The range of vector-valued analytic functions

J. Globevnik*

Introduction

Throughout, Δ (resp. $\bar{\Delta}$) is the open (resp. closed) unit disc in $\mathbf{C}$. If X is a complex Banach space we denote by X' its dual, by $\bar{S}$ the closure of a set $S \subset X$ and by $B_r(X)$ the open ball in X of radius r centered at the origin. The image of $x \in X$ under $u \in X'$ is denoted by $\langle x|u \rangle$. By N we denote the set of all positive integers. As usual, we denote by A the Banach algebra of all complex-valued continuous functions on $\bar{\Delta}$, analytic on Δ, with sup norm, and by H^∞ the Banach space of all bounded complex-valued analytic functions on Δ, with sup norm. We denote by $K(r, z)$ the open disc in C centered at z and having radius r. Suppose that X is a complex Banach space and $f\colon \Delta \to X$ an analytic function. Let $|z_0| = 1$ and denote $T_r = f(K(r, z_0) \cap \Delta)$. According to [3, p. 2] we call $T = \bigcap_{r>0} \bar{T}_r$ the cluster set of f at z_0. It is easy to see that $x \in T$ if and only if there exists a sequence $\{z_n\} \subset \Delta$ converging to z_0 and such that $x = \lim f(z_n)$ (see [3], p. 1).

In [2] it was proved that given a separable complex Banach space X and an $\varepsilon > 0$ there exists an analytic function $f\colon \Delta \to X$ satisfying

$$B_1(X) \subset \overline{f(\Delta)} \subset B_{1+\varepsilon}(X).$$

Later [1, 4] this result was strenghtened by proving that there exists an analytic function $f\colon \Delta \to X$ satisfying $\overline{f(\Delta)} = \overline{B_1(X)}$. It is natural to consider these results as partial solutions to more general problems about the range of vector-valued analytic functions.

Definition. We say that Δ is analytically dense (resp. approximately analytically dense) in a subset F of a separable complex Banach space X if there exists an analytic

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function $f\colon \Delta\to X$ satisfying $\overline{f(\Delta)}=\bar{F}$ (resp. if given any open set O in X containing F there exists an analytic function $f\colon \Delta\to X$ such that $F\subset\overline{f(\Delta)}$ and $f(\Delta)\subset O$).

As the main result of the present paper we prove that Δ is approximately analytically dense in every balanced subset of a separable complex Banach space.

Theorem. *Let F be a balanced set in a separable complex Banach space X and let $O\subset X$ be an open set containing F. There exists a continuous function $f\colon \bar{\Delta}-\{1\}\to X$, analytic on Δ and such that*

(i) $f(\bar{\Delta}-\{1\})\subset O$

(ii) *the cluster set of f at* 1 *is* $\bar{F}$.

Proof. We prove the theorem in several steps.

Step 1. Let $\{\alpha_k\}$, $0<\alpha_k<\pi/2$ be a decreasing sequence converging to zero. Put $z_k=\exp(i\alpha_k)$ $(k\in N)$. By the Rudin—Carleson theorem (see [6], p. 81) there exists a sequence of functions $\{\varphi_k\}\subset A$, bounded by 1 and satisfying $\varphi_k(z_j)=\delta_{kj}$ $(k,j\in N)$ and $\varphi_k(1)=0$ $(k\in N)$. Further (see [6], p. 81) for each $k\in N$ there exists in A a peaking function for z_k, i.e. a function $\psi_k\in A$ satisfying $\psi_k(z_k)=1$, $|\psi_k(z)|<1$ $(z\in\bar{\Delta}, z\notin z_k)$.

Step 2. By the separability of X there exists a sequence $\{y_k\}\subset F$ dense in F and it is easy to construct a sequence $\{x_k\}\subset F$ such that

$$\{y_k;\, k\in N\}\subset\{x_j;\, j\in N, j\geqq n\}\quad (n\in N). \tag{1}$$

Further, if $x\in F$ then $\bar{\Delta}x=\{\alpha x;\, \alpha\in\bar{\Delta}\}\subset F$ since F is balanced. By the compactness of $\bar{\Delta}x$ there exists an $\varepsilon>0$ such that $\bar{\Delta}x+B_\varepsilon(X)\subset O$. Consequently we may assume that there exists a decreasing sequence $\{\varepsilon_n\}$ of positive numbers converging to zero such that

$$\bar{\Delta}x_n+B_{\varepsilon_n}(X)\subset O\quad (n\in N). \tag{2}$$

Step 3. We define inductively a disjoint sequence $\{D_i\}$ of sets of the form $D_i=\overline{K(r_i, z_i)}\cap\bar{\Delta}$ with $r_i>0$ and a sequence $\{\Phi_i\}$ of functions from $\bar{\Delta}$ to X with the following properties

$$\left.\begin{array}{ll}
\text{(a)} & \Phi_i \text{ is continuous on } \bar{\Delta} \text{ and analytic on } \Delta \text{ for } i\in N\\
\text{(b)} & \Phi_i(\bar{\Delta})\subset\bar{\Delta}x_i\quad (i\in N)\\
\text{(c)} & \Phi_i(z_j)=\delta_{ij}x_i\quad (i,j\in N)\\
\text{(d)} & \Phi_i(1)=0\quad (i\in N)\\
\text{(e)} & \|\Phi_i(z)\|<\varepsilon_i/2^i\quad (i\in N,\ z\in\bar{\Delta}-D_i)\\
\text{(f)} & \left\|\sum_{j=1}^{i}\Phi_j(z)\right\|<\varepsilon_{i+1}/2\quad (i\in N,\ z\in D_{i+1}).
\end{array}\right\} \tag{3}$$

If $n=1$ let $r_1>0$ be such that $\overline{K(r_1, z_1)}$ does not contain z_2 (and consequently does not contain any z_k, $k>2$) and put $D_1=\overline{K(r_1, z_1)}\cap\bar{\Delta}$. By the properties of φ_1 and ψ_1 a positive integer p_1 exists such that

$$\|\psi_1(z)^{p_1}\varphi_1(z)x_1\| < \varepsilon_1/2 \quad (z\in\bar{\Delta}-D_1).$$

Define $\Phi_1(z)=\psi_1(z)^{p_1}\varphi_1(z)x_1$ $(z\in\bar{\Delta})$ and observe that by the properties of ψ_1 and φ_1 Φ_1 satisfies (3) (a—e).

Let $n\in N$ be arbitrary and assume that we have constructed a disjoint family $\{D_i, 1\leqq i\leqq n\}$ of the form $D_i=\overline{K(r_i, z_i)}\cap\bar{\Delta}$ with $r_i>0$ such that D_n does not contain z_{n+1}, and a family $\{\Phi_i, 1\leqq i\leqq n\}$ of functions satisfying (3) (a—e). Since $\Phi_1+\Phi_2+\ldots+\Phi_n$ is continuous on $\bar{\Delta}$ and equals 0 at z_{n+1} there exists an $r_{n+1}>0$ such that $D_{n+1}=\overline{K(r_{n+1}, z_{n+1})}\cap\bar{\Delta}$ is disjoint from D_n and also from D_k, $k<n$ and does not contain z_{n+2} and such that (3) (f) holds for $i=n$. Further, let $p_{n+1}\in N$ be sufficiently large, so that

$$\|\psi_{n+1}(z)^{p_{n+1}}\varphi_{n+1}(z)x_{n+1}\| < \varepsilon_{n+1}/2^{n+1} \quad (z\in\bar{\Delta}-D_{n+1})$$

and define

$$\Phi_{n+1}(z) = \psi_{n+1}(z)^{p_{n+1}}\varphi_{n+1}(z)x_{n+1} \quad (z\in\bar{\Delta}).$$

By the properties of ψ_{n+1} and φ_{n+1} it is easy to see that (3) (a—e) is satisfied if $i=n+1$ and that (3) (f) holds for $i=n$.

Step 4. Define

$$f(z) = \sum_{i=1}^{\infty}\Phi_i(z) \quad (z\in\bar{\Delta}). \tag{4}$$

Denote $D=\bar{\Delta}-\bigcup_{j=1}^{\infty}D_j$. We prove first that the series (4) converges uniformly on D and that $f(D)\subset O$. To see this, let $z\in D$. By (3) (e) we have $\|\Phi_i(z)\|<\varepsilon_i/2^i\leqq\varepsilon_1/2^i$ hence the series converges uniformly on D. We have

$$\sum_{i=1}^{\infty}\|\Phi_i(z)\| < \varepsilon_1$$

and since by (2) $B_{\varepsilon_1}(X)\subset O$ it follows that $f(D)\subset O$.

Next we prove that given $j\in N$ the series (4) converges uniformly on D_j and that $f(D_j)\subset O$. Let $j\in N$ and let $z\in D_j$. Then by (3) (f)

$$\left\|\sum_{i=1}^{j-1}\Phi_i(z)\right\| < \varepsilon_j/2 \tag{5}$$

and since by (3) (e) $\|\Phi_i(z)\|<\varepsilon_i/2^i$ $(i\geqq j+1)$, the series (4) converges uniformly on D_j and

$$\sum_{i=j+1}^{\infty}\|\Phi_i(z)\| < \sum_{i=j+1}^{\infty}\varepsilon_i/2^i \leqq \varepsilon_j/2. \tag{6}$$

From (5) and (6) it follows that

$$\left\|\sum_{i=1, i\neq j}^{\infty}\Phi_i(z)\right\| < \varepsilon_j$$

which, by (3) (b) and by (2) implies that $f(D_j)\subset O$.

Since each compact subset of $\bar{\Delta}-\{1\}$ misses all but a finite number of the sets D_j ($j\in N$) it follows that the series (4) converges uniformly on compact subsets of $\bar{\Delta}-\{1\}$. Consequently f is continuous on $\bar{\Delta}-\{1\}$, analytic on Δ and, as shown above, satisfies $f(\bar{\Delta}-\{1\})\subset O$.

Step 5. It remains to prove that the cluster set T of f at 1 is $\bar{F}$.

First we prove that $T\subset\bar{F}$ by proving that $T\subset F+B_\varepsilon(X)$ for every $\varepsilon>0$. So let $\varepsilon>0$ be arbitrary. Choose $n\in N$ so large that $\varepsilon_n<\varepsilon/4$. Since by (3) the function $z\mapsto\sum_{i=1}^n\|\Phi_i(z)\|$ is continuous on $\bar{\Delta}$ and equal 0 at $z=1$ there exists $r>0$ such that, if $D=\overline{K(r,1)}\cap\bar{\Delta}$ we have

$$\sum_{i=1}^n\|\Phi_i(z)\|<\varepsilon/4\quad(z\in D). \tag{7}$$

Passing to a smaller $r>0$ if necessary we may assume that $D_j\cap D=\emptyset$ ($j\leqq n$). Now we show that $f(D)\subset F+B_{\varepsilon/2}(X)$. If $z\in D$ and $z\notin\bigcup_{j=n+1}^\infty D_j$ then by (3) (e)

$$\sum_{i=n+1}^\infty\|\Phi_i(z)\|\leqq\sum_{i=n+1}^\infty\varepsilon_i/2^i\leqq\varepsilon_n<\varepsilon/4.$$

This together with (7) implies that $\|f(z)\|<\varepsilon/2$. Since $0\in F$ it follows that $f(z)\in F+B_{\varepsilon/2}(X)$. If $z\in D\cap D_j$ for some $j>n$ then again by (7) and by (3) (e)

$$\sum_{i=1,\,i\neq j}^\infty\|\Phi_i(z)\|<\varepsilon/2.$$

By (3) (b) we have $\Phi_j(z)\in F$ hence it follows that $f(z)\in F+B_{\varepsilon/2}(X)$. So we have proved that for any $\varepsilon>0$ an $r>0$ exists such that $T_r=f(K(r,1)\cap\Delta)\subset F+B_{\varepsilon/2}(X)$ and consequently $\bar{T}_r\subset F+B_\varepsilon(X)$. It follows that

$$T=\bigcap_{r>0}\bar{T}_r\subset F+B_\varepsilon(X)$$

for every $\varepsilon>0$ which proves that $T\subset\bigcap_{\varepsilon>0}[F+B_\varepsilon(X)]=\bar{F}$.

Finally we prove that $\bar{F}\subset T$. Let $x\in\bar{F}$. Then there exists a sequence $\{w_j\}\subset\{y_k\}$ satisfying $\lim w_j=x$. By (1) it follows that there exists a subsequence $\{x_{nj}\}$ of the sequence $\{x_n\}$, converging to x. By the definition of f we have $f(z_k)=x_k$ ($k\in N$) hence it follows that $x=\lim f(z_{nj})$. By the continuity of f on $\bar{\Delta}-\{1\}$ it is trivial to find a sequence $\{\zeta_j\}\subset\Delta$ converging to 1 and satisfying $x=\lim f(\zeta_j)$ which proves that $x\in T$.

Q.E.D.

Corollary 1. *Δ is approximately analytically dense in every balanced set in a separable complex Banach space.*

Corollary 2. *Δ is analytically dense in every open balanced set in a separable complex Banach space.*

Proof. Apply Theorem with $O=F$. Q.E.D.

Corollary 3 (see [1, 4]). *Let X be a separable complex Banach space. There exists an analytic function f: $\Delta \to X$ such that $f(\Delta)$ is contained and dense in the unit ball of X.*

Next we present an application of Corollary 3. By a result of A. Pelczynski [7] the space A is isometrically universal for all separable complex Banach spaces, i.e. every separable complex Banach space is isometrically isomorphic to a subspace of A. We ask whether the space H^∞ has a similar universal property. Recall that given a Banach space X a determining set for X is any subset $S \subset X'$ such that

$$\|x\| = \sup_{u \in S} |\langle x|u\rangle| \quad \text{for all} \quad x \in X.$$

Corollary 4. *The space H^∞ is isometrically universal for all complex Banach spaces posessing countable determining sets.*

Proof. Let X be a complex Banach space such that there exists a countable determining set $\{u_n\} \subset X'$. Obviously $\|u_n\| \leqq 1$ $(n \in N)$. Denote by Y the separable closed subspace of X' generated by $\{u_n\}$. By Corollary 3 there exists an analytic function f: $\Delta \to Y$ such that $f(\Delta)$ is contained and dense in $B_1(Y)$. Clearly f is analytic as a function into X'. Now associate with each $x \in X$ the function φ_x: $\Delta \to C$ where

$$\varphi_x(z) = \langle x|f(z)\rangle \quad (z \in \Delta).$$

It is clear that $x \mapsto \varphi_x$ is a linear map from X to H^∞. Further, by

$$|\varphi_x(z)| \leqq \|x\| \cdot \|f(z)\| \leqq \|x\| \quad (z \in \Delta)$$

and by

$$\sup_{z \in \Delta} |\varphi_x(z)| = \sup_{z \in \Delta} |\langle x|f(z)\rangle| \geqq \sup_{n \in N} |\langle x|u_n\rangle| = \|x\|$$

this map is also an isometry which proves the assertion. Q.E.D.

Remark. Note that l^∞ is also an isometrically universal space for all Banach spaces posessing countable determining sets. Note also that by Theorem the space H^∞ in Corollary 4 can be replaced by its closed subspace of all functions which are continuous on $\bar{\Delta} - \{1\}$.

By Corollary 2 Δ is analytically dense in every open balanced set in a separable complex Banach space X. Let $x \in X$, $x \neq 0$. Then $F = \bar{\Delta}x$ is an example of a set whose interior is empty although Δ is analytically dense in F. However, Δ is not analytically dense in every balanced subset of X. An example is $F = \bar{\Delta}x_1 \cup \bar{\Delta}x_2$ where $x_1, x_2 \in X$ are linearly independent.

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J. Globevnik
Institute of Mathematics,
Physics and Mechanics
University of Ljubljana
Ljubljana, Yugoslavia