $\mathscr{S}_{(s)}(\mathbf{R}^{n'})$ on the subspace that is the closure of those $\psi\in\mathscr{S}(\mathbf{R}^{n'})$ for which $(1+|\xi'|^2)^{n'}\psi(\xi')/Q(\xi')$ is bounded.

Take $0<\varepsilon<1$, $t\in\mathbf{R}^n$ and define $E_\varepsilon=E_\varepsilon(t,P)$ by

$$\check{E}_\varepsilon(u) = (2\pi)^{-n}\int e^{i\langle t,\xi\rangle}(\Pi\hat{u})(\xi)/P(\xi-i\varepsilon N)\,d\xi,\quad u\in\mathscr{S}(\mathbf{R}^n),$$

where $N=(0, N'')$ and $N''\in\operatorname{int}V^*$. From Lemma 4.1.1 in Hörmander [4] we obtain that

$$|P(\xi-i\varepsilon N)| \leqq \tilde{P}(\xi-i\varepsilon N) \leqq C_1(1+|\xi''|)^m\tilde{P}(\xi',\zeta_0'') \leqq$$
$$\leqq C_2(1+|\xi''|)^m|P(\xi',\zeta_0'')| \leqq C_2(1+|\xi''|)^m\tilde{P}(\xi',\zeta_0'') \leqq$$
$$\leqq C_3(1+|\xi''|)^{2m}\tilde{P}(\xi-i\varepsilon N) \leqq C_4\varepsilon^{-m}(1+|\xi''|)^{2m}|P(\xi-i\varepsilon N)|,$$

where $\sim$ (see page 35 in Hörmander [4]) is taken with respect to the ξ'' variables, $m=\deg_{\xi''}P(\xi)$ and the constants are independent of ξ'. Thus, there is a constant $C>0$ such that

$$C^{-1}(1+|\xi''|)^{-m} \leqq |P(\xi',\zeta_0'')/P(\xi-i\varepsilon N)| \leqq C\big((1+|\xi''|)/\varepsilon\big)^m \tag{2}$$

for all $\xi\in\mathbf{R}^n$. Further, we obtain from Lemma 5 that there is a constant C_0 such that

$$\sup_{\xi'}\left|(1+|\xi'|^2)^{n'}(\Pi\hat{u})(\xi)/P(\xi',\zeta_0'')\right| \leqq C_0\|(\Pi\hat{u})(\cdot,\xi'')\|_s \leqq$$
$$\leqq C_0\|\hat{u}(\cdot,\xi'')\|_s = C_0\Big(\int(1+|\xi'|^2)^s\sum_{|\alpha|\leqq s,\,\alpha''=0}|D^\alpha\hat{u}(\xi)|^2\,d\xi'\Big)^{1/2}$$

where the norm is taken with respect to the ξ' variables only. This implies that there are constants C and s_1 such that

$$\sup_{\xi}\left|(1+|\xi''|^2)^{n''+m/2}(1+|\xi'|^2)^{n'}(\Pi\hat{u})(\xi)/P(\xi',\zeta_0'')\right| \leqq C\|u\|_{s_1}.$$

Thus E_ε is well-defined and $|\check{E}_\varepsilon(u)|\leqq C\varepsilon^{-m}\|u\|_{s_1}$. It also follows from (2) and the definition of E_ε that $\check{E}_\varepsilon\big(P(D-i\varepsilon N)u\big)=u(t)$ if $u\in\mathscr{S}(\mathbf{R}^n)$. We finally want to prove that $\operatorname{supp} E_\varepsilon \subset \Gamma - \{t\}$. Let $\theta = (0, \theta'') \in \mathbf{R}^n$, where $\theta'' \in \operatorname{int} V^*$. Take $v \in C_0^\infty(\mathbf{R}^n)$ with $\operatorname{supp} v \subset \{x \in \mathbf{R}^n; \langle x, \theta \rangle > 0\} + \{t\}$. Then

$$\check{E}_\varepsilon(v) = (2\pi)^{-n} \int e^{i\langle t, \xi \rangle} (\Pi \hat{v})(\xi)/P(\xi - i\varepsilon N)\, d\xi =$$

$$= (2\pi)^{-n} \int e^{i\langle t', \xi' \rangle} \Big(\int e^{i\langle t'', \xi'' \rangle} (\Pi \hat{v})(\xi)/P(\xi - i\varepsilon N)\, d\xi'' \Big)\, d\xi'.$$

However, since $P(\xi', D'' - i\varepsilon N'')$ is hyperbolic with respect to V for almost every $\xi' \in \mathbf{R}^{n'}$ we obtain by changing the integration contour that the inner integral is 0 a.e. (Cf. The proof of Theorem 5.6.1 in Hörmander [4].) Thus $\check{E}_\varepsilon(v) = 0$, which proves that $\operatorname{supp} E_\varepsilon \subset \Gamma - \{t\}$.

Proof of Theorem 2. Set $E = E(t, P) = e^{\langle x + t, \varepsilon N \rangle} E_\varepsilon$, where E_ε is the distribution defined above. Then

$$\check{E}(u) = (2\pi)^{-n} \int e^{i\langle t, \xi - i\varepsilon N \rangle} (\Pi \hat{u})(\xi - i\varepsilon N)/P(\xi - i\varepsilon N)\, d\xi, \quad u \in C_0^\infty(\mathbf{R}^n),$$

and since $(\Pi \hat{u})(\xi)$ is analytic with respect to ξ'' we obtain that E is independent of $\varepsilon > 0$. It is also clear that $\operatorname{supp} E \subset \Gamma - \{t\}$ and $\check{E}(P(D)u) = u(t)$ if $u \in C_0^\infty(\mathbf{R}^n)$. Let $0 \leqq \lambda \in C_0^\infty((-2, 2))$ with $\lambda(y) = 1$ for $|y| \leqq 1$ and set

$$\chi_j(x) = \lambda(\langle x, N \rangle + j - 1) / \textstyle\sum_1^\infty \lambda(\langle x, N \rangle + k - 1).$$

Then

$$\check{E}(u) = \textstyle\sum_1^\infty \check{E}(\chi_j u) = \sum_1^\infty e^{\langle t, N/j \rangle} \check{E}_{1/j}(e^{-\langle x, N/j \rangle} \chi_j u) \quad \text{if} \quad u \in C_0^\infty(\mathbf{R}^n).$$

However,

$$|e^{\langle t, N/j \rangle} \check{E}_{1/j}(e^{-\langle x, N/j \rangle} \chi_j u)| \leqq C e^{\langle t, N/j \rangle} j^m \|e^{-\langle x, N/j \rangle} \chi_j u\|_{s_1} \leqq$$

$$\leqq C_1 (e^{\langle t, N \rangle} + 1) j^m \|\chi_j u\|_{s_1} \leqq C_2 (1 + e^{\langle t, N \rangle}) j^{-2} \|u\|_{s'},$$

where $s' = s_1 + m + 2$. This proves that

$$|\check{E}(u)| \leqq C(1 + e^{\langle t, N \rangle}) \|u\|_{s'}, \quad u \in \mathcal{S}(\mathbf{R}^n),$$

so that

$$|\check{E}(u)| \leqq 2C \|u\|_{s'} \quad \text{for all} \quad u \in \mathcal{S}(\mathbf{R}^n) \quad \text{and all} \quad t \in \Gamma_-.$$

Thus if $t \in \Gamma_-$ and $\mathcal{S}(\mathbf{R}^n) \ni \psi = u$ on Γ_-, then

$$|\check{E}(u)| = |\check{E}(\psi)| \leqq 2C \|\psi\|_{s'}$$

which implies that

$$|\check{E}(u)| \leqq 2C \|u\|_{s'}^{\Gamma_-}, \quad u \in \mathcal{S}(\mathbf{R}^n), \quad t \in \Gamma_-.$$

This proves Theorem 2.

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