A martingale that occurs in harmonic analysis

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Introduction

It has been recognised for a long time that the sequence $\{\exp(ir^k\theta);\ k=0, 1, 2, \ldots\}$ with r an integer greater than one and $0 \leqq \theta \leqq 2\pi$, is quite similar to a sequence of independent random variables. That is, many statements that are valid for sums of independent variables are also true for sums of exponentials of the above type. This coincidence may be explained by the observation that the sequence, while not independent, is a martingale difference sequence.

Our purpose in this note is to discuss this type of martingale in the context of the theory of H^p-spaces. In fact, we show that for any positive integer $r>1$ one can find a sequence of σ-fields with respect to which the above lacunary exponentials become martingale differences. Using this, we define H^p-spaces in a manner analogous to what has been done in the classical case (cf. [2]). These H^p-spaces are translation invariant subspaces of $L^1(T)$ that coincide with $L^p(T)$ for $p>1$.

The most interesting case is when $p=1$; here the spaces which we denote by H_r^1 are translation invariant subspaces of $L^1(\mathbf{T})$, distinct from the classical Hardy space H^1. The space H_r^1 may be characterised as follows: $f \in H_r^1$ if and only if f and its "conjugate" $\tilde{f}_r$ belong to $L^1(\mathbf{T})$. Here $\tilde{f}_r$ is, of course, not the harmonic conjugate function; nevertheless, it is obtained from f by a Fourier multiplier taking the values ± 1.

The spaces H_r^1 and their associated conjugate functions are closely related to some results of Taibleson and Chao [3]; we indicate this in some detail in § 3.

We also use these ideas to obtain some recent results on lacunary series. We discuss these applications in § 2.

For background on martingale theory and H^p-spaces, we cite [1] and the excellent exposition by Garsia [4].

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1. Statement of results

Let $f(\theta)=\sum \hat{f}(n)\exp(in\theta)$ be a trigonometric polynomial. For any positive integer $r>1$, we define two auxiliary functions

$$f_r^*(\theta) = \sup_{k\geqq 0}\left|\sum_{j=0}^{r^k-1} f\left(\theta+\frac{j}{r^k}\right) r^{-k}\right|$$

$$S_r(f)(\theta) = \{|\hat{f}(0)|^2+\sum_{k=0}^{\infty}|\sum_{r\nmid j}\hat{f}(jr^k)\exp(ijr^k\theta)|^2\}^{1/2}.$$

Theorem 1. *Let f be a trigonometric polynomial. If $0<p<\infty$, we have*

$$c_p\|S_r(f)\|_p \leqq \|f_r^*\| \leqq C_p\|S_r(f)\|_p.$$

If $1<p<+\infty$, then we may replace f_r^* by f in the above inequalities.

Let $r>0$ be an odd integer with r we can associate the following partition of the integers $\mathbf{Z}$

$$A = A_r = \left\{r^{n+1}s+r^n q;\ s\in\mathbf{Z},\ q=1,2,\ldots\frac{r-1}{2},\ n\geqq 0\right\}$$

$$B = B_r = \left\{r^{n+1}s+r^n q;\ s\in\mathbf{Z},\ q=\frac{r+1}{2},\ldots,r-1,\ n\geqq 0\right\}.$$

Theorem 2. *Let f be a trigonometric polynomial then*

$$c(\|\sum_{\nu\in A}\hat{f}(\nu)e^{i\nu\theta}\|_1+\|\sum_{\nu\in B}\hat{f}(\nu)e^{i\nu\theta}\|_1) \leqq \|f_r^*\|_1 \leqq$$
$$\leqq C(\|\sum_{\nu\in A}\hat{f}(\nu)e^{i\nu\theta}\|_1+\|\sum_{\nu\in B}\hat{f}(\nu)e^{i\nu\theta}\|_1)$$

where c, $C>0$ are two constants that depend only on r.

For r even, an analogue of Theorem 2 holds, but the integers have to be partitioned into more sets. For details cf. § 3

Theorem 3. *Let f be an L^1-function; then*

$$(\sum_{k=0}^{\infty}|\hat{f}(r^k)|^2)^{1/2} \leqq C_r\|f_r^*\|_1.$$

An elaboration of the method of proof of Theorem 1 leads to the following corollary.

Corollary 1. *Let f be an L^p-function on the circle and $a_1, a_2, \ldots, a_m$ be integers with $a_i>1$, $i=1,2,\ldots,m$. Then*

$$(\sum_{n_1,\ldots,n_m}|\hat{f}(a_1^{n_1}a_2^{n_2}\ldots a_m^{n_m})|^2)^{1/2} \leqq C_p\|f\|_p$$

for $1<p<+\infty$.

Theorem 1 is a consequence of general martingale inequalities proved in [1]. This is made explicit in § 2. Using these inequalities we can obtain a class of L^p-Fourier multipliers due originally to Peyrière and Spector [5].

Theorem 3 is a well known result of R.E.A.C. Paley when $p>1$ (cf. [7]). Corollary 1 is an extension of Paley's inequality and provides an answer to a problem proposed by Neuwirth, (Neuwirth had already proved the special case $m=2$). This inequality was first proved by Bonami and Peyrière using the results of [5].

2. A backwards martingale

For $r \geqq 1$ a positive integer, let $\mathscr{F}_r$ denote the σ-field of all $2\pi r^{-1}$-periodic Borel sets of the circle $\mathbf{T}$. Then the conditional expectation of a function $f \in L^1(\mathbf{T})$ with respect to $\mathscr{F}_r$ is given by

$$E(f \| \mathscr{F}_r)(\theta) = \sum_{j=0}^{r-1} f\left(\theta + \frac{2\pi j}{r}\right) r^{-1}.$$

If we expand f in a Fourier series

$$f(\theta) \cong \sum_{n=-\infty}^{\infty} \hat{f}(n) \exp(in\theta)$$

then

$$E(f \| \mathscr{F}_r)(\theta) \cong \sum_{n=-\infty}^{\infty} \hat{f}(nr) \exp(in \cdot r\theta).$$

Fix $r>1$ and consider the decreasing sequence of σ-fields $\mathscr{F}_{r^n}$, $n=0, 1, 2, \ldots$; the sequence

$$f_n = E(f \| \mathscr{F}_{r^n}) = \sum_{j=0}^{r^n-1} f\left(\theta + \frac{2\pi j}{r^n}\right) \cdot r^{-n}$$

is a backwards martingale in the sense that for each $N \geqq 1$,

$$f_N, f_{N-1}, \ldots, f_1, f_0 = f$$

is a martingale. If we set $f_\infty = \lim f_n = \hat{f}(0)$ we may verify quite easily that the functions $S_r(f)$ and f_r^* are the martingale square function and maximal function, respectively, corresponding to the above sequence of σ-fields. Furthermore, this martingale has the following regularity property: If $d_n = f_n - f_{n+1}$, then

$$d_n = \sum_{q=1}^{r-1} v_n^{(q)} \varrho_n^{(q)} \tag{1}$$

where $v_n^{(q)}$, $q=1, \ldots, r-1$ are measurable with respect to the σ-field $\mathscr{F}_{r^{n+1}}$ and where

$$E(\varrho_n^{(q)} \| \mathscr{F}_{r^{n+1}}) = 0;$$

$$|\varrho_n^{(q)}| = 1;$$

$$E(\varrho_n^{(k)} \bar{g}_n^{(j)} \| \mathscr{F}_{r^{n+1}}) = \delta_{jk}$$

for $j, k = 1, 2, \ldots, r-1$.

In fact, if we expand d_n in its Fourier series, we see that, as a $2\pi r^{-n}$-periodic function with $E(d_n \| \mathscr{F}_{r^{n+1}})=0$, it can be written

$$d_n(\theta) = \sum_{q=1}^{r-1} g_q(r^{n+1}\theta) \exp(iqr^n\theta)$$

where g_q are uniquely determined. The functions $g_q(r^{n+1}\theta)=v_n^{(q)}(\theta)$ and $\exp(iqr^n\theta)= =\varrho_n^{(q)}(\theta)$ fulfil our requirements. This regularity property allows us to apply the techniques of [1]. (In [1], the regularity condition states that $d_n=v_n\varrho_n$ with somewhat more general conditions on ϱ_n. The extension of the results of [1] to our case presents no problems.)

Thus, Theorem 1 is simply a special case of Theorem 5.1 of [1].

To prove Theorem 3, we have to introduce the space BMO relative to our martingales. We know that the space BMO is the dual to H_1 with respect to our martingale, by Fefferman's theorem. Here, of course,

$$H_1 = \{f \in L^1(\mathbf{T}) : f_r^* \in L^1(T)\}.$$

A function $g \in L^1(\mathbf{T})$ belongs to BMO if

$$E(|g-g_n|^2 \| \mathscr{F}_{r^n}) \leqq C$$

for some constant and all $n \geqq 0$. (This condition coincides with the standard one in [4] because of our regularity condition).

In terms of the Fourier series, the function $g \in \mathrm{BMO}$ provided

$$\left|\sum_{r^n \nmid \nu,\mu;\ r^n | \nu-\mu} \hat{g}(\nu)\cdot\overline{\hat{g}(\mu)}\right| \leqq C \quad \text{for all} \quad n \geqq 1.$$

From this, we see that if

(2) $$g(\theta) \cong \sum_{n=0}^{\infty} \hat{g}(n) \exp(ir^n\theta)$$

then

(3) $$\|g\|_{\mathrm{BMO}} \leqq C\|g\|_2 = C\left(\sum |\hat{g}(n)|^2\right)^{1/2}.$$

Theorem 3 is proved using a duality argument as follows: if f is a trigonometric polynomial, then

$$\left(\sum_{n=0}^{\infty} |f(r^n)|^2\right)^{1/2} = \sup_{\varphi} \int f \cdot \varphi \, dt$$

where the sup is taken over all φ of the form (2) with $\|\varphi\|_2=1$. On the other hand, the duality of H_1 and BMO imply that

$$\int f\varphi \, dt \leqq C\|f_r^*\|_1 \|\varphi\|_{\mathrm{BMO}}.$$

Finally, the inequality (3) shows that $\|\varphi\|_{\mathrm{BMO}} \leqq C$, so that we obtain Theorem 2. We shall finally outline the proof of Corollary. For this, we shall need some facts concerning martingales with a several-dimensional time parameter. For simplicity we limit ourselves to the case where the time parameter runs over the lattice (n, m), $n>0$, $m>0$. Let $\mathcal{F}_n$ and $\mathcal{G}_m$ be two monotone sequences of σ-fields (for simplicity assume they are increasing sequences) and let us suppose that the conditional expectations w.r.t. $\mathcal{F}_n$ and $\mathcal{G}_m$ commute i.e.

$$f_{nm} = E\big(E(f\|\mathcal{F}_n)\|\mathcal{G}_m\big) = E\big(E(f\|\mathcal{G}_m)\|\mathcal{F}_n\big), \quad n, m > 0$$

is said to be a martingale with time parameter (n, m). The S-function associated with the martingale is given by

$$S^2(f) = \sum\nolimits_{n,m} d_{n,m}^2$$

where

$$d_{n,m} = f_{n,m} - f_{n,m-1} - f_{m-1,n} + f_{n-1,m-1}.$$

Lemma 1. *For $1<p<\infty$, the following inequality holds:*

$$\|S(f)\|_p \leqq \tilde{C}_p \|f\|_p.$$

Proof. Define the l^2-valued function

$$F = (d_1, d_2, \ldots)$$

with $d_k = E(f\|\mathcal{F}_k) - E(f\|\mathcal{F}_{k-1})$. Then the L^p-norm of F satisfy

$$\|F\|_p \leqq C_p \|f\|_p$$

by Burkholder's inequalities. See 2.5. on page 256 of [1]. Observe that

$$d_{n,m} = E(d_n\|\mathcal{G}_m) - E(d_n|\mathcal{G}_{m-1})$$

and that

$$E(F\|\mathcal{G}_m) = \big(E(d_1\|\mathcal{G}_m), E(d_2\|\mathcal{G}_m), \ldots\big)$$

is a l^2-valued martingale. Therefore we may apply the Burkholder inequalities again to obtain the desired result.

As a special case, let $\mathcal{F}_n$ be the collection of all p^n-periodic Borel sets of $\mathbf{T}$, and $\mathcal{G}_m$ the collection of all q^m-periodic sets, where p and q are primes. By a small calculation one can verify that the "double" S-function, corresponding to these sequences, is given by

$$S^2(f) = \sum\nolimits_{m,n} \Big|\sum\nolimits_{p \nmid \lambda,\, q \nmid \lambda} a_{\lambda p^m \cdot q^n} e^{i\lambda p^m q^n \theta}\Big|^2$$

so that Corollary 1 follows from the lemma and a standard duality argument in the special case of two primes. However, the process can be iterated finitely many times, so the above argument leads to a proof of Corollary 1. If we replace the primes p and q by by arbitrary integers a and b, then the expression for $S(f)$ becomes more complicated. However, we can simply decompose the numbers $a_i = 1, 2, \ldots, M$ into their prime factors and use the inequality obtained for primes to conclude the proof.

Remarks. As we noted above, Theorem 3 is a well-known inequality due to Paley when $p>1$. However, there are some differences when $p=1$.

Paley's inequality differs from ours in two ways: (a) instead of the sequence r^n, $n=0, 1, \ldots$, his result holds for any lacunary sequence of integers λ_n, such that $\inf_n (\lambda_{n+1}/\lambda_n)>1$; (b) for the case $p=1$, the correct result is obtained by using the classical H^1-norm of the function rather than its L^1-norm. On the other hand, our results are restricted to geometric sequences in an essential way. In fact, it is quite easy to see that one can find functions $f\in L^1$ such that $\|f_r^*\|_1<\infty$ for some $r>1$, but $\|f\|_{H^1}=\infty$ and vice versa.

The L_p-norm in Corollary 1 cannot be replaced by the H_1-norm. In fact, it is known [6] that if $\varepsilon_k=0, 1, k=0, 1, 2, \ldots$ is a sequence of Fourier multipliers for H_1 to L_2 if and only if $\varepsilon_k=1$ for a finite number of indices k, $2^n\leqq k\leqq 2^{n+1}$, independent of n; this condition is not satisfied for example, by the multipliers corresponding to a "double" lacunary sequence $p^n q^m$, $n, m=0, 1, 2, \ldots$.

Paley's theorem and Corollary 1 suggest the following question: given two lacunary sequences m_k and n_j, is it true that

$$\left(\sum\nolimits_{k,j} |\hat{f}(m_k n_j)|^2\right)^{1/2} \leqq C_p \|f\|_p$$

for all $1<p<\infty$? The answer is negative, however. In fact let $m_k=2^k$, $k=0, 1, \ldots, N$. For n_j, choose $n_j=2^N(M-j)/2^j$, $j=0, 1, \ldots, N$. If we choose the coefficients $\hat{f}(m_k\cdot n_j)=1$ or 0 provided $k=j$ or not, we obtain, essentially, the M^{th} partial sum of the Fourier series of a unit mass at $\theta=0$.

3. The "conjugate" functions

We now prove the inequalities given in Theorem 2. The following variant of $S_r(f)$ is useful here:

$$s(f) = s_r(f) = [|\hat{f}(0)|^2+\sum\nolimits_{n=0}^{\infty} E(|d_n|^2 \| \mathscr{F}_{r^{n+1}})]^{1/2} =$$
$$= [|f(0)|^2+\sum\nolimits_{n=0}^{\infty} \sum\nolimits_{q=1}^{r-1} |v_n^{(q)}|^2]^{1/2}$$

where the functions $v_n^{(q)}$ are those defined in (1).

Lemma 2. *For any $r>1$ and $0<p<\infty$, we have*

$$c_p \|s_r(f)\|_p \leqq \|S_r(f)\|_p \leqq C_p \|s_r(f)\|_p.$$

This lemma is a variant of a result from [1] (see Theorems 5.3).

To prove the left-hand inequality in Theorem 2, it is sufficient to observe that if

$$f_A(\theta) = \sum\nolimits_{N\in A} \hat{f}(n) \exp(in\theta)$$

then

$$s(f_A)(\theta) \leqq s(f)(\theta),$$

so that for $0<p<\infty$,

$$\begin{aligned}\|f_A\|_p &\leqq \|f_A^*\|_p \\ &\leqq c_p\|s(f_A)\|_p \\ &\leqq C_p\|s(f)\|_p \\ &\leqq C_p\|f^*\|_p.\end{aligned}$$

Here we have used Theorem 1, Lemma 2, and the pointwise inequality just mentioned. The same argument may be applied to the function f_B. Therefore, we have, in fact, shown that the left-hand inequality of Theorem 2 holds for all $0<p<\infty$.

The right-hand side inequality of Theorem 2 depends on a theorem due to Taibleson and Chao ([3], Theorem 2).

Lemma 3. *Let f be a trigonometric polynomial such that $f=f_A$ (that is, the spectrum of f is zero outside A). There exists an $\alpha_0=\alpha_0(r)<1$ such that for all $\alpha>\alpha_0$, the sequence $|f|^\alpha$, $|f_1|^\alpha$, ... obtained from the (backwards) martingale $f, f_1, f_2, \ldots$ is a submartingale. That is,*

$$E(|f_n|^\alpha \| \mathscr{F}_{r^{n+1}}) \geqq |f_{n+1}|^\alpha$$

for all $n=0, 1, \ldots$.

Before giving the proof of Lemma 3, let us indicate the proof of the right-hand inequality of Theorem 2. In fact, Lemma 3 implies that

$$\|f^*\|_1 \leqq C_\alpha \sup_n \|f_n\|_1;$$

since $|f_n|^\alpha$ is a submartingale that is $L^{\alpha^{-1}}$-bounded, with $\alpha^{-1}>1$. (Here we have used the maximal inequalities for submartingales (see Garsia [4]).

Let us examine the proof of Lemma 3. The function f_A, written out in terms of its martinagale differences $d_n(\theta)$, is of the form.

$$\begin{aligned}d_n(\theta) = {} & \textstyle\sum_{q=1}^{(r-1)/2} [u_q^{(n)}(r^{n+1}\theta)\cos(qr^n\theta) - v_q^{(n)}(r^{n+1}\theta)\sin(qr^n\theta)] + \\ & + i\textstyle\sum_{q=1}^{(r-1)/2} [u_q^{(n)}(r^{n+1}\theta)\sin(qr^n\theta) + v_q^{(n)}(r^{n+1}\theta)\cos(qr^n\theta)] = R_n(\theta)+iI_n(\theta),\end{aligned}$$

where the functions $u_q^{(n)}$, $v_q^{(n)}$ are real-valued. Here we have simply used the representation (1) and split the functions there into their real and imaginary parts.

The important point in this decomposition is that

$$\begin{aligned} E(R_nI_n \| \mathscr{F}_{r^{n+1}}) &= 0, \\ E(R_n^2 \| \mathscr{F}_{r^{n+1}}) &= E(I_n^2 \| \mathscr{F}_{r^{n+1}}), \end{aligned} \tag{4}$$

a fact which can be verified rather easily. Furthermore, the conditional expectation $E(\ \| \mathscr{F}_{r^{n+1}})$ acting on an r^n-periodic function, reduces to a simple average of r quantities, as the reader can easily verify. These circumstances allow us to apply Theorem 2 of Taibleson and Chao [3] to the martingale differences R_n, I_n; the conclusion of their theorem, stated in our terms, is precisely Lemma 3.

As we said in the introduction there is a version of Theorem 2 for the even integers. It seems to be more complicated.

Let us define:

$$A_1 = \{m : m = 4^n(4s+1);\ s\in\mathbf{Z}\ \ n \geqq 0\}$$

$$A_2 = \{m : m = 4^n(4s+3);\ s\in\mathbf{Z}\ \ n \geqq 0\}$$

$$A_3 = \{m : m = 4^n(8s+2);\ s\in\mathbf{Z}\ \ n \geqq 0\}$$

$$A_4 = \{m : m = 4^n(8s+6);\ s\in\mathbf{Z}\ \ n \geqq 0\}.$$

It is easy to verify that $A_1, \ldots, A_4$ is a partition of the integers.

Let also $r=2t$ be an even integer and let us define

$$B_1 = \left\{m : m = (rs+q)r^n;\ s\in\mathbf{Z}\ \ n \geqq 0 \quad q = 1, \ldots, \frac{r}{2}-1\right\}$$

$$B_2 = \left\{m : m = (rs+q)r^n;\ s\in\mathbf{Z}\ \ n \geqq 0 \quad q = \frac{r}{2}+1, \ldots, r-1\right\}$$

$$C = \left\{m : m = \left(rs+\frac{r}{2}\right)r^n;\ s\in\mathbf{Z}\ \ n \geqq 0\right\}.$$

B_1, B_2 and C is then a partition of $\mathbf{Z}$ that depends on r; a general element of C can now be written in the form

$$m = \left(r(rl+v)+\frac{r}{2}\right)r^n$$

where $l\geqq 0$, $n\geqq 0$ and $v=1, 2, \ldots r-1$. We can partition C now into four subsets C_1, C_2, C_3, C_4 by demanding that n and v stays in a fixed class mod 2 (i.e. takes only even or odd values).

B_1, B_2, C_1, C_2, C_3, C_4 is then a partition of $\mathbf{Z}$ that depends on r.

Let now $f\in L^1(\mathbf{T})$

$$f \sim \sum \hat{f}(n) \exp(in\theta)$$

and let us denote in general

$$f_A \sim \sum\nolimits_{n\in A} \hat{f}(n) \exp(in\theta)$$

for any $A\subset\mathbf{Z}$ subset of the integers. We have then

Theorem 4. *Let f be an L^1-function and let r be some even integer then*

a) *if $r=2^k$ is a power of 2 then $f\in H^1_r$ if and only if $f_{A_1}, f_{A_2}, f_{A_3}, f_{A_4}\in L^1(\mathbf{T})$.*

b) *In general (when r is not necessarily a power of 2 but even) $f\in H^1_r$ if and only if $f_{B_1}, f_{B_2}, f_{C_1}, f_{C_2}, f_{C_3}, f_{C_4}\in L^1(\mathbf{T})$.*

c) *The spaces $H^1_{2^k}$ $k=1, 2, \ldots$ are all identical.*

Part c) is of course an immediate consequence of a) but we shall need to obtain an independent proof since the proof of a) is based on c). That independent proof follows from the following two pointwise inequalities:

$$S_2(f)(\theta) \leqq C_k s_{2^k}(f)(\theta)$$

and

$$f_{2^k}^*(\theta) \leqq f_2^*(\theta)$$

which are easy to verify, and Lemma 2. (There is nothing special about 2 in (c), in general $H_{r^k}^1$ $k=1, 2, \ldots$ are all identical spaces).

This point being settled we can now prove

(a) by proving that $f \in H_4^1$ if and only if $f_{A_1}, f_{A_2}, f_{A_3}, f_{A_4} \in L^1(\mathbf{T})$. The proof runs on strictly identical lines as the proof of Theorem 3 and will therefore be omitted. The proof of the general case (b) also follows the same lines, and will be omitted. The thing to be observed here is that the two sets B_1, B_2 behave like the two sets A and B of Theorem 3, and that the four sets $C_1, \ldots, C_4$ behave, like the four sets $A_1, \ldots$ $\ldots, A_4$ of part (a) of Theorem 4. The general case combines, in some sense, Theorem 3 and the special case $r=2^k$.

From Theorem 2, it follows that if $\tau(n)=\pm 1$, according to whether $n \in A$ or not, then τ is a Fourier multiplier that characterizes H_r^1 for r odd: the function $f \in H_r^1$ if and only if $\tau(f)\big(=\tau(n)\hat{f}(n)\big)$ and f belongs to $L^1(\mathbf{T})$.

For the case r even, Theorem 4 says that at most five multipliers are needed. We have not been able to decide whether fewer are sufficient, and we leave this as an open problem.

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