On the L^2 continuity of a class of pseudo differential operators

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Introduction

Pseudo differential operators are often defined by means of the formula:

$$Au(x) = \int e^{i\langle x-y,\xi\rangle} a(x, y, \xi) u(y)\, dy\, d\xi \tag{0-1}$$

where $a(x, y, \xi)$ satisfies on $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n$ the inequalities:

$$|D_x^\alpha D_y^\beta D_\xi^\gamma a(x, y, \xi)| \leqq c_{\alpha,\beta,\gamma} \chi(|\xi|) \varphi^{-|\alpha|-|\beta|}(|\xi|) \Phi^{-|\gamma|}(|\xi|)$$

with χ, φ, Φ fixed weight functions on $\overline{\mathbf{R}}_+ = \{\xi \in \mathbf{R},\ \xi \geqq 0\}$. Our aim is to give necessary and sufficient conditions for the weight functions in order that the operators (0-1) are continuous on $L^2(\mathbf{R}^n)$. As a matter of fact, we shall restrict ourselves to the one-dimensional case, $n=1$, and we shall introduce some hypotheses on χ, φ, Φ. In the first section we enunciate the results and we give some applications. Particularly we obtain for the classes of Hörmander $S^m_{\varrho,\delta}$ on $\mathbf{R} \times \mathbf{R} \times \mathbf{R}$ the results in Calderōn—Vaillancourt [3], Hörmander [6] and also a result of Ching [4]. Another application refers to the classes of pseudo differential operators in Beals—Fefferman [1].

In the second section we give the proofs.

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1. Results and applications

Let χ, φ, Φ be strictly positive smooth functions on $\overline{\mathbf{R}}_+$ with the properties:

(i) $\chi(\xi) \leqq c_1, \quad c_2 \leqq \Phi(\xi) \leqq c_3(1+\xi), \quad \varphi(\xi) \leqq c_4\Phi^{-1}(\xi)$

(ii) $|D^\alpha\chi| \leqq c_5^{(\alpha)}\chi\Phi^{-\alpha}, \quad |D^\alpha\Phi| \leqq c_6^{(\alpha)}\Phi^{1-\alpha}, \quad |D^\alpha\varphi| \leqq c_7^{(\alpha)}\varphi\Phi^{-\alpha}$

where c_1, c_2, c_3, c_4, $c_5^{(\alpha)}$, $c_6^{(\alpha)}$, $c_7^{(\alpha)}$, $\alpha=1, 2, \dots$, are positive constants. Consider the functions $a(x, y, \xi)$ on $\mathbf{R}\times\mathbf{R}\times\mathbf{R}$ which satisfy the inequalities:

$$|D_x^\alpha D_y^\beta D_\xi^\gamma a(x, y, \xi)| \leqq c_{\alpha,\beta,\gamma}\chi(|\xi|)\varphi^{-\alpha-\beta}(|\xi|)\Phi^{-\gamma}(|\xi|). \tag{1-1}$$

We want to study the continuity of the operator A, defined as in (0-1). Let us define for ξ and η in $\overline{\mathbf{R}}_+$:

$$f(\xi, \eta) = \min\left\{1, \frac{\varphi^{-1}(\xi)+\varphi^{-1}(\eta)}{|\xi-\eta|}\right\}. \tag{1-2}$$

For each integer $N\geqq 0$, we set:

$$F_N(\eta) = \int_0^\infty f^N(\xi, \eta)\chi(\xi)\Phi^{-1}(\xi)\, d\xi. \tag{1-3}$$

In particular:

$$F_0 = \int_0^\infty \chi(\xi)\Phi^{-1}(\xi)\, d\xi. \tag{1-4}$$

Theorem 1-1. *Let χ, φ, Φ satisfy* (i), (ii) *and let the function $F_N(\eta)$ be bounded, for some integer N. Then, if $a(x, y, \xi)$ satisfies the inequalities* (1-1) *for $\alpha\leqq 2N$, $\beta\leqq 2N$, $\gamma\leqq 2$, the operator A in* (0-1) *is continuous from $L^2(\mathbf{R})$ to $L^2(\mathbf{R})$.*

In particular, let χ and Φ satisfy the properties in (i), (ii) and let the integral (1-4) be convergent. Then, if:

$$|D_\xi^\gamma a(x, y, \xi)| \leqq c_{0,0,\gamma}\chi(|\xi|)\Phi^{-\gamma}(|\xi|) \tag{1-5}$$

for $\gamma\leqq 2$, we can conclude that the operator A is bounded, without any requirement on the derivatives with respect to x and y.

Now we introduce the following property.

(iii) *One of the following two conditions is satisfied: either* $\lim_{\xi\to\infty} d(\varphi^{-1})/d\xi=0$ *or* $d(\varphi^{-1})/d\xi\geqq 1$.

Let us define the subset of $\overline{\mathbf{R}}_+$:

$$V_\eta = \{\xi \geqq 0,\ |\xi-\eta| \leqq \varphi^{-1}(\xi)+\varphi^{-1}(\eta)\}, \quad \eta\in\overline{\mathbf{R}}_+ \tag{1-6}$$

and

(1-7) $$F_\infty(\eta) = \int_{V_\eta} \chi(\xi)\Phi^{-1}(\xi)\,d\xi.$$

Theorem 1-2. *Let χ, φ, Φ satisfy* (i), (ii), (iii) *and suppose that $F_\infty(\eta)$ is no bounded. Then there exists $a(x, y, \xi)$ which satisfies the inequalities* (1-1) *for all α, β, γ and such that the operator A in* (0-1) *is not continuous from $L^2(\mathbf{R})$ to $L^2(\mathbf{R})$.*

Actually, if $\lim_{\xi\to\infty} d(\varphi^{-1})/d\xi=0$, for η sufficiently large V_η is a closed finite interval and the hypothesis of theorem 1-2 is equivalent to the assumption of the existence of a sequence $\eta_1 \geqq \eta_2 \geqq \dots$ such that $\lim_{j\to\infty} F_\infty(\eta_j)=\infty$. If $d(\varphi^{-1})/d\xi \geqq 1$, we have $V_\eta = \overline{\mathbf{R}}_+$ for all η and hence $F_\infty = F_0$. In this case, when we say that F_∞ is not bounded, we mean that the integral (1-4) is not convergent.

From theorem 1-1 and 1-2 we shall deduce the following corollary, by means of a direct evaluation of the integrals in (1-3), (1-7).

Corollary 1-3. *Let χ, φ, Φ satisfy:*

(i)* $\quad \chi(\xi) \leqq c_1, \quad c_2 \leqq \Phi(\xi) \leqq c_3(1+\xi), \quad c_4(1+\xi)^{\varepsilon-1} \leqq \varphi(\xi) \leqq c_5\Phi^{-1}(\xi)$

(ii)* $\quad |D^\alpha\chi| \leqq c_6^{(\alpha)}\chi\varphi^\alpha, \quad |D^\alpha\Phi| \leqq c_7^{(\alpha)}\Phi\varphi^\alpha, \quad |D^\alpha\varphi| \leqq c_8^{(\alpha)}\varphi(1+\xi)^{-\alpha}$

where $\varepsilon, c_1, c_2, c_3, c_4, c_5, c_6^{(\alpha)}, c_7^{(\alpha)}, c_8^{(\alpha)}, \alpha=1,2,\dots$, are positive constants. Let the function $G(\eta)=\chi(\eta)\Phi^{-1}(\eta)\varphi^{-1}(\eta)$ be bounded. Then, if $a(x, y, \xi)$ satisfies the inequalities (1-1) *for $\gamma\leqq 2$, $\alpha \leqq 2M$, $\beta\leqq 2M$, where M is the least integer such that $\varepsilon M>1$, the operator A in* (0-1) *is continuous from $L^2(\mathbf{R})$ to $L^2(\mathbf{R})$.*

Otherwise, if $G(\eta)$ is not bounded, there exists $a(x, y, \xi)$ which satisfies the inequalities (1-1) *for all α, β, γ and such that the operator A is not continuous.*

Now we shall give some applications. At first take $\chi(\xi)=(1+\xi)^m$, $\Phi(\xi)=(1+\xi)^\varrho$, $\varphi(\xi)=(1+\xi)^{-\delta}$, $m\leqq 0$, $0\leqq\varrho\leqq\delta\leqq 1$. Then the inequalities (1-1) define the class $S^m_{\varrho,\delta}$ of Hörmander on $\mathbf{R}\times\mathbf{R}\times\mathbf{R}$ (see Hörmander [5], [6]). We write here $L^m_{\varrho,\delta}$ for the class of operators in (0-1) with symbol of this form.

If we assume in addition $\delta<1$, all the hypotheses of corollary 1-3 are satisfied, with $G(\eta)=(1+\eta)^{m-\varrho+\delta}$. We can conclude that every operator in $L^m_{\varrho,\delta}$, $m\leqq 0$, $0\leqq\varrho\leqq$ $\leqq\delta<1$, is continuous on $L^2(\mathbf{R})$ if and only if $m\leqq\varrho-\delta$. The generalization of this result to the n-dimensional case is proved in Calderōn—Vaillancourt [3], Hörmander [6].

On the other hand, if we assume $\delta=1$, the second condition in (iii) is satisfied and we have:

$$F_0 = F_\infty = \int_0^\infty (1+\xi)^{m-\varrho}\,d\xi.$$

From theorem 1-1 and theorem 1-2 we deduce that every operator in $L^m_{\varrho,\delta}$, $m\leqq 0$, $0\leqq\varrho\leqq\delta$, $\delta=1$, is continuous on $L^2(\mathbf{R})$ if and only if $m<\varrho-1$. In particular, when $m=0$, $\varrho=\delta=1$, we obtain a result of Ching [4], who gave an example of an operator in $L^0_{1,1}$ which is not continuous.

In the final application, we assume in the corollary 1-3 $\chi=1$, $\Phi=\varphi^{-1}$. Then $G(\eta)$ is certainly bounded and if $a(x, y, \xi)$ satisfies the inequalities (1-1) the corresponding operator in (0-1) is continuous. A similar result in the n-dimensional case is proved in Beals—Fefferman [1].

2. Proofs

The proof of theorem 1-1 will be given by a modification of the method used Calderōn—Vaillancourt [3]. Particularly we shall use the following lemma (for the proof see for example Calderōn—Vaillancourt [2]).

Lemma 2-1. *Let $\xi\to A(\xi)$ be a smooth map from the interval $I=\{\xi, 0\leqq\xi\leqq\omega\}$ to continuous operators on $L^2(\mathbf{R})$. Let $h(\xi,\eta)$ be a positive continuous function on $I\times I$ such that*

$$\|A^*(\xi)A(\eta)\| \leqq h^2(\xi,\eta), \quad \|A(\xi)A^*(\eta)\| \leqq h^2(\xi,\eta) \tag{2-1}$$

and for all s

$$\int_{I^{2s}} h(\xi_1,\xi_2)h(\xi_2,\xi_3)\dots h(\xi_{2s-1},\xi_{2s})\,d\xi_1\dots d\xi_{2s} \leqq k\lambda^{2s} \tag{2-2}$$

where the constants k and λ do not depend on s. Then $\|\int_I A(\xi)\,d\xi\|\leqq\lambda$.

Proof of theorem 1-1. A standard limiting argument reduces matters to the task of proving:

$$\|Au\| \leqq c\|u\|$$

for $u\in\mathcal{S}(\mathbf{R})$ and $a(x,y,\xi)$ of compact support, with c depending only on the constants $c_{\alpha,\beta,\gamma}$ and on χ, φ, Φ. We can also suppose without loss of generality $a(x,y,\xi)=0$ for $\xi\leqq0$; hence, for ω sufficiently large, the support of $a(x,y,\xi)$ with respect to the variable ξ is included in $I=\{\xi, 0\leqq\xi\leqq\omega\}$.

We begin by obtaining a different representation of the operator A in (0-1). For this purpose note that

$$[1+\Phi^2(\xi)(x-y)^2]^{-1}[1+\Phi^2(\xi)D_\xi^2]e^{i(x-y)\xi} = e^{i(x-y)\xi}.$$

Substituting in (0-1) and integrating by parts we obtain

$$Au(x) = \int e^{i(x-y)\xi} b(x,y,\xi)u(y)\,dy\,d\xi \tag{2-3}$$

where

$$b(x,y,\xi) = [1+D_\xi^2\Phi^2(\xi)]\{a(x,y,\xi)[1+\Phi^2(\xi)(x-y)^2]^{-1}\}. \tag{2-4}$$

We consider the following representation of A:

$$A = \int_I A(\xi) d\xi$$

where

$$A(\xi)u(x) = \int e^{i(x-y)\xi} b(x, y, \xi) u(y)\, dy.$$

Let us apply lemma 2-1 to $A(\xi)$. The kernel of $A^*(\xi)A(\eta)$ is given by

$$\int e^{-i(\xi-\eta)z+ix\xi-iy\eta} \bar{b}(z, x, \xi) b(z, y, \eta)\, dz. \tag{2-5}$$

Observing that

$$|\xi-\eta|^{-2N}(D_z^2)^N e^{-i(\xi-\eta)z} = e^{-i(\xi-\eta)z}$$

substituting and integrating by parts (2-5) becomes

$$\int e^{-i(\xi-\eta)z+ix\xi-iy\eta} |\xi-\eta|^{-2N}(D_z^2)^N[\bar{b}(z, x, \xi) b(z, y, \eta)]\, dz.$$

Now we use the inequalities:

$$|D_x^\alpha b(x, y, \xi)| \leqq c\chi(\xi)\varphi^{-\alpha}(\xi) H[\Phi(\xi)(x-y)] \tag{2-6}$$

where H is an integrable function and, from now onwards, we shall use the letter c to denote constants depending on χ, φ, Φ and $c_{\alpha,\beta,\gamma}$. Admitting the (2-6) for a moment, it follows that the kernel of $A^*(\xi)A(\eta)$ is majorized by the convolution kernel

$$c\chi(\xi)\chi(\eta)\left[\frac{\varphi^{-1}(\xi)+\varphi^{-1}(\eta)}{|\xi-\eta|}\right]^{2N} \int H[\Phi(\xi)(z-x)] H[\Phi(\eta)(z-y)]\, dz$$

and we have

$$\|A^*(\xi)A(\eta)\| \leqq c\chi(\xi)\chi(\eta)\Phi^{-1}(\xi)\Phi^{-1}(\eta)\left[\frac{\varphi^{-1}(\xi)+\varphi^{-1}(\eta)}{|\xi-\eta|}\right]^{2N}. \tag{2-7}$$

On the other hand, if we majorize (2-5) directly by using the inequality (2-6) with $\alpha=0$, we obtain

$$\|A^*(\xi)A(\eta)\| \leqq c\chi(\xi)\chi(\eta)\Phi^{-1}(\xi)\Phi^{-1}(\eta). \tag{2-8}$$

It follows from (2-7) and (2-8) that:

$$\|A^*(\xi)A(\eta)\| \leqq h^2(\xi, \eta) = c\chi(\xi)\chi(\eta)\Phi^{-1}(\xi)\Phi^{-1}(\eta) f^{2N}(\xi, \eta)$$

where f is defined as in (1-3). Similarly we can prove: $\|A(\xi)A^*(\eta)\| \leqq h^2(\xi, \eta)$.

According to lemma 2-1 we consider

$$\int_{I^{2s}} h(\xi_1, \xi_2) h(\xi_2, \xi_3) \ldots h(\xi_{2s-1}, \xi_{2s})\, d\xi_1 \ldots d\xi_{2s} =$$

$$= c^{2s-1} \int_{I^{2s}} \chi^{1/2}(\xi_1)\chi^{1/2}(\xi_2)\Phi^{-1/2}(\xi_1)\Phi^{-1/2}(\xi_2) f^N(\xi_1, \xi_2) \ldots \tag{2-9}$$

$$\ldots \chi^{1/2}(\xi_{2s-1})\chi^{1/2}(\xi_{2s})\Phi^{-1/2}(\xi_{2s-1})\Phi^{-1/2}(\xi_{2s}) f^N(\xi_{2s-1}, \xi_{2s})\, d\xi_1 \ldots d\xi_{2s}.$$

Using the estimates

$$\chi^{1/2}(\xi_1)\Phi^{-1/2}(\xi_1) \leqq c_1^{1/2}c_2^{-1/2}, f^N(\xi_1, \xi_2) \leqq 1,$$
$$\chi^{1/2}(\xi_{2s})\Phi^{-1/2}(\xi_{2s}) \leqq c_1^{1/2}c_2^{-1/2},$$

where c_1, c_2 are the constants in the property (i), and integrating with respect to ξ_1 and ξ_{2s}, we see that the right-hand side in (2-9) is majorized by

$$\omega^2 c_1 c_2^{-1} c^{2s-1} \int_{I^{2s-2}} f^N(\xi_2, \xi_3)\chi(\xi_2)\Phi^{-1}(\xi_2)\dots$$
$$f^N(\xi_{2s-1}, \xi_{2s})\chi(\xi_{2s-1})\Phi^{-1}(\xi_{2s-1})\, d\xi_2 \dots d\xi_{2s-1}.$$

It follows from the definition of $F_N(\eta)$ in (1-3) that the condition (2-2) is satisfied with $\lambda = c \max_{\eta\in I} F_N(\eta)$. By using the lemma 2-1, in view of the boundedness of $F_N(\eta)$ we can conclude that the norm of A is majorized by a constant depending only on χ, φ, Φ and $c_{\alpha,\beta,\gamma}$. It remains to prove (2-6). To this end we first observe that for a polynomial P and an integer h:

$$\text{(2-10)} \qquad D_x^\alpha\{P[\Phi(\xi)x][1+\Phi^2(\xi)x^2]^{-h}\} = \Phi^\alpha(\xi)Q_\alpha[\Phi(\xi)x][1+\Phi^2(\xi)x^2]^{-k_\alpha}$$

for some polynomials Q_α and integer k_α, with $2k_\alpha - \deg Q_\alpha > 2h - \deg P$. In particular, if $P(z)(1+z^2)^{-h}$ is integrable, $Q_\alpha(z)(1+z^2)^{-k_\alpha}$ is integrable for all α. The (2-10) is readily verified by induction on α. Now we have for $\alpha \leqq 2N$, $\beta \leqq 2$:

$$\text{(2-11)} \qquad D_x^\alpha D_\xi^\beta\{\Phi^2(\xi)[1+\Phi^2(\xi)x^2]^{-1}\} \leqq \Phi^{2+\alpha-\beta}(\xi)H[\Phi(\xi)x]$$

where H is an integrable function. To prove this, we compute directly in (2-11) the derivatives with respect to ξ and then we use the (2-10) and the inequalities in the property (ii). Finally, using (2-11) and the inequalities (1-1) to estimate (2-4), we obtain (2-6). This completes the proof of theorem 1-1.

To prove theorem 1-2, we shall use a modification of the methods in Hörmander [6] and Ching [4]. At first we consider pseudo differential operators of the form

$$\text{(2-12)} \qquad Au(x) = (2\pi)^{-1}\int e^{ix\xi}a(x, \xi)\hat{u}(\xi)\, d\xi$$

where $\hat{u}$ is the Fourier transform of u and

$$\text{(2-13)} \qquad |D_x^\alpha D_\xi^\beta a(x, \xi)| \leqq c_{\alpha,\beta}\chi(|\xi|)\varphi^{-\alpha}(|\xi|)\Phi^{-\beta}(|\xi|).$$

We assume that χ, φ, Φ satisfy (i), (ii), (iii). Let us introduce

$$\text{(2-14)} \qquad E_\infty(\eta) = \int_{V'_\eta} \chi^2(\xi)\Phi^{-1}(\xi)\, d\xi$$

where

$$\text{(2-15)} \qquad V'_\eta = \{\xi \geqq 0, |\xi-\eta| \leqq \varphi^{-1}(\xi)\}, \eta \in \overline{\mathbf{R}}_+.$$

If the first condition in (iii) is satisfied, for η sufficiently large V'_η is a closed finite interval. If the second condition in (iii) is satisfied, $V'_\eta=\{\xi, \xi\geqq\omega\}$ for a positive ω depending on η. The proof of theorem 1-2 will be obtained as a consequence of the following theorem.

Theorem 2-2. *Let* χ, φ, Φ *satisfy* (i), (ii), (iii) *and suppose that* $E_\infty(\eta)$ *is not bounded. Then there exists* $a(x, \xi)$ *which satisfies the inequalities* (2-13) *for all* α, β *and such that the operator* A *in* (2-12) *is not continuous from* $L^2(\mathbf{R})$ *to* $L^2(\mathbf{R})$.

Proof of theorem 2-2. First take a multiple of Φ as a new weight function, so that we can suppose in (ii): $c_6^{(1)}=1/4$, $c_5^{(1)}=c_7^{(1)}=1/8$. We begin by introducing some notations. Set:

$$\theta(\xi) = \int_0^\xi \Phi^{-1}(\tau)\,d\tau, \quad \xi\geqq 0$$

and let $\psi(\theta)$ denote the inverse function of $\theta(\xi)$: $\psi(\theta(\xi))=\xi$ for $\xi\geqq 0$. We define for $m=0, 1, \ldots$:

$$I_m = \{\xi\geqq 0, \psi(2m)\leqq\xi\leqq\psi(2m+2)\}$$

$$\Delta_m = \psi(2m+2)-\psi(2m)$$

$$\xi_m = \psi(2m+1)$$

and

$$\chi_m = \chi(\xi_m), \quad \varphi_m = \varphi(\xi_m), \quad \Phi_m = \Phi(\xi_m).$$

Observe that

$$\Delta_m \leqq \max_{I_m}\Phi(\xi) \tag{2-16}$$

and, since $c_6^{(1)}=1/4$:

$$\max_{I_m}\Phi(\xi)\leqq 2\min_{I_m}\Phi(\xi). \tag{2-17}$$

Since $c_5^{(1)}=c_7^{(1)}=1/8$, it follows that

$$\max_{I_m}\chi(\xi)\leqq 2\min_{I_m}\chi(\xi), \quad \max_{I_m}\varphi(\xi)\leqq 2\min_{I_m}\varphi(\xi). \tag{2-18}$$

From (2-16), (2-17), (2-18) we can deduce that

$$S_m = \Delta_m\max_{I_m}[\chi^2(\xi)\Phi^{-1}(\xi)]\leqq 16\chi_m^2\leqq 16c_1^2 \tag{2-19}$$

where c_1 is the constant in the property (i). These inequalities will be used later.

Now take $p\in C^\infty(\mathbf{R})$, $p(\theta)=0$ when $|\theta|\geqq 1$ and $p(\theta)=1$ when $|\theta|\leqq 1/2$. We define, for $m=0, 1, \ldots$:

$$q_m(\xi) = \begin{cases} p[\theta(\xi)-2m-1] & \text{if } \xi\geqq 0 \\ 0 & \text{if } \xi\leqq 0. \end{cases}$$

Observe that supp $q_m \subset I_m$ and

$$|D_\xi^\beta q_m(\xi)| \leqq c_\beta \Phi^{-\beta}(\xi) \tag{2-20}$$

with constants c_β which do not depend on m. (2-20) is easily verified by induction on β, by using the inequalities in the property (ii). Initially we suppose that the first condition in (iii) is satisfied. Then there exists a sequence η_j, $j=1, 2, \dots$, in $\overline{\mathbf{R}}_+$ such that

$$\lim_{j\to\infty} E_\infty(\eta_j) = \infty. \tag{2-21}$$

We can assume that V'_{η_j}, $j=1, 2, \dots$, defined as in (2-15), are finite closed disjoint intervals.

Observe that if V'_{η_j} does not include at least one of the intervals I_m it can be covered by two of these intervals: $V'_{\eta_j} \subset I_m \cup I_{m+1}$, for a convenient m depending on j. From (2-14) and (2-19) it follows that

$$E_\infty(\eta_j) \leqq S_m + S_{m+1} \leqq 32c_1^2.$$

In view of (2-21), this inequality can be satisfied only for a finite set of indices j. Therefore, by restricting attention to sufficiently large j, we can suppose that each V'_{η_j} includes at least one of the intervals I_m.

Let m_j be the least integer such that $I_{m_j} \subset V'_{\eta_j}$ and denote by h_j the greatest integer such that $I_{m_j+h_j} \subset V'_{\eta_j}$. Let us define:

$$a(x, \xi) = \sum_j \sum_{i=0}^{h_j} \chi_{m_j+i} e^{ix(\eta_j - \xi_{m_j+i})} q_{m_j+i}(\xi). \tag{2-22}$$

This function satisfies the inequalities in (2-13). In fact, using the definition in (2-15) and the second inequality in (2-18), we have

$$|\eta_j - \xi_{m_j+i}| \leqq 2 \min_{I_{m_j+i}} \varphi^{-1}(\xi), \quad 0 \leqq i \leqq h_j.$$

Hence, by using (2-20):

$$|D_x^\alpha D_\xi^\beta \{e^{ix(\eta_j - \xi_{m_j+i})} q_{m_j+i}(\xi)\}| \leqq 2^\alpha c_\beta \varphi^{-\alpha}(\xi) \Phi^{-\beta}(\xi), \quad 0 \leqq i \leqq h_j.$$

Since all terms in the double sum (2-22) have disjoint supports, in view of the first inequality in (2-18) we can conclude that $a(x, \xi)$ satisfies the (2-13) for all α, β.

We shall prove that the corresponding operator A, defined by (2-12), is not continuous from $L^2(\mathbf{R})$ to $L^2(\mathbf{R})$. Assume the contrary that for some constant c

$$\|Au\|^2 \leqq c\|u\|^2 \tag{2-23}$$

for all $u \in \mathscr{S}(\mathbf{R})$ and test the continuity of A in the following way

Choose $0 \neq f \in \mathscr{S}(\mathbf{R})$ with $\hat{f}(\xi)=0$ when $|\xi| \geqq c_2/4$, where c_2 is the constant in the property (i), and set

$$\hat{u}_j(\xi) = \sum_{i=0}^{h_j} b_i \hat{f}(\xi - \xi_{m_j+i}), \tag{2-24}$$

where the b_i are complex numbers. Note that we have the inclusions

$$\operatorname{supp} \hat{f}(\xi - \xi_m) \subset \{\xi, \psi(2m+1/2) \leqq \xi \leqq \psi(2m+3/2)\} \subset I_m \tag{2-25}$$

and hence the terms in the sum (2-24) have disjoint support. Therefore

$$\|u_j\|^2 = \left(\sum_{i=0}^{h_j} |b_i|^2\right) \|f\|^2. \tag{2-26}$$

On the other hand, using (2-25) we have

$$a(x, \xi)\hat{u}_j(\xi) = \sum_{i=0}^{h_j} b_i \chi_{m_j+i} e^{ix(\eta_j - \xi_{m_j+i})} \hat{f}(\xi - \xi_{m_j+i}).$$

Hence we see that:

$$Au_j(x) = \left(\sum_{i=0}^{h_j} b_i \chi_{m_j+i}\right) e^{ix\eta_j} f(x).$$

Now the (2-23) and the (2-26) give:

$$\left(\sum_{i=0}^{h_j} b_i \chi_{m_j+i}\right)^2 \leqq c \sum_{i=0}^{h_j} |b_i|^2,$$

which implies that

$$\sum_{i=0}^{h_j} \chi_{m_j+i}^2 \leqq c. \tag{2-27}$$

Since

$$E_\infty(\eta_j) \leqq S_{m_j-1} + S_{m_j+h_j+1} + \sum_{i=0}^{h_j} S_{m_j+i},$$

in view of (2-19) we have:

$$16 \sum_{i=0}^{h_j} \chi_{m_j+i}^2 \geqq E_\infty(\eta_j) - 32c_1^2. \tag{2-28}$$

The inequalities (2-27) and (2-28) contradict our hypothesis in (2-21). Therefore the operator A is not continuous on $L^2(\mathbf{R})$.

Now we assume that in (iii) the second condition is satisfied. Since $\int_0^\infty \chi^2(\xi)\Phi^{-1}(\xi)d\xi$ is not convergent, we can construct a sequence η_j, $j=1, 2, \dots$, with $\eta_{j+1} \geqq \eta_j$ in $\overline{\mathbf{R}}_+$ such that

$$\lim_{j\to\infty} \int_{U_j} \chi^2(\xi)\Phi^{-1}(\xi)\, d\xi = \infty$$

where

$$U_j = \{\xi \geqq 0, \eta_j \leqq \xi \leqq \eta_{j+1}\}.$$

Let m_j be the least integer such that $I_{m_j} \subset U_j$ and denote by h_j the greatest integer such that $I_{m_j+h_j} \subset U_j$. We define $a(x, \xi)$ as in (2-22). Then, by observing that $U_j \subset V'_{\eta_j}$ we can repeat all the preceding arguments and obtain the same conclusions. The proof of theorem 2-2 is complete.

In the proofs of theorem 1-2 and corollary 1-3 we shall use the following lemma.

Lemma 2-3. *Let* χ, φ, Φ *satisfy* (i), (ii) *and suppose* $\lim_{\xi\to\infty} d(\varphi^{-1})/d\xi = 0$. *Then the boundedness of* $F_\infty(\eta)$ *is equivalent to the boundedness of each of the functions of* η:

$$(2\text{-}29)\qquad F_\infty(\delta, \eta) = \int_{V_{\delta,\eta}} \chi(\xi)\Phi^{-1}(\xi)\, d\xi, \quad F'_\infty(\delta, \eta) = \int_{V'_{\delta,\eta}} \chi(\xi)\Phi^{-1}(\xi)\, d\xi$$

$$F''_\infty(\delta, \eta) = \int_{V''_{\delta,\eta}} \chi(\xi)\Phi^{-1}(\xi)\, d\xi$$

where δ *is a fixed constant and*

$$(2\text{-}30)\qquad V_{\delta,\eta} = \{\xi \geqq 0,\ |\xi - \eta| \leqq \delta[\varphi^{-1}(\xi) + \varphi^{-1}(\eta)]\}$$

$$V'_{\delta,\eta} = \{\xi \geqq 0,\ |\xi - \eta| \leqq \delta\varphi^{-1}(\xi)\}, \quad V''_{\delta,\eta} = \{\xi \geqq 0,\ |\xi - \eta| \leqq \delta\varphi^{-1}(\eta)\}.$$

Observe that for η sufficiently large $V_{\delta,\eta}$, $V'_{\delta,\eta}$, $V''_{\delta,\eta}$ are closed finite intervals. With the notations in (1-6), (2-15): $V_{1,\eta} = V_\eta$, $V'_{1,\eta} = V'_\eta$.

Proof of lemma 2-3. At first we note that

$$(2\text{-}31)\qquad F''_\infty(\delta/3, \eta) \leqq F'_\infty(\delta, \eta) \leqq F''_\infty(3\delta, \eta),$$

$$F''_\infty(\delta/3, \eta) \leqq F_\infty(\delta, \eta) \leqq F''_\infty(3\delta, \eta).$$

In fact for η large we have the inclusions

$$(2\text{-}32)\qquad V''_{\delta/3,\eta} \subset V'_{\delta,\eta} \subset V''_{3\delta,\eta}, \quad V''_{\delta/3,\eta} \subset V_{\delta,\eta} \subset V''_{3\delta,\eta}.$$

To check that $V''_{\delta/3,\eta} \subset V'_{\delta,\eta}$, observe that for η sufficiently large

$$\left|\max_{V''_{\delta/3,\eta}} \frac{d}{d\xi}(\varphi^{-1})\right| \leqq \frac{1}{\delta}.$$

Hence $\varphi^{-1}[\eta \pm \delta\varphi^{-1}(\eta)/3] \geqq 2\varphi^{-1}(\eta)/3$, and the points $\eta \pm \delta\varphi^{-1}(\eta)/3$ are in $V'_{\delta,\eta}$. Thus $V''_{\delta/3,\eta} \subset V'_{\delta,\eta}$. Similarly we can obtain the other inclusions in (2-32).

Secondly we prove that the boundedness of $F''_\infty(\gamma, \eta)$ implies the boundedness of $F''_\infty(\delta, \eta)$, for every $\delta > \gamma$. Observe that for η_0 sufficiently large

$$\left|\max_{\eta \geqq \eta_0} \frac{d}{d\xi}(\varphi^{-1})\right| \leqq \frac{1}{4\delta}.$$

If we restrict attention to the η such that $\eta-\delta\varphi^{-1}(\eta)\geqq\eta_0$, it follows that

(2-33) $$\min_{V''_{\delta,\eta}}\varphi^{-1}(\xi)\geqq\frac{1}{2}\varphi^{-1}(\eta).$$

Set now:

$$\eta_h=\eta+(h\gamma-\delta)\varphi^{-1}(\eta)$$

$h=0, 1, \dots, r$, where r is the least integer such that $(r+1)\gamma>2\delta$. In view of (2-33), $\varphi^{-1}(\eta_h)\geqq\varphi^{-1}(\eta)/2$ for all h and we have

$$V''_{\delta,\eta}\subset\bigcup_{h=1}^{r}V''_{\gamma,\eta_h}.$$

It follows that

$$F''_\infty(\delta,\eta)\leqq r\max_{\eta\geqq\eta_0}F''_\infty(\gamma,\eta).$$

Therefore we can deduce the boundedness of $F''_\infty(\delta,\eta)$ from the boundedness of $F''_\infty(\gamma,\eta)$. If we apply this to the inequalities (2-31) we have the proof of lemma 2-3.

Proof of theorem 1-2. At first note that, if $F_\infty(\eta)$ is not bounded, also the function of η:

(2-34) $$\int_{V'_\eta}\chi(\xi)\Phi^{-1}(\xi)\,d\xi$$

with V'_η defined as in (2-15), is not bounded. This is obvious if the second condition in (iii) is satisfied and it follows from lemma 2-3 if $\lim_{\xi\to\infty}d(\varphi^{-1})/d\xi=0$.

Since $\chi^{1/2}$ is still a weight function, by using theorem 2-2 we can find $a(x,\xi)$ which satisfies the inequalities

$$|D_x^\alpha D_\xi^\beta a(x,\xi)|\leqq c_{\alpha,\beta}\chi^{1/2}(|\xi|)\varphi^{-\alpha}(|\xi|)\Phi^{-\beta}(|\xi|)$$

for all α, β and such that the operator A in (2-12) is not continuous on $L^2(\mathbf{R})$. Consider now the operator AA^*: it is not continuous on $L^2(\mathbf{R})$ and, if we write it in the form (0-1), its symbol

$$(2\pi)^{-1}a(x,\xi)\bar{a}(y,\xi)$$

satisfies on $\mathbf{R}\times\mathbf{R}\times\mathbf{R}$ the inequalities (1-1) for all α, β, γ. Theorem 1-2 is proved.

Proof of corollary 1-3. In this proof we shall use the letter c to denote constants depending on χ, φ, Φ. Initially observe that φ satisfies the first condition in (iii) since, in view of (i)*, (ii)*, we have:

$$\left|\frac{d}{d\xi}(\varphi^{-1})\right|\leqq c(1+\xi)^{\varepsilon}.$$

Now, if we choose δ sufficiently small, in view of (ii)* we can assume

$$\text{(2-35)}\qquad \frac{1}{2}\max_{V''_{\delta,\eta}}[\chi(\xi)\Phi^{-1}(\xi)] \leqq \chi(\eta)\Phi^{-1}(\eta) \leqq 2\min_{V''_{\delta,\eta}}[\chi(\xi)\Phi^{-1}(\xi)]$$

where $V''_{\delta,\eta}$ is defined in (2-30). Since

$$2\delta\varphi^{-1}(\eta)\min_{V''_{\delta,\eta}}[\chi(\xi)\Phi^{-1}(\xi)] \leqq \int_{V''_{\delta,\eta}}\chi(\xi)\Phi^{-1}(\xi)\,d\xi \leqq 2\delta\varphi^{-1}(\eta)\max_{V''_{\delta,\eta}}[\chi(\xi)\Phi^{-1}(\xi)]$$

it follows from (2-35) that the boundedness of $G(\eta)$ is equivalent to the boundedness of $F''_\infty(\delta,\eta)$, defined as in (2-29), and hence, in view of lemma 2-3, to the boundedness of $F_\infty(\eta)$. Then theorem 1-2 gives immediately the proof of the second part of corollary 1-3. To prove the first part, we shall check that also the difference $F_M(\eta)-F_\infty(\eta)$ is bounded, if we assume $G(\eta)$ bounded. In fact, in view of the inclusions (2-32), we have for η large:

$$F_M(\eta)-F_\infty(\eta) \leqq \int_{\bar{\mathbf{R}}\setminus V''_{1/2,\eta}} g_M(\xi,\eta)\,d\xi$$

where

$$g_M(\xi,\eta) = c\varphi(\xi)\left[\frac{\varphi^{-1}(\xi)+\varphi^{-1}(\eta)}{|\xi-\eta|}\right]^M.$$

Now we introduce the three sets: $W^1_\eta=\{\xi,\ 0\leqq\xi\leqq(1-\gamma)\eta\}$, $W^2_\eta=\{\xi,\ \xi\geqq(1+\gamma)\eta\}$, $W^3_\eta=\bar{\mathbf{R}}_+\setminus(W^1_\eta\cup V''_{1/2,\eta}\cup W^2_\eta)$, where in view of (ii)* we can choose the positive constant γ so small that

$$\frac{1}{2}\max_{W^3_\eta}\varphi(\xi) \leqq \varphi(\eta) \leqq 2\min_{W^3_\eta}\varphi(\xi).$$

By observing that in W^3_η we have

$$g_M(\xi,\eta) \leqq c[|\xi-\eta|\varphi(\eta)]^{-M}\varphi(\eta)$$

and that in view of (i)* $g_M(\xi,\eta)$ is majorized by a multiple of $\eta^{-\varepsilon M+1}$ in W^1_η and by a multiple of $\xi^{-\varepsilon M}$ in W^2_η, a direct computation shows that:

$$F_M(\eta)-F_\infty(\eta) \leqq \sum_{h=1}^{3}\int_{W^h_\eta} g_M(\xi,\eta)\,d\xi \leqq c$$

if $M\varepsilon>1$. We can conclude that $F_M(\eta)$ is bounded and the first part of corollary 1-3 follows from theorem 1-1. The proof is now complete.

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