An extension of Nachbin's theorem to differentiable functions on Banach spaces with the approximation property

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§ 1. Introduction

Let E and F be two real Banach spaces, with $F\neq\{0\}$. When $U\subset E$ is an open subset, we shall denote by $C^m(U; F)$ the vector space of all maps $f\colon U\to F$ which are of class C^m in U. This space will be endowed with the topology τ_c^m defined by the family of seminorms of the form

$$p_{K,L}(f) = \max_{0\leqq k\leqq m}\left\{\sup\{\|D^k f(x)v^k\|;\ x\in K, v\in L\}\right\}$$

where $K\subset U$ and $L\subset E$ are compact subsets and $Tv^k=T(v, \dots, v)$, when T is a k-linear map. When E is finite-dimensional, $\tau_c^m=\tau_u^m$, the compact-open topology of order m.

When $F=\mathbf{R}$, the space $C^m(U; F)$ is an algebra, denoted simply by $C^m(U)$.

When $E=\mathbf{R}^n$ and $F=\mathbf{R}$, Nachbin proved in [3] necessary and sufficient conditions for a subalgebra $A\subset C^m(U)$ to be dense in the topology τ_u^m $(m\geqq 1)$, extending the Stone—Weierstrass theorem to the differentiable case. In fact, he proved that the following are necessary and sufficient conditions for A to be dense in $(C^m(U), \tau_u^m)$:

(1) for every $x\in U$, there exists $f\in A$ such that $f(x)\neq 0$;

(2) for every pair $x, y\in U$, with $x\neq y$, there exists $f\in A$ such that $f(x)\neq f(y)$;

(3) for every $x\in U$ and $v\in E$, with $v\neq 0$, there exists $f\in A$ such that $Df(x)v\neq 0$.

In [1], Lesmes gave sufficient conditions for a subalgebra $A\subset C^m(E)$ to be dense in $(C^m(E), \tau_u^m)$, when $m=1$, and E is a real separable Hilbert space. In fact, he proved that (1), (2), (3) (with $U=E$) and

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** Partially supported by FINEP (SPPR-Brasil) through a grant to the Instituto de Matemática — UFRJ.

(4) there is an $M \in \mathbf{N}$ such that, for every integer $n \geqq M$, if $f \in A$ then $f \circ P_n \in A$; are sufficient for A to be τ_u^1-dense, where P_n is the orthogonal projection of E onto the span of $\{e_1, \ldots, e_n\}$, if $\{e_i;\ i \in \mathbf{N}\}$ is some orthonormal basis of E.

In [4], Prolla studied Nachbin's result for $m \geqq 1$ and the topology τ_c^m. He extended Nachbin's theorem for polynomial algebras $A \subset C^m(\mathbf{R}^n; F)$ and applied this extension to prove the analogue of Lesmes' result for polynomial algebras $A \subset C^m(E; F)$ in the τ_c^m topology, and E a real separable Hilbert space.

In [2], Llavona announced the following result. If E is a real Banach space with the approximation property, and $A \subset C^m(E; F)$ is a polynomial algebra satisfying (1), (2), (3) (with $U=E$) and

(4′) $A \circ G \subset A$

where $G \subset E^* \otimes E$ is some subset such that $i_E \in \bar{G}$, then A is dense in $(C^m(E; F), \tau_c^m)$. Here i_E is the identity map on E and $\bar{G}$ denotes the closure of G in $(L(E; E), \tau_u^0)$.

In this paper, we extend the results of [4] to cover the case of open subsets $U \subset E$, when E is a real Banach space with the approximation property, and polynomial algebras $A \subset C^m(U; F)$. For each integer $n \geqq 1$, $P_f^n(E; F)$ denotes the vector subspace of $C^\infty(E; F)$ generated by the set of all maps of the form $x \to [\varphi(x)]^n v$, where $\varphi \in E^*$, the topological dual of E, and $v \in F$. The elements of $P_f^n(E; F)$ are called n-homogeneous continuous polynomials of finite type from E into F. A vector subspace $A \subset C^0(E; F)$ is called a *polynomial algebra* if, for every integer $n \geqq 1$, $p \circ g \in A$ for all $g \in A$ and all $p \in P_f^n(F; F)$.

§ 2. Finite-dimensional case

In this section E is a finite-dimensional real Banach space and F is any real Banach space, with $F \neq \{0\}$. We may assume that $E = \mathbf{R}^n$.

Lemma 2.1. *$C^m(U) \otimes F$ is τ_u^m-dense in $C^m(U; F)$.*

Proof. Let $f \in C^m(U; F)$, $K \subset U$ a compact subset and $\varepsilon > 0$. Since $D^m(U; F)$ is τ_u^m-dense in $C^m(U; F)$, we may assume $f \in D^m(U; F)$. (We recall that $D^m(U; F)$ is the vector subspace of all $f \in C^m(U; F)$ which have compact support in U.) Let L be the support of f. Let M be a compact neighborhood of $K \cup L$ in U. Let $\varphi \in D(\mathbf{R}^n)$ be such that $\varphi(x)=1$ for all $x \in M$. Define $g \in D^m(\mathbf{R}^n; F)$ by $g(x)=\varphi(x)f(x)$ for all $x \in M$ and $g(x)=0$ for all $x \notin M$. Since $D(\mathbf{R}^n) \otimes F$ is τ_u^m-dense in $D^m(\mathbf{R}^n; F)$ (see Schwartz [6]), there exists $h \in D(\mathbf{R}^n) \otimes F$ such that $p_K(g-h) < \varepsilon$. Since $g=f$ in a neighborhood of K, and $k=h|U \in C^m(U) \otimes F$, we see that $p_K(f-k) < \varepsilon$. Hence $C^m(U) \otimes F$ is τ_u^m-dense in $C^m(U; F)$.

Theorem 2.2. *Let $A\subset C^m(U;F)$ be a polynomial algebra. Then A is τ_u^m-dense in $C^m(U;F)$ if, and only if, conditions* (1)—(3) *of* § 1 *are verified.*

Proof. The necessity of the conditions is easily verified. Conversely, assume that the polynomial algebra A satisfies (1)—(3) of § 1.

Let $M=\{\varphi\circ f;\ \varphi\in F^*, f\in A\}$. By Lemma 2.2 of [5], M is a subalgebra of $C^m(U)$ such that $M\otimes F\subset A$. By Nachbin's theorem (see [3], page 1550), M is τ_u^m-dense in $C^m(U)$. Hence $M\otimes F$ is τ_u^m-dense in $C^m(U)\otimes F$. By Lemma 2.1, $M\otimes F$ is τ_u^m-dense in $C^m(U;F)$, and since $M\otimes F\subset A$, this ends the proof.

§ 3. Infinite-dimensional case

In this section E and F are real Banach spaces and E has the approximation property, i.e. given $K\subset E$ compact and $\varepsilon>0$ there exists $u\in E^*\otimes E$ such that $\|u(x)-x\|<\varepsilon$ for all $x\in K$.

Lemma 3.1. *Let $f\in C^m(U;F)$; let $K\subset U$ and $L\subset E$ be compact subsets and let $\varepsilon>0$. There exists a map $u\in E^*\otimes E$ and an open subset $V\subset U$ such that $K\subset V$, $u(V)\subset U$ and $p_{K,L}(f|V-f\circ(u|V))<\varepsilon$.*

Proof. Since $(x,v)\mapsto D^kf(x)v^k$ is continuous for all $0\leqq k\leqq m$, and $K\times L$ is compact, we can find a real number $\delta>0$ such that $0<\delta<\operatorname{dist}(K, E\setminus U)$ and

(i) $\|f(x)-f(y)\|<\varepsilon$, and

(ii) $\|D^kf(x)v^k-D^kf(y)w^k\|<\varepsilon,\quad 1\leqq k\leqq m,$

for all $(x,v)\in K\times L$, $(y,w)\in U\times E$ such that $\|x-y\|<\delta$ and $\|v-w\|<\delta$.

By the approximation property, we can find $u\in E^*\otimes E$ such that

(iii) $\|u(x)-x\|<\delta/2$, for all $x\in K$, and

(iv) $\|u(v)-v\|<\delta$, for all $v\in L$.

Let $r=\delta/(2(\|u\|+1))$. For each $x\in K$, let $B(x;r)=\{t\in E;\ \|t-x\|<r\}$. By compactness of K we can find $x_1,\dots,x_n\in K$ such that $K\subset V=B(x_1;r)\cup\dots\cup B(x_n;r)$. Since $r<\operatorname{dist}(K,E\setminus U)$ it follows that $V\subset U$. Let $t\in V$. There exists some index i, with $1\leqq i\leqq n$, such that $t\in B(x_i;r)$. Hence $\|t-x_i\|<r$. Therefore $\|u(t)-x_i\|\leqq\leqq\|u(t)-u(x_i)\|+\|u(x_i)-x_i\|<\delta/2+\delta/2=\delta<\operatorname{dist}(K,E\setminus U)$. This shows that $u(t)\in U$, i.e. $u(V)\subset U$. Therefore the composition $f\circ(u|V)$ is defined and (i) and (iii) imply

(v) $\|f(x)-f\circ(u|V)(x)\|<\varepsilon$

for all $x\in K$. Similarly, (ii), (iii) and (iv) imply

(vi) $\|D^kf(x)v^k-D^kf(u(x))u(v)^k\|<\varepsilon$

for all $x \in K$ and $v \in L$. By the chain rule, $D^k f(u(x))u(v)^k = D^k(f \circ (u|V))(x)v^k$, and therefore (v) and (vi) show that $p_{K,L}(f|V - f \circ (u|V)) < \varepsilon$.

Lemma 3.2. *Let E_n be a finite-dimensional subspace of E, and $U_n \subset E_n \cap U$ a non-empty open subset. Let $T_n: C^m(U; F) \to C^m(U_n; F)$ be the map $f \mapsto f|U_n$. If $A \subset C^m(U; F)$ is a polynomial algebra satisfying conditions* (1)—(3) *of* § 1, *then $T_n(A)$ is τ_u^m-dense in $C^m(U_n; F)$.*

Proof. $T_n(A)$ is a polynomial algebra contained in $C^m(U_n; F)$ and satisfying conditions (1)—(3) of § 1. By Theorem 2.2, $T_n(A)$ is τ_u^m-dense in $C^m(U_n; F)$.

Theorem 3.3. *Let $A \subset C^m(U; F)$ be a polynomial algebra such that, for any $u \in E^* \otimes E$ and any open subset $V \subset U$ with $u(V) \subset U$,*

$$g \in A \Rightarrow g \circ (u|V) \in A|V. \tag{5}$$

Then A is τ_c^m-dense in $C^m(U; F)$ if, and only if, conditions (1)—(3) *of* § 1 *are verified.*

Proof. The necessity of the conditions is easily verified. For sufficiency, let $f \in C^m(U; F)$; let $K \subset U$ and $L \subset E$ be compact subsets; and let $\varepsilon > 0$ be given. By Lemma 3.1, there exists $u \in E^* \otimes E$ and $V \subset U$ an open subset such that $K \subset V$, $u(V) \subset U$ and $p_{K,L}(f|V - f \circ (u|V)) < \varepsilon/2$. Let E_n be the finite-dimensional subspace $u(E) \subset E$. Let $U_n = E_n \cap U$. Since $u(K) \subset U_n$ is a compact subset, by Lemma 3.2 there exists $g \in A$ such that $P_{u(K)}(T_n g - T_n f) < \delta$, for a given $\delta > 0$. Choose $\delta > 0$ such that $\delta < \varepsilon/(2(r+1)^k)$, for all $0 \leqq k \leqq m$, where $r = \sup\{\|u(v)\|; v \in L\}$. Then, the following is true:

(i) $\|g(u(x)) - f(u(x))\| < \varepsilon/2$

(ii) $\|D^k(T_n g)(u(x)) - D^k(T_n f)(u(x))\| < \delta$

for all $x \in K$. Since

$$D^k(T_n g)(u(x)) = D^k g(u(x))\,|\,E_n^k$$

and

$$D^k(T_n f)(u(x)) = D^k f(u(x))\,|\,E_n^k$$

for all $x \in K$, it follows from (ii) that

(iii) $\|D^k g(u(x))u(v)^k - D^k f(u(x))u(v)^k\| \leqq \delta r^k < \varepsilon/2$

for all $x \in K$ and $v \in L$. By the chain rule,

$$D^k g(u(x))u(v)^k = D^k(g \circ (u|V))(x)v^k$$

and

$$D^k f(u(x))u(v)^k = D^k(f \circ (u|V))(x)v^k.$$

Hence (i) and (iii) show that

$$p_{K,L}(f \circ (u|V) - g \circ (u|V)) < \varepsilon/2.$$

By condition (5), there exists $h \in A$ such that $g \circ (u|V) = h|V$. Therefore $p_{K,L}(f|V - h|V) < \varepsilon$.

Since V is an open neighborhood of K, the last inequality is equivalent to $p_{K,L}(f-h) < \varepsilon$.

Hence A is dense in $C^m(U; F)$.

Corollary 3.4. *The polynomial algebra $P_f(U; F)$ is τ_c^m-dense in $C^m(U; F)$.*

Proof. $P_f(U; F)$ is the set $P_f(E; F)|U$, where $P_f(E; F)$ is the vector space generated by the union of $P_f^n(E; F)$ for all $n \geqq 1$ and the constant maps. One easily verifies that $P_f(U; F)$ satisfies conditions (1)—(3), (5).

Corollary 3.5. *The following polynomial algebras are τ_c^m-dense in $C^m(U; F)$:*
(a) $P(U; F)$
(b) $C^\infty(U; F)$
(c) $C^r(U; F)$, $r \geqq m$.

Proof. Just notice that

$$P_f(U; F) \subset P(U; F) \subset C^\infty(U; F) \subset C^r(U; F).$$

Corollary 3.6. *The following polynomial algebras are τ_c^m-dense in $C^m(U; F)$:*
(a) $P_f(U) \otimes F$,
(b) $P(U) \otimes F$,
(c) $C^\infty(U) \otimes F$,
(d) $C^r(U) \otimes F$, $r \geqq m$.

Proof. Just notice that $P_f(U) \otimes F = P_f(U; F)$.

Remark 3.7. The proof of Lemma 3.1 shows that we can choose there $u \in E^* \otimes E$ in any subset $G \subset E^* \otimes E$, such that $i_E \in \bar{G}$, the τ_u^0-closure of G in $L(E; E)$. Hence Theorem 3.3 remains true if $A \subset C^m(U; F)$ is a polynomial algebra such that, there exists $G \subset E^* \otimes E$ as above, and for any $u \in G$ and any open subset $V \subset U$ with $u(V) \subset U$, then (5) is true. When $U = E$, this condition becomes $A \circ G \subset A$, i.e. the following result is true and generalizes Llavona's result [2].

Theorem 3.8. *Let $A \subset C^m(U; F)$ be a polynomial algebra. Suppose that there exists $G \subset E^* \otimes E$ whose τ_u^0-closure in $L(E; E)$ contains i_E, and for any $u \in G$ and for any open subset $V \subset U$ with $u(V) \subset U$,*

(6) $$g \in A \Rightarrow g \circ (u|V) \in A|V.$$

Then A is τ_c^m-dense in $C^m(U; F)$, if and only if A satisfies conditions (1)—(3) *of* § 1.

Example 3.9. Suppose that E satisfies the metric approximation property and that A satisfies (6) with $G = \{u \in E^* \otimes E;\ \|u\| \leqq 1\}$.

Example 3.10. Suppose that E is a real separable Banach space with a Schauder basis $\{x_n, x_n^*; n\in\mathbf{N}\}$, and that A satisfies (6) with $G=\{P_k; k\in\mathbf{N}\}$, where P_k is the map

$$x\mapsto \sum_{i\leqq n_k} x_i^*(x)x_i$$

for each $k\in\mathbf{N}$, and $\{n_k\}$ is a subsequence of $\mathbf{N}$, i.e. $n_1<n_2<\ldots<n_k<\ldots$.

In particular, suppose that E is a real separable Hilbert space and $\{x_n; n\in\mathbf{N}\}$ is an orthonormal basis for E and P_k is the orthogonal projection of E onto the span of $\{x_1, \ldots, x_{n_k}\}$.

§ 4. The role of the approximation property

In this section we study the converse of Corollary 3.6. More generally, we study the relation between the approximation property for *real* Banach spaces E and the spaces $(C^m(U; F), \tau_c^m)$, for $U\subset E$ open and $m\geqq 1$.

For the case of *complex* Banach spaces, $C^m(U; F)=\mathscr{H}(U; F)$, if $m\geqq 1$. The relationship between the approximation property for E and several spaces of holomorphic mappings and topologies on them, has been studied by Aron and Schottenloher. (See [0], in particular Theorems 2.2, 4.1 and 4.3 of [0].)

Theorem 4.1. *Let E be a real Banach space; then the following properties are equivalent:*

(1) *E has the approximation property.*

(2) *For every $m\geqq 1$, for every non-void open subset $U\subset E$, and for every Banach space F, $C^m(U)\otimes F$ is τ_c^m-dense in $C^m(U; F)$.*

(3) *For every $m\geqq 1$, for every real Banach space F, and for every non-void open subset $V\subset F$, $C^m(V)\otimes E$ is τ_c^m-dense in $C^m(V; E)$.*

(4) *For every $m\geqq 1$, $C^m(E)\otimes E$ is τ_c^m-dense in $C^m(E; E)$.*

(5) *For every $m\geqq 1$, the identity map on E belongs to the τ_c^m-closure of $C^m(E)\otimes E$ in $C^m(E; E)$.*

(6) *The identity map on E belongs to the τ_c^1-closure of $C^1(E)\otimes E$ in $C^1(E; E)$.*

Proof. Part (d) of Corollary 3.6 states that (1)$\Rightarrow$(2).

(1)$\Rightarrow$(3). Let $K\subset V$ and $L\subset F$ be compact subsets, let $\varepsilon>0$, and let $f\in C^m(V; E)$, $m\geqq 1$. Since the mappings $(x, v)\mapsto D^k f(x)v^k$ are continuous, for all $0\leqq k\leqq m$, and $K\times L$ is compact, the sets

$$A_k = \{D^k f(x)v^k, (x, v)\in K\times L\}\subset E$$

are compact. Let $A=A_0\cup A_1\cup\ldots\cup A_m$. Since E has the approximation property, we can find $u_i \in E^*$ and $e_i \in E$, $1 \leqq i \leqq n$, such that

$$\left\| y - \sum_{i=1}^n u_i(y) e_i \right\| < \varepsilon$$

for all $y \in A$. Hence, for all $(x, v) \in K \times L$,

$$\left\| D^k f(x) v^k - \sum_{i=1}^n u_i\big(D^k f(x) v^k\big) e_i \right\| < \varepsilon$$

for all $0 \leqq k \leqq m$. By the chain rule,

$$u_i\big(D^k f(x) v^k\big) = D^k (u_i \circ f)(x) v^k,$$

since u_i is linear. Therefore

$$\left\| D^k f(x) v^k - D^k\big(\sum_{i=1}^n (u_i \circ f) \otimes e_i\big)(x) v^k \right\| < \varepsilon$$

for all $0 \leqq k \leqq m$ and $(x, v) \in K \times L$. It remains to notice that $(u_i \circ f) \otimes e_i \in C^m(V) \otimes E$, $1 \leqq i \leqq n$.

By setting $U=E=F$ and $V=F=E$, we see that (2)⇒(4), and (3)⇒(4), respectively.

(4)⇒(5)⇒(6). Obvious.

(6)⇒(1). Let $K \subset E$ be compact and $\varepsilon > 0$. Since $\{0\} \subset E$ is compact, the seminorm

$$f \mapsto \sup \{\|Df(0)x\|, x \in K\}$$

is τ_c^1-continuous. By (6), there is a function f belonging to $C^1(E) \otimes E$ such that

$$\|x - Df(0)x\| < \varepsilon$$

for all $x \in K$. Assume

$$f = \sum_{i=1}^n f_i \otimes e_i, \quad f_i \in C^1(E), \quad e_i \in E, \quad 1 \leqq i \leqq n.$$

Let $u_i = Df_i(0) \in E^*$, $1 \leqq i \leqq n$. Then

$$\left\| x - \sum_{i=1}^n u_i(x) e_i \right\| < \varepsilon$$

for all $x \in K$, and (1) obtains.

Remarks 4.2. (a) The above Theorem 4.1 generalizes the results announced in [2] by Llavona: (1)⇔(2′)⇔(5), where

(2′) For every $m \geqq 1$, and for every Banach space F, $C^m(E) \otimes F$ is τ_c^m-dense in $C^m(E; F)$.

Indeed, (2)⇒(2′) by setting $U=E$, and (2′)⇒(4) by setting $F=E$.

(b) The condition (6) cannot be changed to $m=0$. Indeed, by Corollary 4.3 of [5], the identity map on E belongs to the τ_c^0-closure=compact-open closure of $C^0(E) \otimes E$ in $C^0(E, E)$, for *any* real Banach space E. However (6) could be weakened to

(6′) The identity map on E belongs to the τ_c^0-closure of $C^1(E) \otimes E$ in $C^1(E; E)$.

We do not know if (6′)⇒(1). (See also the remark after proof of Theorem 4.1 of [0].)

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Received August 29, 1975

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