The convergence of Padé approximants to series of Stieltjes

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1. Introduction

A series of Stieltjes is a (formal) power series $f(z)=\sum_{n=0}^{\infty} c_n(-z)^n$ where $c_n=\int_0^\infty t^n\,d\alpha(t)$ for some real, bounded, nondecreasing function $\alpha(t)$ assuming infinitely many values on $t\geqq 0$. These functions were first studied by Stieltjes who proved that the moment problem on $[0, \infty[$ associated with $\{c_n\}_{n=0}^\infty$ is determinate if and only if the corresponding continued fractions expansion of $f(z)$ converges except on the negative real axis. (Stieltjes [10].) This theorem is also of significance for the theory of Padé approximation (Padé [8]).

Definition. The rational function $P_n(z)/Q_m(z)$ is called the $[n, m]$ Padé approximant to the formal power series $f(z)$ if P_n and Q_m are polynomials of degree at most n and m, respectively, $Q_m\not\equiv 0$ and the formal product $Q_m\cdot f-P_n$ only contains terms of degree greater than $n+m$. In the sequel the $[n, m]$ Padé approximant to f is denoted $f[n, m](z)$.

If $f(z)$ is a series of Stieltjes then the $2n$:th convergent of the corresponding continued fractions expansion is just $f[n-1, n](z)$ and thus Stieltjes' theorem gives a condition assuring that

$$f[n-1, n](z) \to g(z) = \int_0^\infty \frac{d\alpha(t)}{1+zt}$$

except on the negative real axis. The convergence of $\{f[n-1, n](z)\}_0^\infty$ and the determinacy of the moment problem are also connected with the convergence problem of the Gauss—Jacobi quadrature procedure for the measure $d\alpha$. This connection is indicated by the fact that $f[n-1, n](z)$ is exactly the n:th order Gaussian quadrature approximation to $g(z)$ defined above (see e.g. Perron [9] p. 200—201).

Following Stieltjes, many authors have discussed conditions on $\{c_n\}_0^\infty$ ensuring that $f[n-1, n](z) \to g(z)$. We mention here Carleman [4], who proved that $\sum_0^\infty (1/c_n)^{1/2n} = \infty$ is sufficient. If $f(z)$ has a positive radius of convergence R, $\alpha(t)$ is constant for $t > 1/R$ and $f(z) = \int_0^{1/R} d\alpha(t)/(1+zt)$. In this case Carleman's condition is clearly fulfilled and $f[n-1, n](z) \to f(z)$. It is also possible in this case to estimate the rate of convergence; Gragg [6] has proved for z not in the interval $I =]-\infty, -R]$

$$|f(z) - f[n-1, n](z)| \leqq C(z) \left| \frac{1-\sqrt{1+z/R}}{1+\sqrt{1+z/R}} \right|^{2n}$$

where $C(z)$ is bounded in any compact set disjoint from I.

The theory presented here was inspired by a variety of problems and the results on Padé approximation are without exception implicit in the respective work. It therefore seems worth while to give a coherent presentation of the theory in the language of Padé approximation. Parts of such a program have been carried out by Baker [3] using determinant theory and recently by Allen et al [1, 2] using the Schwinger variational principle and the theory of generalized matrix inverses. In this paper we present a unified and complete approach to the subject based on the presumably more well-known theory of orthogonal polynomials. Most of the results are well-known and we have not been able in all cases to trace them to their first appearance. In these cases we give reference to some easily available source.

2. Main results

In this section we state the main results of the paper. Proofs are given in the following sections.

We first collect the necessary algebraic properties of the Padé approximant.

Theorem 1. *Let $f(z)$ be a series of Stieltjes, i.e. $f(z) = \sum_{n=0}^\infty c_n(-z)^n$ where $c_n = \int_0^\infty t^n d\alpha(t)$ for some real, bounded, nondecreasing function $\alpha(t)$ taking infinitely many values on $t \geqq 0$. Let $P_n(z)$ be the n:th orthogonal polynomial with respect to $d\alpha$ and γ_n the leading coefficient of P_n. Put $g(z) = \int_0^\infty d\alpha(t)/(1+zt)$, $z \notin R_- = \{z \mid z \leqq 0\}$. Then*

(a) $$f[n-1, n](z) = \sum_{i=1}^n \frac{\lambda_{in}}{1+\alpha_{in} z}$$

where $\{\alpha_{in}\}_{i=1}^{n}$ *are the zeros of* $P_n(z)$ *and* $\{\lambda_{in}\}_{i=1}^{n}$ *the corresponding Christoffel numbers.*

(b) $$g(z)-f[n-1,n](z) = \frac{1}{P_n\left(-\frac{1}{z}\right)}\cdot\int_0^\infty \frac{P_n(t)\,d\alpha(t)}{1+zt} = \frac{1}{P_n^2\left(-\frac{1}{z}\right)}\int_0^\infty \frac{P_n^2(t)\,d\alpha(t)}{1+zt},$$

$z \notin R_-$.

(c) $$f[n-1,n](z)-f[n,n+1](z) = \frac{\gamma_{n+1}}{\gamma_n}\cdot\frac{1}{z\cdot P_n\left(-\frac{1}{z}\right)\cdot P_{n+1}\left(-\frac{1}{z}\right)}, \quad z\notin R_-.$$

As mentioned above (a) can be found in Perron [9], as can (c) (p. 193) and the first error expression in (b) (p. 194). The simple but useful reformulation that gives the last equality in (b) seems to be new.

(a) has a corollary worth noting:

Corollary 1. *The poles of* $f[n,n+1](z)$ *are all simple and located on the negative real axis. Between any two neighbouring poles lies one pole of* $f[n-1,n](z)$ *and also one zero of* $f[n,n+1](z)$.

The basic convergence theorem is the following due to Stieltjes [10] and Carleman [4].

Theorem 2. *Under the conditions of Theorem* 1, $\{f[n-1,n](z)\}_{n=1}^{\infty}$ *converges to a holomorphic function uniformly on compact sets disjoint from* R_-. *If* $\sum_{n=0}^{\infty}(1/c_n)^{1/2n}=\infty$ *the limit function is* $g(z)$.

When the radius of convergence is positive we can estimate the rate of convergence by means of the following theorem.

Theorem 3. *Suppose* $f(z)$ *is a series of Stieltjes with radius of convergence* $R>0$. *Let* $|d\alpha|$ *be the total mass of* $d\alpha$, $d(\cdot,\cdot)$ *denote the distance function and*

$$\varphi(z) = \frac{1-\sqrt{1+z/R}}{1+\sqrt{1+z/R}}, \quad z\notin I =]-\infty, -R]$$

where $\sqrt{\ }$ *denotes the principal branch of the square root. Then for* $z\notin I$

$$|f(z)-f[n-1,n](z)| \leqq \frac{|d\alpha|\cdot|\varphi(z)|^{2n-2}}{R^2\cdot|z|\cdot d\left(-\frac{1}{z},\left[0,\frac{1}{R}\right]\right)^3\cdot|1+\varphi^{2n-2}(z)|}.$$

Remark 1. The function $C(z)$ in Gragg's theorem is not the same as in Theorem 3. A comparison between the two results shows that neither of the error bounds obtained is generally superior to the other. Theorem 3 gives a smaller bound for z close to zero while Gragg's result is better for z near the cut.

That the geometric degree of convergence implied by Theorem 3 cannot in general be improved is shown by the following theorem which seems to be new (the particular case $R=1$, $\alpha(t)=t$ is in Gragg [6]).

Theorem 4. *Suppose that $R>0$ and that $d\alpha$ is absolutely continuous with respect to Lebesgue measure. Put $h(\theta)=\alpha'(1+\cos\theta)/2R\cdot|\sin\theta|$. Suppose also that $\int_{-\pi}^{\pi}|\log h(\theta)|\,d\theta<\infty$. Let $\varphi(z)$ have the same meaning as in Theorem 3. Then for $z\notin I$*

$$g(z)-f[n-1,n](z)=\frac{\pi\cdot D^2(\varphi(z))}{R\sqrt{1+z/R}}\cdot\varphi(z)^{2n}\cdot(1+\varepsilon_n(z))$$

where

$$D(z)=\exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi}\log h(\theta)\frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}\,d\theta\right\}\quad\text{and}\quad\varepsilon_n(z)\to 0\quad\text{as}\quad n\to\infty,$$

uniformly on compact sets disjoint from I.

Remark 2. Theorems 1—4 are true with obvious modifications of statements and proofs if the lower limit 0 in the integrals defining c_n is replaced by any number $a>-\infty$ (for $\varphi(z)$ in Theorems 3 and 4 should be taken that conformal mapping of $\{z|-1/z\notin[a,1/R]\}$ into the interior of the unit circle which maps 0 to 0 and is real for real z).

When $f(z)$ is a series of Stieltjes simple relations exist between $f[n-1,n](z)$ and $f[n+j,n](z)$, $j\geqq 0$. These relations permit extension of most of the above results to other sequences $\{f[n+j,n]\}_{n=0}^{\infty}$ of Padé approximants. These extensions are discussed in Baker [3]. For the sake of completeness we collect the results in the following theorem.

Theorem 5. *Let $f(z)$ be a series of Stieltjes and let $j>-1$. Then*

(a) *Corollary 1 and Theorem 2 are still valid if $f[n-1,n](z)$ is replaced by $f[n+j,n](z)$ throughout.*

(b) *If $R>0$ then*

$$|f(z)-f[n+j,n](z)|\leqq C_j(z)\cdot\left|\frac{1-\sqrt{1+z/R}}{1+\sqrt{1+z/R}}\right|^{2n}$$

where $C_j(z)$ is bounded in any compact set disjoint from I.

(c) *The following inequalities hold for $z>0$:*

(i) $(-1)^{j+1}(f[n+j+1,n+1]-f[n+j,n])>0,$

(ii) $(-1)^{j+1}(f[n+j+1,n-1]-f[n+j,n])<0.$

(iii) $(-1)^{j+1}(g-f[n+j,n])>0.$

3. Auxiliary results on orthogonal polynomials

In the proof of Theorems 1—5 we shall make extensive use of known properties of orthogonal polynomials. We collect some necessary estimates in the following lemma.

Lemma 1. *Let $\alpha(t)$ be a real, bounded, nondecreasing function taking infinitely many values on* $[0, 1/R]$. *Put*

$$\Psi(z) = \frac{1-\sqrt{1-1/Rz}}{1+\sqrt{1-1/Rz}} \quad z \notin [0, 1/R].$$

Then with the notation of Theorem 1—4 *we have*

(a) $|P_n(z)| \geqq R \cdot |d\alpha|^{-1/2} \cdot d(z, [0, 1/R]) \cdot |\Psi(z)^{n-1} + \Psi(z)^{-(n-1)}|$

(b) *if* $\int_{-\pi}^{\pi} |\log h(\theta)|\, d\theta < \infty$, *then*

$$P_n(z) = \sqrt{\frac{R}{\pi}} \cdot \frac{1+r_n(z)}{D(\Psi(z)) \cdot \Psi(z)^n}$$

where $r_n(z) \to 0$ *uniformly on any closed set disjoint from* $[0, 1/R]$. *Furthermore,* $\gamma_{k+1}/\gamma_k \to 4R$ *as* $k \to \infty$.

Proof. (b) is a classical result by Szegö (see Szegö [11] pp. 297 and 309). (a) is just a more informative version of Theorem III. 7.1 in Freud [5]. Just keeping track of the various constants appearing in Freud's proof gives the statement of the lemma.

Remark. We note here that the function $\varphi(z)$ of Theorems 3 and 4 is defined so that $\varphi(z) = \Psi(-1/z)$.

4. Proof of Theorem 1

(a) is a consequence of the fact that the Gauss—Jacobi quadrature formula $Q_n(d\alpha; f) = \sum_{i=1}^{n} \lambda_{in} \cdot f(\alpha_{in}) \approx \int f\, d\alpha$ is exact for polynomials of degree $\leqq 2n-1$. Thus

$$Q_n\left(d\alpha;\ \frac{1}{1+zt}\right) = Q_n\left(d\alpha;\ \sum_{i=0}^{2n-1} (-zt)^i + \frac{(-zt)^{2n}}{1+zt}\right) =$$

$$= \sum_{i=0}^{2n-1} (-z)^i \cdot Q_n(d\alpha;\ t^i) + z^{2n} \cdot Q_n\left(d\alpha;\ \frac{t^{2n}}{1+zt}\right) = \sum_{i=0}^{2n-1} c_i(-z)^i + O(z^{2n}), \quad z \to 0.$$

Since $Q_n(d\alpha; 1/(1+zt))$ is a rational function of type $(n-1, n)$ and interpolates to f of order $2n-1$ at the origin it is the $[n-1, n]$ Padé approximant (this proof is also given in Allen et al [1]).

(b) We prove the first error formula. Put $f[n-1, n](z)=R_{n-1}(z)/Q_n(z)$, $Q_n(z)=\sum_{k=0}^n \beta_k z^k$. From (a) we conclude that $P_n(t)=k_n\cdot t^n\cdot Q_n(-1/t)= =k_n \sum_{k=0}^n \beta_k(-1)^k\cdot t^{n-k}$ for some constant k_n. Now R_{n-1} must have the same Taylor coefficients as $Q_n\cdot f$ in the first n slots:

$$R_{n-1}(z) = \sum_{\nu=0}^{n-1} z^\nu \cdot \sum_{k=0}^{\nu} (-1)^{\nu-k}\cdot c_{\nu-k}\cdot\beta_k = \sum_{\nu=0}^{n-1} z^\nu\cdot\sum_{k=0}^{\nu}\beta_k\cdot\int_0^\infty (-t)^{\nu-k}\,d\alpha(t)=$$

$$=\int_0^\infty \sum_{\nu=0}^{n-1}\sum_{k=0}^{\nu}\beta_k\cdot z^\nu\cdot(-t)^{\nu-k}\,d\alpha(t) = \int_0^\infty \sum_{k=0}^{n-1}\sum_{\nu=k}^{n-1}\beta_k\cdot z^\nu\cdot(-t)^{\nu-k}\,d\alpha(t) =$$

$$=\int_0^\infty \sum_{k=0}^{n}\beta_k\cdot z^k\frac{1-(-zt)^{n-k}}{1+zt}\,d\alpha(t) = Q_n(z)\cdot g(z)-z^n\int_0^\infty\frac{\sum_{k=0}^n\beta_k(-t)^{n-k}\,d\alpha(t)}{1+zt}.$$

Thus

$$g(z)-R_{n-1}(z)/Q_n(z) = \frac{1}{P_n\left(-\frac{1}{z}\right)}\int_0^\infty\frac{P_n(t)\,d\alpha(t)}{1+zt}.$$

That the second error formula is valid follows immediately from the first and the observation that $S(t)=(P_n(t)-P_n(-1/z))/(1+zt)$ is a polynomial of degree $n-1$ in t and thus $\int_0^\infty P_n(t)\cdot S(t)\,d\alpha(t)=0$.

(c) follows from (b) and the Christoffel—Darboux summation formula

$$\sum_{i=0}^k P_i(t)\cdot P_i(w) = \frac{\gamma_k}{\gamma_{k+1}}\,\frac{P_{k+1}(t)P_k(w)-P_k(t)P_{k+1}(w)}{t-w}.$$

Integrating both sides with respect to $d\alpha(t)$ and dividing by $(\gamma_k/\gamma_{k+1})P_k(w)\cdot P_{k+1}(w)$ yields

$$\frac{\gamma_{k+1}}{\gamma_k}\cdot\frac{1}{P_k(w)\cdot P_{k+1}(w)} = \frac{1}{P_{k+1}(w)}\int_0^\infty\frac{P_{k+1}(t)\,d\alpha(t)}{t-w}-\frac{1}{P_k(w)}\int_0^\infty\frac{P_k(t)\,d\alpha(t)}{t-w}.$$

Now put $w=-1/z$ to get the desired conclusion.

As to the corollary the assertion concerning the poles is just a reformulation of the corresponding well-known property of the zeros of orthogonal polynomials (see Szegö [11], section 3.3). The interlacing property of the zeros is a consequence of the fact that all $\lambda_{in}>0$.

5. Proof of Theorem 2

We first prove that the sequence $\{f[n-1,n](z)\}_1^\infty$ is uniformly bounded in any compact set F disjoint from R_-.

$$\max_{z\in F}\left|\sum_{i=1}^n \frac{\lambda_{in}}{1+\alpha_{in}z}\right| \leqq \max_{z\in F}\max_{t\geqq 0}\frac{1}{|1+tz|}\cdot\sum_{i=1}^n \lambda_{in}.$$

Put $r=\max_{z\in F}|z|$. Now if $t<1/2r$ and $z\in F$ $|1/(1+tz)|<2$ and if $t\geqq 1/2r$

$$\left|\frac{1}{1+tz}\right| = \frac{1}{|t|\cdot\left|\frac{1}{t}+z\right|} \leqq \frac{2r}{d(R_-,F)}.$$

Thus

$$\max_{z\in F}|f[n-1,n](z)| \leqq \max\left(2,\frac{2r}{d(R_-,F)}\right)\cdot|d\alpha|.$$

If we can prove that $\{f[n-1,n](z)\}$ converges at a set of points having a limit point in the interior of the complement of R_- we can apply Vitali's theorem (Titchmarsh [12], p. 168) to conclude that the sequence converges uniformly on compact sets disjoint from R_-. But for $z>0$ it follows from Theorem 1 (b) and (c) that $f[n-1,n](z)<f[n,n+1](z)<g(z)$. Thus for these z the sequence converges being a bounded, increasing sequence of real numbers. For the second part of the theorem, suppose $\sum_{n=0}^\infty(\gamma_{n+1}/\gamma_n)^{1/2}=\infty$ and fix $z_0>0$. Put $w_0=-1/z_0$. Schwarz' inequality yields

$$\sum\left(\frac{\gamma_{n+1}}{\gamma_n}\right)^{1/2} \leqq \left\{\sum\left(-\frac{\gamma_{n+1}}{\gamma_n}\cdot\frac{1}{P_n(w_0)P_{n+1}(w_0)}\right)\cdot\sum\left(-P_n(w_0)P_{n+1}(w_0)\right)\right\}^{1/2}.$$

From the first part of the theorem and Theorem 1 (c) follows that the first series to the right converges, which forces $\sum P_n(w_0)P_{n+1}(w_0)=\infty$. Now $-P_n(w_0)P_{n+1}(w_0)\leqq$ $\leqq 1/2(P_n(w_0)^2+P_{n+1}(w_0)^2)$ implies $\sum P_n(w_0)^2=\infty$. From Theorem 1 (b) we get

$$P_n(w_0)\cdot(g(z_0)-f[n-1,n](z_0)) = \int_0^\infty \frac{P_n(t)\,d\alpha(t)}{1+z_0t}$$

which defines the nth Fourier coefficient of $1/(1+z_0t)$. Next we invoke Bessel's inequality:

$$\sum P_n^2(w_0)\cdot(g(z_0)-f[n-1,n](z_0))^2 \leqq \int_0^\infty \frac{d\alpha(t)}{(1+z_0t)^2}<\infty.$$

From this inequality and the fact that $\sum P_n^2(w_0)=\infty$ we conclude that at least a subsequence of $\{f[n-1,n](z_0)\}_{n=0}^\infty$ converges to $g(z_0)$. Since this holds for any $z_0>0$

and we know that the whole sequence converges to a holomorphic function except on the negative real axis we must have $f[n-1, n](z) \to g(z)$ for $z \notin R_-$.

It remains to prove that $\sum (1/c_n)^{1/2n} = \infty$ implies $\sum (\gamma_{n+1}/\gamma_n)^{1/2} = \infty$. We first note that $1/\gamma_n = \int_0^\infty t^n P_n(t)\, d\alpha(t) \leqq \sqrt{c_{2n}}$ by Schwarz' inequality. We also need Carleman's inequality (Carleman [4]; see also Hardy, Littlewood, Polya [8], p. 249): If $u_n \geqq 0$, $n = 1, 2, \ldots$ then

$$\sum_{n=1}^\infty (u_1 \cdot u_2 \cdot \ldots \cdot u_n)^{1/n} \leqq e \sum_{n=1}^\infty u_n .$$

Hence

$$\sum_1^\infty \left(\frac{\gamma_n}{\gamma_{n-1}}\right)^{1/2} \geqq \frac{1}{e} \sum_{n=1}^\infty \left(\frac{\gamma_1}{\gamma_0} \cdot \frac{\gamma_2}{\gamma_1} \cdot \ldots \cdot \frac{\gamma_n}{\gamma_{n-1}}\right)^{1/2n} = \frac{1}{e} \sum_1^\infty \left(\frac{\gamma_n}{\gamma_0}\right)^{1/2n} \geqq \text{const} \cdot \sum_1^\infty \left(\frac{1}{c_{2n}}\right)^{1/4n} .$$

But $\sum (1/c_n)^{1/2n}$ and $\sum (1/c_{2n})^{1/4n}$ diverge simultaneously. This holds since $c_n \leqq \sqrt{c_{n-1} c_{n+1}}$ by Schwarz' inequality and thus $\{c_n\}_0^\infty$ is either bounded or increasing for large n. In the latter case

$$\sum \left(\frac{1}{c_{2n+1}}\right)^{1/4n+2} \leqq \sum \left\{\left(\frac{1}{c_{2n}}\right)^{1/4n}\right\}^{1-1/(2n+1)} < \infty \quad \text{if} \quad \sum \left(\frac{1}{c_{2n}}\right)^{1/4n} < \infty .$$

This completes the proof of Theorem 2.

6. Proofs of Theorems 3 and 4

Theorems 3 and 4 are both established by inserting an estimate of $P_n(z)$ into one of the error formulae of Theorem 1.

Proof of Theorem 3. We use Theorem 1 (b) and Lemma 1 (a):

$$|g(z) - f[n-1, n](z)| = \frac{1}{\left|P_n^2\left(-\frac{1}{z}\right)\right|} \left| \int_0^{1/R} \frac{P_n^2(t)\, d\alpha(t)}{1+zt} \right| \leqq$$

$$\leqq \frac{|d\alpha|}{R^2 \cdot d^2\left(-\frac{1}{z}, \left[0, \frac{1}{R}\right]\right) \cdot \left|\Psi\left(-\frac{1}{z}\right)^{n-1} + \Psi\left(-\frac{1}{z}\right)^{-(n-1)}\right|^2} \cdot \max_{t \in [0, 1/R]} \left|\frac{1}{1+tz}\right| \cdot$$

$$\cdot \int_0^{1/R} P_n^2(t)\, d\alpha(t) = \frac{|d\alpha|}{R^2 \cdot |z| \cdot d^3\left(-\frac{1}{z}, \left[0, \frac{1}{R}\right]\right) \cdot |\varphi(z)^{n-1} + \varphi(z)^{-(n-1)}|^2} .$$

Proof of Theorem 4. From Lemma 1 (b) and Theorem 1 (b) we conclude that it is enough to prove that $\int_0^\infty P_n^2(t)\,d\alpha(t)/(1+zt) \to 1/\sqrt{1+z/R}$ uniformly on compact subsets disjoint from I. Theorem 1 (c) implies that

$$I_n(z) = \int_0^\infty \frac{P_n^2(t)\,d\alpha(t)}{1+zt} = -\sum_{k=n}^\infty \frac{\gamma_{k+1}}{\gamma_k}\cdot\frac{P_n^2\left(-\frac{1}{z}\right)}{z\cdot P_k\left(-\frac{1}{z}\right)\cdot P_{k+1}\left(-\frac{1}{z}\right)}.$$

But Lemma 1 (b) implies that for any $\varepsilon>0$ the inequality

$$\left|\frac{\gamma_{k+1}}{\gamma_k}\cdot\frac{P_n^2\left(-\frac{1}{z}\right)}{z\cdot P_k\left(-\frac{1}{z}\right)\cdot P_{k+1}\left(-\frac{1}{z}\right)}-\frac{4R\varphi(z)^{2(k-n)+1}}{z}\right| \leqq \frac{4R\varepsilon M^{2(k-n)+1}}{\min\limits_{z\in D(M)}|z|}$$

holds uniformly in $D(M)=\{z\,\big|\,|\varphi(z)|\leqq M<1\}$ for large enough n if $k>n$. Thus

$$\left|I_n(z)-\sum_{k=n}^\infty 4Rz^{-1}\cdot\varphi(z)^{2(k-n)+1}\right| \leqq \frac{4R\varepsilon M}{\min\limits_{z\in D(M)}|z|\cdot(1-M^2)}$$

and we have that

$$I_n(z) \to -\sum_{k=0}^\infty \frac{4R\varphi(z)^{2k+1}}{z} = -\frac{4R\varphi(z)}{z(1-\varphi^2(z))} = \frac{1}{\sqrt{1+z/R}}$$

uniformly on compact sets disjoint from I since any such set is contained in $D(M)$ for some $M<1$.

This completes the proof of Theorem 4.

7. Proof of Theorem 5

Put $f_j(z)=\sum_{k=0}^\infty c_{k+j+1}\cdot(-z)^k$. Then $f_j(z)$ is also a series of Stieltjes since its power series coefficients are moments of the measure $t^{j+1}d\alpha(t)$. Simple calculations show that for $k\geqq 0$ and $j\geqq -1$

$$f[n+j+k, n] = \sum_{k=0}^{j} c_k(-z)^k+(-z)^{j+1}\cdot f_j[n+k-1, n], \tag{1}$$

$$f-f[n+k+j, n] = (-z)^{j+1}(f_j-f_j[n+k-1, n]). \tag{2}$$

Using (2) for $k=0$ we conclude that (a) and (b) follow immediately from Corollary 1 and Theorem 2 and 3. As to (c), it is enough to prove (i) and (iii) for $j=-1$. In this case (i) follows from Theorem 1 (b) and (iii) from Theorem 1 (c). It remains to prove (ii) which by means of (2) for $k=0$ and $k=2$ can be reduced to proving

$$f[n-1,n](z)-f[n,n-1](z)>0 \quad \text{for} \quad z>0.$$

To this end put $f[n-1,n](z)=A_{n-1}(z)/B_n(z)$ and $f[n,n-1](z)=C_n(z)/D_{n-1}(z)$ with $B_n(0)=D_{n-1}(0)=1$. Then

$$\frac{A_{n-1}(z)}{B_n(z)}-\frac{C_n(z)}{D_{n-1}(z)}=\frac{A_{n-1}D_{n-1}-B_nC_n}{B_nD_{n-1}}=O(z^{2n}), \quad z\to 0.$$

But $A_{n-1}D_{n-1}-B_nC_n$ is a polynomial of degree $2n$ and thus

$$f[n-1,n](z)-f[n,n-1](z)=\frac{k_n\cdot z^{2n}}{B_n(z)D_{n-1}(z)}.$$

Since B_n and D_{n-1} only have negative zeros this expression does not change sign for $z>0$. But $f[n-1,n](z)\to 0$ for $z\to\infty$ and we must prove that $f[n,n-1](z)\to-\infty$, i.e. that the leading coefficient of C_n is negative. Put $D_{n-1}(z)=\sum_0^{n-1}d_kz^k$, $d_0=1$ and let $P_n(d\beta;t)$ with leading coefficient $\gamma_n(d\beta)>0$ be the nth orthogonal polynomial with respect to $d\beta=t^2d\alpha$. By reasoning as in the proof of Theorem 1 (b) we find that the leading coefficient of C_n equals

$$\sum_{k=0}^{n-1}d_k(-1)^{n-k}c_{n-k}=-\int_0^\infty\sum_{k=0}^{n-1}d_k(-t)^{n-1-k}t\,d\alpha(t)=$$

$$=\frac{(-1)^n}{\gamma_{n-1}(d\beta)}\int_0^\infty P_{n-1}(d\beta;t)t\,d\alpha(t)=\frac{(-1)^n}{\gamma_{n-1}(d\beta)\cdot P_{n-1}(d\beta;0)}\int_0^\infty P_{n-1}^2(d\beta;t)t\,d\alpha(t).$$

Since the last integral is positive and sign $P_{n-1}(d\beta;0)=(-1)^{n-1}$ we find that the leading coefficient of C_n is negative which completes the proof of Theorem 5.

Note added in proof. After submission of this paper an article by G. Freud appeared, in which he proves essentially our Theorem 3, stated for Padé approximants at infinity, using methods very similar to ours. (Reference: FREUD, G., An estimate of the error of Padé approximants, *Acta Math. Sci. Hungar.* **25** 1974, (213—221).)

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