Random measures on planar curves

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Two theorems in this paper investigate Fourier—Stieltjes transforms of measures on curves in the plane; the first is more general in respect of the curves investigated and more precise in the inequalities obtained for Fourier transforms, while the second introduces a more powerful and flexible technique. The first is proved by elementary methods, insofar as it avoids the theory of stochastic processes and martingales.

A survey of Gaussian processes in related extremal problems is given by Kahane [3, ch. 13—15], who extends some results to stable processes in [4]. A set of the type S occurring in the first theorem is often called a *Salem set* of dimension β [8], [5, VIII]. Non-linear transformations and Hausdorff dimension are studied in [6].

1. Let Γ be a curve of class C^2 in the plane $\mathbf{R}^2$, of positive curvature. An elementary proof is provided for the following *theorem:*

For each β in $(0, 1)$ there exists a compact set $S \subseteqq \Gamma$, of Hausdorff dimension β, and a positive measure μ on S, such that

$$|\hat{\mu}(u)| = o(1)\,\|u\|^{-\beta/2} \quad \textit{for} \quad u \quad \textit{in} \quad \mathbf{R}^2.$$

For definiteness we suppose that Γ is described by co-ordinates $-2 \leqq x \leqq 2$, and $y=y(x)$, where y is of class C^2 and $y''>0$. Whenever λ is a measure on $[-1, 1]$ we denote by μ its transform by the mapping of $[-2, 2]$ onto Γ. For present purposes, μ is best defined by its Fourier—Stieltjes transform on $\mathbf{R}^2$:

$$\hat{\mu}(u_1, u_2) = \int \exp\left(-iu_1x - iu_2y(x)\right)\lambda(dx), \quad u \in \mathbf{R}^2.$$

Lemma. *Let f be an element of $C^2[-2, 2]$ and $f=0$ outside $(-1, 1)$. For any $\varepsilon>0$ and integer $N>N_\varepsilon$ there is an element $g_1 \geqq 0$ of $C^2[-2, 2]$ so that*

(i) *The closed support of g_1 is covered by $10N^\beta \log^2 N$ intervals of length N^{-1}.*

(ii) *The measures μ and μ_1 corresponding to $f(x)dx$ and $g_1(x)f(x)dx$ fulfil the inequality*

$$|\hat{\mu}(u) - \hat{\mu}_1(u)| \leqq \varepsilon(1 + \|u\|)^{-\beta/2}, \quad u \in \mathbf{R}^2.$$

In the proof we use a special partition $1=\sum h_k$, in which $0\leqq h_k\leqq 1$, $h_k\in C^2(-\infty,\infty)$, $|h_k'|\leqq CN$ and $h_k=0$ outside $((k-1)/2N,(k+1)/2N)$. The existence of a partition $1=\sum H_k$, adapted to $N=1$, is well known, and we have only to choose $h_k(x)\equiv H_k(Nx)$. In the construction of $g_1=\sum \xi_k h_k$, with certain random variables ξ_k, we need bounds for the partial integrals

$$b_k(u)=\int h_k(x)f(x)\exp(-iu_1x-iu_2y(x))\,dx.$$

In fact we have the inequalities

(a) $|b_k(u)|\leqq CN^{-1}$ for all u.

(b) $\sum |b_k(u)|^2\leqq C\|u\|^{-1}$ when $\|u\|\geqq N$.

Here C depends on f and $y(x)$ but not on N; it is easy to see that $b_k(u)=0$ if $|k|>2N$, and (a) is a consequence of the properties of h_k. As for (b), we can easily dispose of the special case $|u_1|\geqq 2A|u_2|$, where $A=\sup\{|y'(x)|\}$. For the analysis leading to van der Corput's inequality [9, p. 116] yields $|b_k(u)|\leqq C\|u\|^{-1}$, and then $\sum|b_k(u)|^2\leqq \leqq C\|u\|^{-2}N\leqq C\|u\|^{-1}$. Henceforth, in the proof of (b) we assume $|u_1|\leqq 2A|u_2|$.

We let m_k be the minimum of $|u_1+u_2y'(x)|$ on the interval $((k-1)/2N,(k+1)/2N)$, and obtain $|b_k(u)|\leqq Cm_k^{-1}$ by the argument mentioned before. We also have the bounds $|b_k(u)|\leqq CN^{-1}$ and $|b_k(u)|\leqq C'|u_2|^{-1/2}\leqq C\|u\|^{-1/2}$ by [9, p. 116]. Suppose now that m_k attains a minimum at $k=p$; then $|m_{k+p}|\geqq C'p|u_2|N^{-1}\geqq Cp\|u\|N^{-1}$ if (say) $|p|\geqq 3$. When $N\leqq\|u\|\leqq N^2$ we use the bound $|b_{k+p}(u)|\leqq CN^{-1}$ for the range $|p|\leqq \leqq 3\|u\|^{-1}N^2$, and the lower bound on m_k for other values of p. When $N^2\geqq\|u\|$ we use the bound $|b_k(u)|\leqq C\|u\|^{-1/2}$ for $|p|\leqq 3N\|u\|^{-1/2}$, and Cm_k^{-1} outside.

To construct g_1 from all these preliminaries we set $p_N=N^{\beta-1}\log^2 N$, and take independent random variables ξ_k with law

$$P(\xi_k=p_N^{-1})=p_N,\quad P(\xi_k=0)=1-p_N.$$

Then $g_1=\sum\xi_k h_k$.

For each fixed u in $\mathbf{R}^2$, $\hat\mu_1(u)-\hat\mu(u)=\sum(\xi_k-1)b_k(u)$, a sum of independent terms of magnitude $CN^{-1}p_N^{-1}$ and total variance $Cp_N^{-1}\sum|b_k^2(u)|$. Classical bounds for expected values of $\exp(t|\mathrm{Re}\,\hat\mu_1-\mathrm{Re}\,\hat\mu|)$ and $\exp(t|\mathrm{Im}\,\hat\mu-\mathrm{Im}\,\hat\mu_1|)$ are valid provided $|t|\leqq C^{-1}Np_N$. Choosing then $t=\eta N^{1-\beta/2}p_N$ for a small constant $\eta>0$ we conclude

$$P(|\hat\mu_1(u)-\hat\mu(u)|\geqq \varepsilon N^{-\beta/2})\leqq 4\exp(-\eta' N^{1-\beta}p_N)\leqq 4\exp(-\eta'\log^2 N).$$

This inequality governs $\hat\mu_1(u)-\hat\mu(u)$ for each u in the ball $\|u\|\leqq N$, but we can obtain a similar, uniform, inequality for the entire ball by checking $\hat\mu_1-\hat\mu$ at N^C points u; C is a constant whose exact value is immaterial in the presence of the strong bound on P. This remark is valid for the remaining estimation of $\hat\mu_1-\hat\mu$.

In the range $N \leqq \|u\| \leqq N^2$ somewhat more care is required in the choice of t. The expected value of $\exp t|\hat{\mu}_1-\hat{\mu}|$ is bounded by $4\exp(Ct^2p_N^{-1}\|u\|^{-1})$ if $0 \leqq t \leqq \eta N p_N$. For these numbers t we have

$$P\big(|\hat{\mu}_1(u)-\hat{\mu}(u)| \geqq \varepsilon\|u\|^{-\beta/2}\big) \leqq 4\exp(Ct^2p_N^{-1}\|u\|^{-1}-\varepsilon t\|u\|^{-\beta/2}).$$

The infimum, for unrestricted values t, obtained at $t_0=\varepsilon(2C)^{-1}p_N\|u\|^{1-\beta/2}$, equals $4\exp(-\delta p_N\|u\|^{1-\beta}) \leqq 4\exp(-\delta\log^2 N)$. This is of course a sufficient bound, and can be used if $t_0 \leqq \eta N p_N$, that is, $\|u\|^{1-\beta/2} \leqq \eta_1 N$. If we assume the opposite inequality, $\|u\|^{1-\beta/2} \geqq \eta_1 N$, and choose $t=\eta_2 N p_N$, then the negative term exceeds twice the positive, and the bound becomes $\exp(-\eta_3 N p_N\|u\|^{-\beta/2}) \leqq \exp(-\eta_3 N p_N N^{-\beta}) = \exp(-\eta_3\log^2 N)$.

In the range $\|u\|>N^2$, we note that $|b_k(u)| \leqq C\|u\|^{-1/2}$ by the arguments used before in estimating variances, so the exponential bounds are valid for $0 \leqq t \leqq$ $\leqq C^{-1}\|u\|^{1/2}p_N$. Now $\|u\|^{1/2}p_N\|u\|^{-\beta/2} \geqq \log^2 N$ for these values u, and the bounds of $\hat{\mu}-\hat{\mu}_1$ can be extended to the ball $\|u\| \leqq N^C$, $C=4(1-\beta)^{-1}$. But $\sum \xi_k < 5N$ with probability near 1, so we have $\sum|\xi_k-1|\cdot|b_k(u)| < C\|u\|^{-1/2}5N$ and this is $O(\|u\|^{-\beta/2})$ when $\|u\|^{1-\beta}>N^3$, for example. Thus we obtain the required inequalities for $\hat{\mu}(u)-\hat{\mu}_1(u)$ in four different regions, and the lemma is proved.

To prove the theorem in its entirety we begin with $f \in C^2[-2, 2]$, $f=0$ outside of $(-1, 1)$, $f>0$ on $(-1, 1)$ and apply the modification of the lemma successively, with numbers $\varepsilon_j=3^{-j}\int f(x)\,dx$ so that the limit measure is positive. Its support S has finite Hausdorff measure for the function $h(t)=t^\beta\log^2 t^{-1}$, hence $\dim S \leqq \beta$. The method can be improved to cover any measure function $h(t)$ such that $t^\beta\log t^{-1}=$ $=o(h)$ as $t \to 0+$. This approaches the theorem of Kahane [3, p. 13—15] in precision. In a certain sense, discussed briefly at the conclusion of [6], the set S is much more massive than some Salem sets of dimension β.

To obtain $o(1)$ in place of $O(1)$, we observe that each measure in our construction belongs to C^2, so its Fourier transform is $O(\|u\|^{-2})$.

2. In the next theorem y is a function of class $C^\infty(-\infty, \infty)$ and $y''>0$ everywhere. Moreover λ is a probability measure on $[0, 1]$ with a Lipschitz condition $\lambda(a, a+h) \leqq Ch^\alpha$ for a certain α in $(0, 1/2)$. The measure μ is now defined to be the image of λ by the mapping $t \to (X(t), y \circ X(t))$ so that μ is carried by the graph of y in the plane. X denotes Brownian motion.

Theorem. *For almost all paths X, $|\hat{\mu}(u)| \leqq C\|u\|^{\delta-\alpha}$ for $u \in \mathbf{R}^2$, and each $\delta>0$.*

For the proof we fix a function Φ of class C^∞ and compact support and investigate, instead of μ, the measure $\Phi(X(t))\lambda(dt)$, and its transform μ_1. Our theorem will be a simple consequence of the next assertion:

(M) For each $p=1, 2, 3, \ldots$ the p-th moment of $\hat{\mu}_1(u)$ admits a bound $\|\hat{\mu}_1(u)\|_p^p \leqq$ $\leqq C\|u\|^{1-\alpha p}$.

We observe that $\Phi(x)=0$ outside a certain interval $|x|\leqq A$, and on this interval we have $|y'|\leqq A_1$. Now the transform $\hat{\mu}_1(u)$, $u=(u_1, u_2)$, involves a mapping $u_1x+ +u_2y(x)$, whose derivative has absolute value $\geqq |u_1|-A_1|u_2|$. As we shall soon observe, the estimation of $\hat{\mu}_1(u)$ becomes much simpler if $|u_1|\geqq 2A_1|u_2|$; a similar easy case occurred in the proof completed above. In the opposite case, $|u_2|>(2A_1)^{-1}|u_1|$, we can write $u_1x+u_2y(x)=\pm\|u\|\,g_u(x)$, where $g_u''\geqq c>0$ and all derivatives of g_u are bounded on $(-A, A)$ by constants independent of u. Until the conclusion of the demonstration of (M), we keep $g=g_u$ and write v in place of $\|u\|$.

First of all we partition the interval $0<t<1$ to isolate the small values of $g'(X(t))$. For this purpose we construct a function L of class C^∞, vanishing outside (1, 4) so that $0\leqq L\leqq 1$ and $\sum_{-\infty}^{\infty} L(2^k x)=1$ if $x>0$. A function L can be obtained from an ordinary partition of unity $1\equiv\sum h(y+k)$ by the substitution $x=2^y$. We shall see that $g'(X)=0$ only on a set of λ-measure 0, so we have $1=\sum L(2^k g'(X))+ +\sum L(-2^k g'(X))$ almost everywhere.

Let $0<\varepsilon<(8p)^{-1}$; we intend to neglect all the terms in the sum in which $4^k\geqq v^{1-\varepsilon}$. The error introduced is no larger than

$$\lambda\{t:|g'X(t)|\leqq Cv^{\varepsilon/2}v^{-1/2}\}\quad\text{or}$$

$$\lambda\{t:|X(t)-x_0|\leqq C'v^{\varepsilon/2}v^{-1/2}\}\quad\text{since}$$

$g''\geqq c>0$. The random variable $h(r)=\sup\lambda\{t\colon |X(t)-x_0|\leqq r\}$ has p-th moment $\leqq Cr^{-1+2\alpha p}$ [7], and therefore these terms can indeed be omitted from further calculations. We observe now that the number of terms remaining, in which $L(2^k g'(X))\Phi(X)$ doesn't vanish identically, is at most $C\log v$. We complete the analysis for the integral containing $L(2^k g')$; the method for $L(-2^k g')$ is the same.

For each index k not already excluded we define $r=r(k, v)=4^k v^{\varepsilon-2}$ and divide the interval [0, 1] into adjacent subintervals of length r, denoting by $I_j=I_j(k, v)$ the corresponding partial integrals. (Precisely, I_j is extended over the range $jr\leqq t\leqq \leqq(j+1)r$). We shall use the theory of martingales to bound the moments of $\sum I_{2j}$; $\sum I_{2j+1}$ is handled in the same way. Now I_{2j} is measurable over the Borel field $F_{2j+1}=F\{X(t);\, t\leqq(2j+1)r\}$. Thus we obtain a series of martingale differences by writing $\sum I_{2j}-E(I_{2j}|F_{2j-1})$, with the convention that $F_{-1}=F_0$ is the trivial field. According to the Markov property $E(I_{2j}|F_{2j-1})=E(I_{2j}|X(2jr-r))$, and we shall give a bound for this, uniform with respect to all values $b=X(2jr-r)$. We use the observation that when $2jr\leqq t\leqq(2j+1)r$, $X(t)$ has conditional distribution $b+\lambda X(1)$, with $r\leqq\lambda^2\leqq 2r$. We are led to integrals of the form

$$\int\Phi(b+\lambda s)L(2^k g'(b+\lambda s))\exp(-ivg(b+\lambda s))e^{-s^2/2}\,ds/\sqrt{2}\pi.$$

The factor $L(\cdot)$ vanishes outside the interval defined by the inequality $1<2^k g'(b+\lambda s)<4$, and on this interval the derivative of $vg(b+\lambda s)$ falls in the interval

$v2^{-k}\lambda < D < 4v2^{-k}\lambda$. The successive derivatives of $\Phi(b+\lambda s)$ remain bounded because $\lambda \leqq 2r^{1/2} = 2^k v^{-1} v^{\varepsilon/2}$, and we excluded all indices k for which $4^k \geqq v^{1-\varepsilon}$. The m-th derivative of $L(2^k g'(b+\lambda s))$ remains bounded for $m=1, 2, 3, \ldots$, for the following reasons. For $m=1$ we have the inequality $2^k\lambda \cdot v2^{-k}\lambda = v\lambda^2 \leqq 2vr \leqq 4^{k+1} v^{\varepsilon-1} < 4$. From $m \geqq 2$ we observe that $2^k \lambda^2 \leqq 2^{k+1} r < 8^{k+1} v^{\varepsilon-2} < 8v^{3/2} v^{\varepsilon-2} < 8$.

We intend to use $z = 2^k \lambda^{-1} g(b+\lambda s)$ as a new variable; throughout the domain of integration $1 < z' < 4$. Moreover $|z''| \leqq C2^k\lambda \leqq C2^{k+1} r^{1/2} \leqq 2C4^k v^{\varepsilon/2} v^{-1} < 2Cv^{-\varepsilon/2}$. Similar estimates apply to the higher derivatives of z; the relations $vg(b+\lambda s) = v\lambda 2^{-k} z$, $v\lambda 2^{-k} > vr^{1/2} 2^{-k} = v \cdot 2^k v^{\varepsilon/2} v^{-1} 2^{-k} = v^{\varepsilon/2}$, allow us to obtain the bound $E(I_{2j}|F_{2j-1}) \leqq C_B \mu(2jr, (2j+1)r) v^{-B}$ for any constant B.

We are now in sight of the moment inequality (M). We have just obtained a uniform bound on $\sum E(I_{2j}|F_{2j-1})$, and it remains to obtain bounds on the sum $\sum I_{2j} - E(I_{2j}|F_{2j-1})$. By an inequality of Burkholder [1, 2], in case $p>1$, it will be sufficient to obtain bounds for $\sum |I_{2j}|^2$ and $\sum |E(I_{2j}|F_{2j-1})|^2$. The second "square-function" is of course covered by the uniform estimates with a large exponent B. For the first sum we use the inequality $(\sum |I|^2) \leqq \max I \cdot \sum |I_j|$. Recalling the dependance on k and v, we have $\max I_j \leqq Cr^\alpha = C4^{\alpha k} v^{(\varepsilon-2)\alpha}$. The sum $\sum I_j$ doesn't exceed $\int L(2^{-k} g' X(t)) \Phi(X(t))\, d\lambda$, and we saw before that this has p-th moment $\leqq C4^{-k\alpha p} 2^k$. We find, then, that $(\sum |I_j|^2)^{1/2}$ has p-th moment $\leqq C2^k v^{(\varepsilon-2)\alpha p} < Cv^{-2\alpha p} v$. Summation with respect to k involves a factor $\log^p v$, but this can be absorbed by Schwarz' inequality. Thus (M) is proved.

Now (M) easily implies that $|\hat{\mu}_1(u)| \leqq C\|u\|^{\delta-\alpha}$ for characters u of the special form $(\pm n_1^{1/2}, \pm n_2^{1/2})$, $(n_1, n_2 = 0, 1, 2, \ldots)$. But μ_1 has compact support, and a device from Fourier analysis [3, p. 165] enables us to extend an inequality of this type to all of $\mathbf{R}^2$. By a suitable choice of a sequence $\Phi_1, \Phi_2, \ldots$, we then obtain the theorem for $\hat{\mu}(u)$.

The inequality of Burkholder concerns martingales Y_n, $1 \leqq n \leqq N$, their L^p-norm $\|Y_N\|_p$, and the L^p-norm of the square-function S, defined by $S^2 = |Y_1|^2 + |Y_2 - Y_1|^2 + \ldots + |Y_N - Y_{N-1}|^2$. For $1 < p < \infty$, $\|S\|_p$ and $\|Y_N\|_p$ are equivalent norms, with constants depending only on p.

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