On the existence of capacitary strong type estimates in $\mathbf{R}^n$

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Introduction

This paper is a direct outgrowth of a conversation the author had with Professor V. G. Maz'ya in the Spring of 1974 concerning the existence of certain L_p estimates for the restriction of Riesz potentials of L_p functions to small sets. The main open question concerned the validity of the estimate $\|I_m f\|_{p,\mu} \leqq C\|f\|_p$ for positive integers $m>2$ ($m<n$, $1<p<n/m$) given that the measure μ satisfies $\mu(K) \leqq C \cdot R_{m,p}(K)$ for all compact sets K in $\mathbf{R}^n$. (See section 1 for the definitions.) Maz'ya has shown ([11] and [12]) that this is valid for $m=1, 2$. We now establish this estimate for the remaining integer values of m — Theorem 4 of section 4. The proof relies on the following theorem, which is the principal estimate of this paper.

Theorem 1. *If m=positive integer$<n$, $1<p<n/m$, then*

$$\int_0^\infty R_{m,p}(\{x\colon |I_m f(x)| \geqq t\})\, dt^p \leqq C\|f\|_p^p,$$

for all $f \in L_p(\mathbf{R}^n)$. C is a constant depending only on n, p and m.

Of course the "weak type" estimate

$$R_{m,p}(\{x\colon |I_m f(x)| \geqq t\}) \leqq C\|f\|_p^p t^{-p}$$

is trivially valid (with $C=1$) and so what is new is the existence of what we prefer to call a "strong type" capacitary inequality. These weak and strong type capacitary estimates oppose each other in much the same way as do the usual weak and strong L_p estimates.

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1. Preliminaries

The Riesz potential of order m, $0<m<n$, is

$$I_m f(x) = \int |x-y|^{m-n} f(y)\, dy \tag{1.1}$$

for any f for which the integral converges absolutely for almost every (a.e.) $x\in\mathbf{R}^n$. Integration is taken over $\mathbf{R}^n$. In particular, this is always the case whenever $f\in L_p(\mathbf{R}^n)$, $1\leqq p<n/m$, the usual Lebesgue p-th power integrable functions on $\mathbf{R}^n$, $\|f\|_p=(\int|f(x)|^p dx)^{1/p}<\infty$. For a measure μ other than Lebesgue measure we write $\|f\|_{p,\mu}=(\int|f(x)|^p d\mu(x))^{1/p}$. The Sobolev inequality for Riesz potentials is

$$\|I_m f\|_{p^*} \leqq C\|f\|_p, \quad 1<p<n/m,$$

for all $f\in L_p(\mathbf{R}^n)$, C a constant depending only on n, p and m.

The Bessel potential of order $m>0$ is defined as convolution over $\mathbf{R}^n$ with the $L_1(\mathbf{R}^n)$ function $g_m(x)$ whose Fourier transform is $(1+|\xi|^2)^{-m/2}$, $\xi\in\mathbf{R}^n$, i.e. $G_m f(x)=\int g_m(x-y)f(y)dy$. Also $g_m(x)\leqq C\cdot|x|^{m-n}$ for all $x\in\mathbf{R}^n$, provided $0<m<n$.

The Riesz capacity of order m and degree p is defined by

$$R_{m,p}(A) = \inf\{\|f\|_p^p : f\in L_p^+(\mathbf{R}^n) \text{ and } I_m f(x) \geqq 1 \text{ on } A\}$$

for any set $A\subset\mathbf{R}^n$. Here the "+" denotes the non-negative elements in L_p. Some properties of $R_{m,p}$ will be useful ([13]):

(i) monotonicity: $R_{m,p}(A_1)\leqq R_{m,p}(A_2)$, $A_1\subset A_2$

(ii) subadditivity: $R_{m,p}(\bigcup_i A_i)\leqq\sum_i R_{m,p}(A_i)$

(iii) $R_{m,p}(\emptyset)=0$, $\emptyset=$empty set

(iv) $R_{m,p}(A)=0$ if and only if there is an $f\in L_p^+(\mathbf{R}^n)$ such that $I_m f(x)=+\infty$ on A. Also

$$R_{m,p}(\{x: |I_m f(x)| \geqq t\}) \leqq t^{-p}\|f\|_p^p. \tag{1.2}$$

(v) $R_{m,p}$ is an outer capacity, i.e. $R_{m,p}(A)=\inf_{G\supset A} R_{m,p}(G)$, $G=$open set in $\mathbf{R}^n$. Furthermore, all analytic sets are capacitable, i.e. $R_{m,p}(A)=\sup_{K\subset A} R_{m,p}(K)$, K compact in $\mathbf{R}^n$ and A analytic.

(vi) If $p\geqq n/m$, there are no sets of positive $R_{m,p}$ capacity.

In particular, property (vi) implies that given any set A, there is an $f\in L_p^+$ such that $I_m f(x)=+\infty$ on A when $p\geqq n/m$. Hence the restriction $p<n/m$ is essential when working with Riesz potentials of L_p functions.

The Bessel capacity of order m and degree p is defined by replacing $I_m f$ by $G_m f$ in the definition of $R_{m,p}$. We write $B_{m,p}$ for the Bessel capacity and note that it too has properties (i)—(v). However, since $g_m(x)$ decays exponentially as $|x|\to\infty$, $B_{m,p}$ does not satisfy (vi). For $1\leqq p<n/m$, $R_{m,p}$ and $B_{m,p}$ are locally equivalent, i.e. there is a constant $c>0$ such that $c^{-1}R_{m,p}(A)\leqq B_{m,p}(A)\leqq cR_{m,p}(A)$ for all sets A contained in a fixed ball (c depends on the radius of this ball).

$C_0^\infty(\mathbf{R}^n)$ denotes the infinitely differentiable functions (C^∞) on $\mathbf{R}^n$ with compact support, $\mathcal{S}=\mathcal{S}(\mathbf{R}^n)$, the class of rapidly decreasing C^∞ functions on $\mathbf{R}^n$. $W^{m,p}(\mathbf{R}^n)$, m = positive integer, denotes the usual Sobo'ev spaces, i.e. those $u\in L_p(\mathbf{R}^n)$ which have distribution derivatives $(\partial/\partial x)^\alpha u$ of orders $\leqq m$ which belong to $L_p(\mathbf{R}^n)$. $\alpha=(\alpha_1, \ldots, \alpha_n)$ a multiindex, α_i=positive integer, $(\partial/\partial x)^\alpha=(\partial/\partial x_1)^{\alpha_1}\ldots(\partial/\partial x_n)^{\alpha_n}$, $x=(x_1, \ldots, x_n)$. $|\alpha|=\alpha_1+\ldots+\alpha_n$.

The Lorentz spaces $L(p, q)(\mu)$ with respect to the measure μ on $\mathbf{R}^n$ consist of those f for which $(0<p, q<\infty)$

$$\|f\|_{L(p,q)(\mu)} = \left(\int_0^\infty [\mu(\{x: |f(x)| \geqq t\})^{1/p} t]^q \frac{dt}{t}\right)^{1/q} < \infty$$

when μ=Lebesgue measure on $\mathbf{R}^n$, we shall write $L(p, q)$. For $q=\infty$, $\|f\|_{L(p,\infty)(\mu)}$ is $\sup_{t>0}\mu[|f|\geqq t]^{1/p}t$.

Throughout the paper, the letter C will denote various constants which may differ from one formula to the next even within a single string of estimates. In general we make no attempt to obtain the best values for these constants.

2. Smooth truncation

If $H(t)$ is uniformly Lipschitz on $\mathbf{R}^1$ with $H(0)=0$ and such that $H'(t)$ has only a finite number of discontinuities, then it is well known that $H(u)\in W^{1,p}(\mathbf{R}^n)$ whenever $u\in W^{1,p}\mathbf{R}(^n)$, $1\leqq p\leqq\infty$. (In particular, because of the choice $H(t)=t$, for $|t|\leqq 1$ and $=\operatorname{sgn} t$, for $|t|>1$, it is natural to refer to $H(u)$ as a general "truncation" operator on $W^{1,p}(\mathbf{R}^n)$.) For higher order derivatives, this is no longer the case, i.e. $H(u)\notin W^{m,p}(\mathbf{R}^n)$, for $m\geqq 2$. Indeed, $\partial/\partial x_j H(u)$ is not, in general, absolutely continuous on lines and hence is not in $W^{m-1,p}(\mathbf{R}^n)$. This failure is due, of course, to the fact that H is not sufficiently differentiable. Consequently, we might ask whether $H(u)\in W^{m,p}(\mathbf{R}^n)$ given $u\in W^{m,p}(\mathbf{R}^n)$ provided $H(t)$ is a sufficiently smooth function which vanishes at the origin, or in otherwords, is the "smooth truncation" operator $H(u)$ bounded on $W^{m,p}$? When $mp\geqq n$, the results is true and well known. One needs only apply the Sobolev inequality $\|D^k u\|_q\leqq C\|D^m u\|_p$, $q=np/(n-(m-k)p)$, when $mp=n$ and the Nirenberg inequality, $\|D^k u\|_{mp/k}\leqq C\|D^m u\|_p^{k/m}\cdot\|u\|_\infty^{1-k/m}$, when $mp>n$ to the various terms that arise in estimating the L_p norms of the derivatives of $H(u)$. However, when $1\leqq p<n/m$, little seems to be known regarding this problem, thongh it seems apparent that one will have to either restrict the class of H's and/or the class of u's in this case.

In this section, we give two results of this type. Note that they both avoid assuming that the u are bounded functions on $\mathbf{R}^n$. The first result, Theorem 2, is the principle tool of this paper. It applies only to functions u of the form $I_m f$, where

$f \in L_p^+$, $1<p<n/m$. The second, Theorem 3, applies only to $u \in W_+^{2,p}$, or, for that matter, to any $u \in W^{2,p}$ which is either bounded above or below. This last result is a simple modification of a technique due to Maz'ya [12]. We mention it here mainly because it does not seem to be generally known. It is exactly the technique of Maz'ya in his proof of Theorem 1 for the case $m=2$. Whether or not theorem 3 remains valid for $m>2$ remains open.

Theorem 2. *Let $H_j(t)$, $j=0, \pm 1, \pm 2, \dots$, be a doubly infinite sequence of $C^m(\mathbf{R}^1)$ functions identically zero for $t<0$ with H_j' having disjoint carriers on $(0, \infty)$ and such that*

$$\sup_{t>0} |t^{k-1} H_j^{(k)}(t)| \leqq M < \infty \tag{2.1}$$

$k=0, 1, \dots, m$. Then for all $f \in L_p^+$, $1<p<n/m$, there is a constant C depending only on n, m, p and M such that

$$\sum_j \left\| \left(\frac{\partial}{\partial x}\right)^{\alpha} H_j(I_m f) \right\|_p^p \leqq C \|f\|_p^p,$$

where α is a multiindex with $|\alpha|=m$.

The proof rests on the following

Lemma 1. *If $u(x)=I_m f(x)$, m=positive integer$<n$, $f \in L_p^+$, then for $1<p<n/m$, and all multi-indices β, $0<|\beta|<m$,*

$$\int \frac{\left|\left(\frac{\partial}{\partial x}\right)^{\beta} u(x)\right|^{p\sigma}}{u(x)^{p(\sigma-1)}}\, dx \leqq C \|f\|_p^p,$$

where $\sigma=m/|\beta|$, and C is independent of f.

Proof. We begin by estimating the Riesz potential $I_m f$ pointwise by the method of Hedberg [10]. For $0<\theta<1$,

$$I_{m\theta} f(x) = \int_{|x-y|<\delta} \dots dx + \int_{|x-y|\geqq\delta} \dots dx = I' + I'',$$

for some $\delta>0$ to be specified.

$$I' = \sum_{k=0}^{\infty} \int_{2^{-k-1}\delta \leqq |x-y| < 2^{-k}\delta} |x-y|^{m\theta-n} f(y)\, dy \leqq$$

$$\leqq C \sum (2^{-k-1}\delta)^{m\theta-n} (2^{-k}\delta)^n Mf(x) = C\delta^{m\theta} Mf(x),$$

where $Mf(x)$ is the usual Hardy—Littlewood maximal function.

$$I'' \leqq \delta^{m\theta-m} I_m f(x).$$

Now choosing $\delta=\delta(x)=[I_m f(x)/Mf(x)]^{1/m}$, we have

$$I_{m\theta} f(x) \leqq C[I_m f(x)]^{\theta}[Mf(x)]^{1-\theta}.$$

Letting $\theta=1/\sigma'=1-1/\sigma$, $\sigma=m/|\beta|$, and noting the estimate

$$\left|\left(\frac{\partial}{\partial x}\right)^{\beta} u(x)\right| \leqq CI_{m-|\beta|} f(x) = CI_{m/\sigma'} f(x),$$

we have

$$\int \frac{\left|\left(\frac{\partial}{\partial x}\right)^{\beta} u(x)\right|^{p\sigma}}{u(x)^{p(\sigma-1)}}\, dx \leqq C\int \frac{[I_{m/\sigma'} f(x)]^{p\sigma}}{[I_m f(x)]^{p(\sigma-1)}}\, dx \leqq C\int [Mf(x)]^p\, dx \leqq C\int f(x)^p\, dx,$$

by the well known L_p estimate for Mf.

Proof of Theorem 2. With $u=I_m f$ and $|\beta|=m$,

$$\left(\frac{\partial}{\partial x}\right)^{\beta} H_j(u) = \sum_{k=1}^{m} H_j^{(k)}(u) \sum C_{\beta}\cdot\left(\frac{\partial}{\partial x}\right)^{\beta^1} u(x) \dots \left(\frac{\partial}{\partial x}\right)^{\beta^k} u(x)$$

where the last sum is over all multi-indices $\{\beta^1, \dots, \beta^k\}$ such that $\beta^1+\dots+\beta^k=\beta$. Upon applying (2.1), we see that the L_p-norm of $(\partial/\partial x)^{\beta} H_j(u)$ does not exceed the p-th root of

$$CM^p \sum_{k=1}^{m} \int_{s_j} \frac{\left|\left(\frac{\partial}{\partial x}\right)^{\beta^1} u(x) \dots \left(\frac{\partial}{\partial x}\right)^{\beta^k} u(x)\right|^p}{u(x)^{p(k-1)}}\, dx$$

where s_j denotes the carrier of $H_j'(u)$ which are a priori disjoint. Summing over j gives

$$\sum_j \left\|\left(\frac{\partial}{\partial x}\right)^{\beta} H_j(u)\right\|_p^p \leqq C \sum_{k=1}^{m} \int \frac{\left|\left(\frac{\partial}{\partial x}\right)^{\beta^1} u(x) \dots \left(\frac{\partial}{\partial x}\right)^{\beta^k} u(x)\right|^p}{u(x)^{p(k-1)}}\, dx. \tag{2.2}$$

For the terms in (2.2) corresponding to $k>1$, we have $0<|\beta^i|<m$, $i=1, \dots, k$. Hence choosing $p_i=m/|\beta^i|$, then $1<p_i<\infty$ and $\sum 1/p_i=1$. Hölder's inequality applied to those terms gives

$$\int \frac{\left|\left(\frac{\partial}{\partial x}\right)^{\beta^1} u(x)\right|^p}{u(x)^{p/p_1'}} \cdots \frac{\left|\left(\frac{\partial}{\partial x}\right)^{\beta^k} u(x)\right|^p}{u(x)^{p/p_k'}}\, dx \quad (1/p_i' = 1-1/p_i)$$

$$\leqq \prod_{i=1}^{k} \left(\int \frac{\left|\left(\frac{\partial}{\partial x}\right)^{\beta^i} u(x)\right|^{pp_i}}{u(x)^{p(p_i-1)}}\, dx\right)^{1/p_i} \leqq C \prod_{i=1}^{k} (\|f\|_p^p)^{1/p_i} = C\|f\|_p^p$$

by Lemma 1. For the terms in (2.2) corresponding to $k=1$, $|\beta^1|=m$. In this case $(\partial/\partial x)^{\beta^1}u=K_0*f$, a Calderón—Zygmund transform of f, and consequently, $\|K_0*f\|_p\leqq C\|f\|_p$ which gives the remaining required estimate to prove the theorem.

Now the result of Maz'ya — we restrict ourselves to a single function H:

Theorem 3. *Let $H(t)$ be a $C^2(\mathbf{R}^1_+)$ function satisfying (2.1) for $k=0, 1, 2$, then for all $u\in W^{2,p}_+(\mathbf{R}^n)$, $1<p<n/2$, there is a constant C depending only on n, p and M such that*

$$\|H(u)\|_{2,p} \leqq C\|u\|_{2,p}.$$

Proof. Because of (2.1), it is clear that we need only concentrate on estimating the L_p-norm of $D_{ij}H(u)=H'(u)\cdot D_{ij}u+H''(u)D_iu\cdot D_ju$, $D_i=\partial/\partial x_i$, $D_{ij}=\partial^2/\partial x_i\partial x_j$. The first term is clearly in L_p. It is the second that requires attention. By Hölders inequality, it suffices to show that $|H''(u)|^{1/2}|D_ju|\in L_{2p}$. To this end we claim

$$\int \frac{|D_ju|^{2p}}{u^p}\,dx \leqq C\|D_{jj}u\|_p^p \tag{2.3}$$

which by (2.1) gives the theorem. To show (2.3), we first consider $u\in C_0^\infty(\mathbf{R}^n)^+$ and show

$$\int \frac{|D_ju|^{2p}}{(1+u)^p}\,dx \leqq C\|D_{jj}u\|_p^p \tag{2.4}$$

with, of course C independent of u. (2.3) will then follow by taking a sequence $u_k\in C_0^\infty(\mathbf{R}^n)^+$ with $D_{jj}u_k\to D_{jj}u$ in L_p and $u_k\to u$ and $D_ju_k\to D_ju$ a.e. (which is always possible for $u\in W^{2,p}$), and then applying Fatou's lemma to the left side of (2.4); and finally replacing u by u/ε, $\varepsilon>0$ giving

$$\int \frac{|D_ju|^{2p}}{(\varepsilon+u)^p}\,dx \leqq C\|D_{jj}u\|_p^p$$

and letting $\varepsilon\to 0$.

To see (2.4) write

$$\begin{aligned} I &= \int |D_ju|^{2p}(1+u)^{-p}\,dx = \int (\operatorname{sgn} D_ju)|D_ju|^{2p-1}(1+u)^{-p}D_j(u+1)\,dx = \\ &= -(2p-1)\int |D_ju|^{2p-2}D_{jj}u(1+u)^{-p+1}\,dx + p\int(\operatorname{sgn} D_ju)|D_ju|^{2p-1}(1+u)^{-p}D_ju\,dx \end{aligned} \tag{2.5}$$

upon integrating by parts. There are no boundary terms since D_ju has compact support and $p>1$. But (2.5) is just

$$I = \frac{(2p-1)}{p-1}\int |D_ju|^{2(p-1)}(1+u)^{-(p-1)}\cdot D_{jj}u\,dx.$$

Hence

$$I \leqq \frac{(2p-1)}{p-1}\cdot I^{1/p'}\|D_{jj}u\|_p^p$$

which gives the result!

3. Proof of Theorem 1

We first show that for $1<p<n/m$, the Riesz capacity $R_{m,p}$ is equivalent to

$$C_{m,p}(K) = \inf\left\{\sum_{|\alpha|=m}\left\|\left(\frac{\partial}{\partial x}\right)^\alpha u\right\|_p^p : u\in C_0^\infty(\mathbf{R}^n),\ u\equiv 1 \text{ in a neighborhood of } K\right\},$$

where K is compact in $\mathbf{R}^n$. In particular, we give a new proof of the following known result (see [11] or the author's paper with J. Polking, Proc. Amer. Math. Soc. 37 (1973)). The proofs there rely on knowledge of the boundedness of the $R_{m,p}$-capacitary potential $I_m f$.

Proposition 1. *There is a constant $c>0$ such that*

$$c^{-1}R_{m,p}(K) \leqq C_{m,p}(K) \leqq cR_{m,p}(K) \tag{3.1}$$

for all compact K, c independent of K.

Proof. For the first inequality in (3.1), recall the representation $(u\in C_0^\infty(\mathbf{R}^n))$

$$u(x) = \sum_{|\alpha|=m} c_\alpha K_\alpha * \left[\left(\frac{\partial}{\partial x}\right)^\alpha u\right](x) \tag{3.2}$$

where K_α is the convolution operator $K_{\alpha_1} * \ldots * K_{\alpha_n}$, $\alpha=(\alpha_1, \ldots, \alpha_n)$, with K_{α_i} the convolution operator whose kernel is $\omega_{n-1}^{-1}\cdot x_i|x|^{-n}$ convolved α_i-times. See [15 p. 125]. Clearly, $|u(x)|\leqq CI_m\left(\sum_{|\alpha|=m}|(\partial/\partial x)^\alpha u|\right)(x)$, hence taking $f=C\cdot\sum_{|\alpha|=m}|(\partial/\partial x)^\alpha u|$, we have $I_m f\geqq 1$ on K which easily gives the result.

For the second inequality in (3.1), we apply Theorem 2 for one H-function. Namely, let $H(t)\equiv 0$, for $t\leqq 1/2$, and $\equiv 1$, for $t\geqq 1$, and $C^\infty(\mathbf{R}^1)$ otherwise. Now choose a $\psi\in C_0^\infty(\mathbf{R}^n)^+$ such that supp $\psi\subset B_1(0)$, $\int\psi\,dx=1$. And set $\psi_h(x)=h^{-n}\psi(x/h)$, $h>0$. Choose an open bounded set $G\supset K$ and any $f\in L_p^+$ such that $I_m f\geqq 1$ on G. Then for sufficiently small h, $\psi_h * I_m f\geqq 1$ on G_1, where, $K\subset G_1\subset G$, G_1 open with $\bar{G}_1\subset G$. Now set $u(x)=H\left(I_m(\psi_h * f)(x)\right)$. Clearly $u\in C_0^\infty(\mathbf{R}^n)$ and $u\equiv 1$ on G_1. And by Theorem 2

$$C_{m,p}(K) \leqq \sum_{|\alpha|=m}\left\|\left(\frac{\partial}{\partial x}\right)^\alpha u\right\|_p^p \leqq C\|\psi_h * f\|_p^p \leqq C\|f\|_p^p.$$

Thus $C_{m,p}(K)\leqq cR_{m,p}(G)$, for all such G. The result now follows since $R_{m,p}$ is an outer capacity.

We extend the definition of $C_{m,p}$ to the class of all subsets of $\mathbf{R}^n$ as an inner capacity, i.e. for any $A\subset\mathbf{R}^n$, $C_{m,p}(A)$ is defined to be $\sup_{K\subset A} C_{m,p}(K)$, where K is compact. Clearly, this definition agrees with our earlier one when A is compact, since $C_{m,p}$ is monotone.

Since $I_m f$, $f \geqq 0$, is lower semi-continuous, the set $\{x: I_m f(x) \geqq t\}$ is a G_δ-set and hence $R_{m,p}$-capacitable. Thus, due to Proposition 1, there is no loss in proving Theorem 1 with $R_{m,p}$ replaced by $C_{m,p}$. Furthermore, it suffices to consider only $f \in \mathcal{S}^+$ in proving Theorem 1, since by taking a sequence $f_k \in \mathcal{S}^+$, $\|f_k - f\|_p \to 0$ as $k \to \infty$, we can write $R_{m,p}(\{x: I_m f(x) \geqq 2t\}) \leqq t^{-p} \|f - f_k\|_p^p + R_{m,p}(\{x: I_m f_k(x) \geqq t\})$. Hence $\lim_{k\to\infty} R_{m,p}(\{x: I_m f_k(x) \geqq t\}) \geqq R_{m,p}(\{x: I_m f(x) \geqq 2t\})$.

With this done, we estimate as follows: with $u = I_m f$,

$$\int_0^\infty C_{m,p}(\{x: u(x) \geqq t\})\, dt^p = \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} C_{m,p}(\{x: u(x) \geqq t\})\, dt^p \leqq$$

$$\leqq 2^p \sum_{-\infty}^{\infty} C_{m,p}(\{x: u(x) \geqq 2^j\}) 2^{jp}.$$

Now set $H(t) \equiv 0$, for $t \leqq 0$ and $\equiv 1$, for $t \geqq 1$, and $C^\infty(\mathbf{R}^1)$ otherwise. Let $H_j(t) = 2^j H(2^{2-j} t - 1)$. Then $H_j(u) \in C_0^\infty(\mathbf{R}^n)$ with $2^{-j} H_j(u) \equiv 1$ on $\{x: u(x) > 2^{j-1}\}$, a neighborhood of $\{x: u(x) \geqq 2^j\}$, a compact set. Thus

$$\sum_{-\infty}^{\infty} C_{m,p}(\{x: u(x) \geqq 2^j\}) 2^{jp} \leqq \sum_{-\infty}^{\infty} \sum_{|\alpha|=m} \left\| \left(\frac{\partial}{\partial x}\right)^\alpha H_j(u) \right\|_p^p \leqq C \|f\|_p^p$$

by Theorem 2, since H_j clearly satisfies (2.1).

Remark 1. Using the Hölder and Sobolev inequalities with $1 < p < n/m$ we can write

$$|A| \leqq \int_A I_m f\, dx \leqq \|I_m f\|_{p^*} |A|^{1-1/p^*} \leqq C \|f\|_p |A|^{1-1/p^*},$$

where A is a Borel set, with its measure $|A| < \infty$ and $f \in L_p^+$, with $I_m f \geqq 1$ on A, $p^* = np/(n - mp)$. Hence

$$|A|^{1-mp/n} \leqq C R_{m,p}(A).$$

This inequality also holds if $|A| = \infty$. Putting this into Theorem 1 gives

$$\int_0^\infty [|\{x: |I_m f(x)| \geqq t\}|^{1/p^*} t]^p \frac{dt}{t} \leqq C \|f\|_p^p,$$

which is O'Neil's Theorem [14],

$$\|I_m f\|_{L(p^*, p)} \leqq C \|f\|_p,$$

$1 < p < p^* < \infty$, a sharpened form of the Sobolev inequality. In the next section, we pursue this idea further.

4. Applications

We first consider "trace" theorems for Riesz potentials.

Theorem 4. *For $1<p<n/m$, $1<q<\infty$, μ a Borel measure, the following are equivalent:*

(a) $\mu(K)\leqq A_1^q R_{m,p}(K)^{q/p}$, *for all compact sets K;*

(b) $\|I_m f\|_{L(q,p)(\mu)}\leqq A_2\|f\|_p$, *for all $f\in L_p$;*

(c) $\|I_m f\|_{L(q,\infty)(\mu)}\leqq A_3\|f\|_p$, *for all $f\in L_p$;*

(d) $\|I_m\mu_K\|_{p'}\leqq A_4\mu(K)^{1/q'}$, *for all compact K. (μ_K denotes μ restricted to K). Furthermore, the constants A_i satisfy: $A_1\leqq A_4\leqq q'A_3\leqq q'A_2\leqq q'p^{-1/p}CA_1$, where C is the constant of Theorem 1.*

In particular, (b) and (c) say that I_m maps L_p into $L_p(\mu)$ continuously if and only if it maps L_p into weak $-L_p(\mu)$, i.e. $f\to I_m f$ is of strong type-p if and only if it is of weak type-p.

Proof. We show (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d)$\Rightarrow$(a). The first of these is immediately obvious by Theorem 1. (b)$\Rightarrow$(c) is clear since $\|I_m f\|_{L(q,\infty)(\mu)}\leqq\|I_m f\|_{L(q,p)(\mu)}$. For (c)$\Rightarrow$(d), we write

$$\int|I_m f|\,d\mu_K\leqq\int_0^\infty\mu_K[|I_m f|\geqq t]\,dt=\int_0^\delta+\int_\delta^\infty.$$

$$\int_\delta^\infty\leqq A_3^q\|f\|_p^q\int_\delta^\infty t^{-q}\,dt=\frac{A_3^q}{q-1}\delta^{1-q}\|f\|_p^q,$$

and

$$\int_0^\delta\leqq\mu(K)\cdot\delta.$$

Choosing $\delta=A_3\|f\|_p/\mu(K)^{1/q}$ gives

$$\left|\int I_m f\,d\mu_K\right|\leqq q'A_3\|f\|_p\mu(K)^{1-1/q}.$$

But since $\int I_m f\,d\mu_K=\int f\cdot I_m\mu_K\,dx$, the result follows by duality.

Finally (d)$\Rightarrow$(a): choose $f\in L_p^+$ such that $I_m f\geqq 1$ on K, then

$$\mu(K)\leqq\int_K I_m f\,d\mu\leqq\|f\|_p\|I_m\mu_K\|_{p'}\leqq\|f\|_p A_4\mu(K)^{1-1/q}.$$

Hence $\mu(K)\leqq A_4R_{m,p}(K)^{1/p}$.

Remark 2. (i) Note that for any F, $\|F\|_{L(q,q)(\mu)}\leqq C\|F\|_{L(q,p)(\mu)}$, when $q\geqq p$, hence (a) is clearly sufficient for

(b)′ $\|I_m f\|_{q,\mu}\leqq A\|f\|_p$, for all $f\in L_p$, $q\geqq p$.

That (a) is also necessary here follows by choosing $f\in L_p^+$ with $I_m f\geqq 1$ on K and writing

$$\mu(K) \leqq \int (I_m f)^q \, d\mu \leqq A^q \|f\|_p^q.$$

Hence

$$\mu(K) \leqq A^q R_{m,p}(K)^{q/p}, \quad 1 < p < n/m.$$

(ii) When $p<q\leqq p^*=np/(n-mp)<\infty$, (a) can be replaced by a simplier condition (for any $m\in(0,n)$)

(a′) $\mu(B_\varrho(x)) \leqq A' \varrho^{(n-mp)q/p}$, for all $\varrho > 0$ and all $x\in \mathbf{R}^n$.

($B_\varrho(x)$ is the ball of radius ϱ centered at x).

The fact that (a)⇒(a)′ is a simple matter of calculating the $R_{m,p}$-capacity of $B_\varrho(x)$. For the implication (a)′⇒(b)′ (and hence (a)) see [3], where estimates of this kind are done in much greater generality.

(iii) Also note that (a) and (a)′ are no longer equivalent when $q=p$. Indeed, we need only take a Borel set K with $0<H^{n-mp}(K)<\infty$ (H^d — Hausdorff mesaure of dimension $d>0$), then by a well known theorem of Frostman [6], there is a Borel measure $\mu\not\equiv 0$ supported by K such that (a)′ holds with $q=p$. But by [13] — see Theorem 21 — $R_{m,p}(K)=0$. Hence there is an $f\in L_p^+$ such that $I_m f=+\infty$ on K. Thus (b)′ fails (with $q=p$).

(iv) In the case $mp=n$, for any $m\in(0,n)$, the condition

$$\mu(B_\varrho(x)) \leqq C\varrho^d, \text{ for all } \varrho > 0 \text{ and } x\in \mathbf{R}^n \text{ and some } d > 0,$$

μ a Borel measure with compact support,
is at least sufficient for

$$\sup_{\|f\|_p\leqq 1} \int \exp(c|g_m * f|^{p'}) \cdot d\mu < \infty.$$

This is shown in [2].

(v) Using Theorem 1, it is not hard to find a sufficient condition of type (a) for (b)′ to hold for $1\leqq q<p$. In fact, the condition

(a)″ $\sup_S \sum_{j=-\infty}^{\infty} [\mu(G_j)^{p/q}/R_{m,p}(G_j)]^{r/(p-r)} < \infty$,

where $S=\{G_j\}$ is any doubly infinite system of nested bounded open sets $G_j\subset\mathbf{R}^n$, implies

(b)″ $\|I_m f\|_{L(q,r)(\mu)} \leqq C\cdot \|f\|_p$, for all $f\in L_p$,

when $1\leqq r<p<n/m$, $1\leqq q<\infty$.

To see this, we need only consider $f\in\mathcal{S}^+$. Set $G_j=\{x: I_m f(x)>2^j\}$, then

$$\int_0^\infty \{\mu[I_m f\geqq t]^{1/q}\cdot t\}^r\frac{dt}{t}\leqq \sum_{-\infty}^{\infty}\mu(G_j)^{r/q}\cdot 2^{jr}\leqq C\big(\sum_{-\infty}^{\infty} R_{m,p}(G_j)\cdot 2^{jp}\big)^{r/p}$$

by Hölder's inequality. Now apply Theorem 1. The interested reader should compare this with the corresponding result in [12].

When $q=r=1$, (a)″ can be replaced by a simpler necessary and sufficient condition for (b)″ as has been pointed out in [12], namely

(d)′ $\|I_m\mu\|_{p'}<\infty$.

Indeed, (d)′⇒(b)″, $q=r=1$, since

$$\int |I_m f|\,d\mu\leqq\int |f|\cdot I_m\mu\,dx\leqq \|f\|_p\cdot\|I_m\mu\|_{p'}.$$

And for (b)″⇒(d)′, one simply uses duality.

(vi) Finally, we remark concerning a relationship between the non-linear potentials of the measure μ,

$$U_{m,p}(\mu;x)=I_m(I_m\mu)^{1/(p-1)}(x)$$

and Theorem 4. These functions have been useful in the study of the pointwise regularity of the linear potentials $I_m f$, $f\in L_p$ — see [5] and [9].

First of all, we note that the condition

(e) $U_{m,p}(\mu;x)\leqq A_5$, for all $x\in\operatorname{supp}\mu$

for any $m\in(0,n)$, is strictly stronger than (b) with $q=p$. To see this, we need only show (e)⇒(d) and this is obvious upon writing

$$\|I_m\mu_K\|_{p'}^{p'}=\int U_{m,p}(\mu;x)\,d\mu_K(x)\leqq A_5\mu(K).$$

However, by Theorem 5 of [2], the measure $d\mu=|x|^{-\alpha p}dx$ satisfies (b) with $q=p$, but $U_{m,p}(\mu;x)$ is unbounded at $x=0$. This last fact follows by estimate (3) of Theorem 2 in [2].

Motivated again by [12], we can, however, obtain a partial converse to the above, namely: (b)″ with $q=p$ implies $U_{mr,p/r}(\mu;x)$ is bounded in $\mathbf{R}^n$, provided

$$1\leqq r<p \quad\text{and}\quad r\left(2-\frac{mr}{n}\right)<p<n/m. \tag{4.1}$$

This will follow by showing that (b)″ with $q=p$ implies

$$\sup_x\int_0^\infty [\varrho^{mp-n}\mu(B_\varrho(x))]^{r/(p-r)}\frac{d\varrho}{\varrho}<\infty,$$

for p satisfying (4.1). See [2].

To that end, let $\varphi \in C_0^\infty(\mathbf{R}^n)$ such that $\varphi(x)=1$, $|x|\leqq 1$, and $\varphi(x)=0$, $|x|\geqq 2$. Set $\varphi_k(x)=\varphi(x-x_0)/2^k$, $x_0\in\mathbf{R}^n$, $k=0, \pm 1, \dots, \pm N$, and $B_k=B_{2^k}(x_0)$. Also let $\psi(x)=(\tau_k-\tau_{k+1})\varphi_k(x)+\tau_{k+1}$, for $x\in B_{k+1}\sim B_k$, $\psi(x)=\tau_{-N}$, for $x\in B_{-N}$, and $\psi(x)=0$, for $x\in\sim B_{N+1}$. Here

$$\tau_k=\sum_{j=k}^N [\mu(B_j)^{r/p}/2^{j(n-mp)}]^{1/(p-r)}.$$

Clearly, $\psi\in C_0^\infty(\mathbf{R}^n)$ and $B_k=\{x\colon \psi(x)>\tau_k\}$. Thus

$$\begin{aligned}\int_0^\infty \{\mu[\psi\geqq t]^{1/p}\cdot t\}^r\frac{dt}{t} &\geqq \frac{1}{r}\sum_{-N}^N \mu(B_k)^{r/p}(\tau_k^r-\tau_{k+1}^r)\geqq\\ &\geqq \frac{1}{r}\sum_{-N}^N \mu(B_k)^{r/p}(\tau_k-\tau_{k+1})^r=\frac{1}{r}\sum_{-N}^N [\mu(B_k)/2^{k(n-mp)}]^{r/(p-r)},\end{aligned}\tag{4.2}$$

since $r\geqq 1$. Also for $|\alpha|=m$,

$$\begin{aligned}\int\left|\left(\frac{\partial}{\partial x}\right)^\alpha\psi\right|^p dx &= \sum_{-N}^N\int_{B_{k+1}\sim B_k}\left|\left(\frac{\partial}{\partial x}\right)^\alpha\psi\right|^p dx=\\ &=\sum_{-N}^N(\tau_k-\tau_{k+1})^p\int\left|\left(\frac{\partial}{\partial x}\right)^\alpha\varphi_k\right|^p dx=C\sum_{-N}^N[\mu(B_k)/2^{k(n-mp)}]^{r/(p-r)}\end{aligned}\tag{4.3}$$

by the choice of φ_k.

But since (b)″ is equivalent to

$$\|\psi\|_{L(p,r)(\mu)}\leqq C\cdot\sum_{|\alpha|=m}\left\|\left(\frac{\partial}{\partial x}\right)^\alpha\psi\right\|_p$$

for all $\psi\in C_0^\infty(\mathbf{R}^n)$, the result follows by substituting in (4.2) and (4.3).

Our next application of Theorem 1 concerns certain maximal operators that arise from sequences of (principal value) convolution operators. In particular, consider a sequence θ_k, $k=1, 2, \dots$ and $T^kf=\theta_k*f$, where f is initially taken to be of class $\mathscr{S}$, say. We then set

$$T^*f(x)=\sup_k|T^kf(x)|.$$

We say that T^* is of strong type-p or weak type-p if T^* can be extended as a continuous sublinear operator from $L_p(\mathbf{R}^n)$ into $L_p(\mathbf{R}^n)$, or respectively from $L_p(\mathbf{R}^n)$ into weak $L_p(\mathbf{R}^n)$, i.e. $L(p, \infty)$. In order for these definitions to be meaningful, it is only necessary to know the value of the distribution function for T^*f, $|\{x\colon T^*f(x)\geqq t\}|$. Thus in analogy with the above definitions, we replace the usual distribution function by the $R_{m,p}$-capacitary distribution function:

$$R_{m,p}(\{x\colon T^*(I_mf)(x)\geqq t\}).\tag{4.4}$$

Then we shall speak of capacitary strong type-(m,p) or capacitary weak type-(m,p), respectively. $0<m<n$, $1<p<n/m$ throughout.

Proposition 2. *If T^* is of strong type-p, then it is also of capacitary strong type-(m, p) for m positive integer. And for any $m\in(0, n)$, T^* of strong type-p implies capacitary weak type-(m, p).*

Proof. For $f\in\mathcal{S}$, $T^k(I_m f) = I_m(T^k f)$, hence $T^*(I_m f) \leqq I_m(T^* f)$. Thus by Theorem 1,

$$\int_0^\infty R_{m,p}(\{x\colon T^*(I_m f)(x) \geqq t\})\, dt^p \leqq C\|T^* f\|_p^p \leqq C\|f\|_p^p,$$

which gives the first statement. For the second, just use (1.2) in place of Theorem 1.

It might be noted here that if $R_{m,p}$ is replaced by $B_{m,p}$, and I_m by G_m in (4.4), then it is known that T^* is of strong type-p, for $1<p_0\leqq p\leqq p_1<\infty$, if and only if it is of capacitary weak type-(m, p) uniformly for $m>0$, and the same values of p. This is shown in [1]. The situation there is somewhat different, however, since $g_m\in L_1(\mathbf{R}^n)$.

Examples of T^* in Proposition 2, might arise from sequences of singular integral operators, or merely from approximations to one. For example, we could take

$$T^* f(x) = \sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon} k(x-y) f(y)\, dy\right|,$$

a Calderón—Zygmund maximal operator, i.e. k is a homogeneous function of degree $-n$ with mean value zero on the unit sphere such that its modulus of continuity there satisfies a Dini type condition (see [15]). For this operator, we can think of Proposition 2, as a capacitary version of the classical Calderón—Zygmund L_p estimates for T^*.

We conclude this section now with a simple example of an operator T^* which is of capacitary strong type-(m, p), $m=(n-1)/2$, but not of strong type-p. The operator is the spherical summation operator T_0^*, first defined for smooth f by

$$T_\lambda^R f(x) = \int_{|\xi|<R} (1-|\xi|^2/R^2)^\lambda \hat{f}(\xi)\, e^{ix\cdot\xi}\, d\xi$$

where $\lambda\geqq 0$ and $\hat{f}$ denotes the Fourier transform of f. $T_0^* f = \sup_R |T_0^R f|$.

C. Fefferman has shown in [7] that T_0 is not bounded on L_p, $1<p<\infty$, unless $p=2$. Hence T_0^* can not be bounded on L_p, $p\neq 2$.

Proposition 3. *T_0^* is of capacitary strong type-$((n-1)/2, p)$, for $1<p<2n/(n-1)$, $n=3, 5, 7, \ldots$.*

Proof. In view of Theorem 1, we need only show that if $I_m(x)=|x|^{m-n}$,

$$T_0^* I_m(x) \leqq C\cdot I_m(x),$$

for all x, C independent of x. To see this, we expand $s^{-\alpha/2}$, $s>0$, in its Taylor expansion with remainder about $s=R^2$:

$$s^{-\alpha/2} = R^{-\alpha}-\frac{\alpha}{2}R^{-\alpha-2}(s-R^2)+\frac{\alpha}{2}\left(\frac{\alpha}{2}+1\right)\int_{R^2}^{s}(s-t)t^{-\alpha/2-2}\,dt.$$

Then

$$\begin{aligned} T_0^R I_m(x) &= c\int_{|\xi|<R}|\xi|^{-m}e^{ix\cdot\xi}\,d\xi = CR^{-m}T_0^R(x)+ \\ &+CR^{-m}T_1^R(x)+c\int_0^{R^2}t^{-m/2-1}T_1^{t^{1/2}}(x)\,dt. \end{aligned} \tag{4.5}$$

Now we need to recall the estimate

$$|T_\lambda^R(x)| \leqq CR^n(1+R|x|)^{-n/2-1/2-\lambda}, \quad \lambda \geqq 0. \tag{4.6}$$

See [16, p. 171]. (4.6) applied to the terms of (4.5) gives:

$$|1^{st}\text{ term}| \leqq CR^{-m}R^n(1+R|x|)^{-n/2-1/2} \leqq C|x|^{m-n},$$

when $m=(n-1)/2$,

$$|2^{nd}\text{ term}| \leqq CR^{n-m} \leqq C|x|^{m-n}$$

when $R|x|\leqq 1$, and $\leqq CR^{-1}|x|^{-n/2-1/2-1} \leqq C|x|^{m-n}$ when $R|x|\geqq 1$,

$$|3^{rd}\text{ term}| \leqq c\int_0^{R^2}t^{(n-m)/2}(1+t^{1/2}|x|)^{-n/2-3/2}\frac{dt}{t} \leqq C|x|^{m-n}$$

since $\int_0^\infty t^{n-m}(1+t)^{-n/2-3/2}\,dt/t<\infty$, for $m=(n-1)/2$.

5. Some open questions

(i) One rather intriguing question that remains open is whether or not Theorem 1 is valid for fractional values of $m<n$. A natural way of approaching this might be via some fractional analogue of Theorem 2. In particular, one might study the L_p boundedness of the operators $I_{-m}H(I_m f)$ using some form of interpolation. Here, for example, one can take $I_{-m}=\sum_{j=1}^n R_j\,\partial/\partial x_j\, I_{1-m}$, for $0<m<1$, where R_j denotes the j-th Riesz transform.

There is, however, an alternative approach that seems to produce some results on this question, at least for $0<m<1$. The idea is to consider the Besov spaces $\Lambda_{m,p}(\mathbf{R}^n)$, i.e. those $u\in L_p(\mathbf{R}^n)$ for which

$$|u|_{m,p} \equiv \left\{\iint |\Delta_h u(x)|^p|h|^{-mp-n}\,dx\,dh\right\}^{1/p}<\infty.$$

Here $\Delta_h u(x) = u(x+h)-u(x)$, $0<m<1$. And if we set

$$S_{m,p}(K) = \inf\{|\varphi|^p_{m,p} : \varphi\in C_0^\infty(\mathbf{R}^n),\ \varphi\equiv 1 \text{ on a neighborhood of } K\},$$

K a compact set of $\mathbf{R}^n$, and extended to all subsets of $\mathbf{R}^n$ as an inner capacity in the usual way, then one easily has

$$\int_0^\infty S_{m,p}(\{x : u(x)\geqq t\})\, dt^p \leqq C|u|^p_{m,p}, \tag{5.1}$$

for all $u\in\mathscr{S}^+$, say. To see this, one merely proceeds as in the proof of Theorem 1, showing

$$\sum_j |H_j(u)|^p_{m,p} \leqq C|u|^p_{m,p}$$

as an analogue of Theorem 2.

The next task is to replace $S_{m,p}$ in (5.1) by $R_{m,p}$, and this is done in the spirit of [4] where similar results were obtained for Bessel capacities. In particular, we can show that there are constants c_1 and c_2 such that

$$R^{(n)}_{m,p}(K) \leqq c_1 R^{(n+1)}_{m+1/p,p}(K) \leqq c_2 S^{(n)}_{m,p}(K) \tag{5.2}$$

for all compact $K\subset\mathbf{R}^n$, $1<p<n/m$. Here the superscripts (n) and $(n+1)$ refer to corresponding capacities for $\mathbf{R}^n$ and $\mathbf{R}^{n+1}$. The second inequality in (5.2) is a consequence of the fact that every function u (which is the limit of C_0^∞ — functions) with $|u|_{m,p}<\infty$ on $\mathbf{R}^n$ can be extended to $\mathbf{R}^{n+1}$ as a Riesz potential of order $m+1/p$, of an L_p function on $\mathbf{R}^{n+1}$. This potential equals u on $\mathbf{R}^n$. The first inequality in (5.2) is trivial.

Thus we can replace (5.1) by the weaker inequality

$$\int_0^\infty R_{m,p}(\{x : u(x)\geqq t\})\, dt^p \leqq C(\|u\|^p_p + |u|^p_{m,p}),$$

for all $u\in\mathscr{S}^+$. And now by the well known fact that $L_{m,p}(\mathbf{R}^n)\subset\Lambda_{m,p}(\mathbf{R}^n)$, $p\geqq 2$, [15, p. 155], we have

$$\int_0^\infty R_{m,p}(\{x : G_m f(x)\geqq t\})\, dt^p \leqq C\|f\|^p_p,$$

for all $f\in L_p^+(\mathbf{R}^n)$, $0<m<1$, $2\leqq p<n/m$.

(ii) Throughout this paper we have adhered to finding capacitary strong type estimates on $\mathbf{R}^n$, the idea being that the global behavior of potentials is of interest as well as their local behavior. And so in this regard it would be of interest to know whether or not the Riesz capacities and Riesz potentials in Theorem 1 can be replaced by the corresponding Bessel capacities and Bessel potentials, even for integer values of m. If so it would shed light on the global behavior of Sobolev functions and perhaps even in the case $mp=n$.

One can, however, treat the local theory in the case $mp=n$ by defining Riesz capacity using test functions $f \in L_p^+$ with supports in a fixed ball. In this respect, Theorem 1 remains valid with minor alterations in its proof.

(iii) And finally, does Theorem 3 have an analogue for integer values of $m>2$? Such a result would most certainly have immediate applications in potential theory and partial differential equations, especially if the restriction to non-negative functions could be dropped altogether.

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