The distribution of the Riesz mass of certain subharmonic functions

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1. Introduction

Let $u(z)$ be a function subharmonic in the plane, let

$$M(r, u) = \max_{|z|=r} u(z),$$

$$m(r, u) = \inf_{|z|=r} u(z).$$

The order ϱ and lower order λ of $u(z)$ are defined by

$$\begin{matrix}\varrho \\ \lambda\end{matrix} = \overline{\underline{\lim}}_{r\to\infty} \frac{\log M(r, u)}{\log r}.$$

Heins [5] has proved the following theorem, which is the analogue for subharmonic functions of the Weierstrass representation formula for integral functions.

Theorem A. *Let $u(z)$ be a function subharmonic in the plane, harmonic at $z=0$ and of order less than one. Then there exists a unique non-negative Borel measure μ which is such that*

$$u(z) = u(0) + \int_{|\zeta|<\infty} \log\left|1-\frac{z}{\zeta}\right| d\mu e_\zeta \tag{1.1}$$

for all z.

The proof of Theorem A depends on a general representation theorem of Riesz [8] for functions subharmonic in a domain. We will be concerned in this paper with the asymptotic behavior of $u(z)$ as $|z|$ tends to infinity and this behavior will remain unaffected if the values of $u(z)$ in a small circle about the origin are replaced by the Poisson integral of the boundary values of $u(z)$ on the circle. This modified function is then harmonic in the small circle (and so at the origin) and subharmonic in the plane and consequently allows the representation (1.1). It will be assumed hereafter that this modification has been carried out and that (1.1) holds.

Anderson [1] has proved:

Theorem B. *Let $u(z)$ be a function subharmonic in the plane and satisfying*

$$m(r, u) < O(1) \quad \text{as} \quad r \to \infty, \tag{1.2}$$

$$\varlimsup_{r\to\infty} \frac{M(r, u)}{r^{1/2}} < \infty. \tag{1.3}$$

Suppose we are given $\delta > 0$, $k > 1$. Then, for all $r > 0$, there exists a continuous function $\theta(r)$ such that, if

$$\mu_0(r) = \mu\{k^{-1}r \leqq |z| < kr, \delta \leqq |\arg ze^{-i\theta}| \leqq \pi\}, \tag{1.4}$$

then

$$\mu_0(r) = o(r^{1/2}) \quad \text{as} \quad r \to \infty. \tag{1.5}$$

Moreover

$$\theta(r) = o(\log r) \quad \text{as} \quad r \to \infty. \tag{1.6}$$

That the conclusion (1.6) concerning the growth of $\theta(r)$ may not be the best possible is suggested by a theorem of Kennedy [6] from a special case of which it follows that, if $u(z)$ is bounded on a receding curve γ and satisfies (1.3), then

$$\lim \frac{\arg z}{(\log |z|)^{1/2}} = 0 \quad \text{as} \quad z \to \infty \quad \text{along} \quad \gamma. \tag{1.7}$$

In Theorem 1 below it will be shown that, under more general hypotheses than (1.2) and (1.3), the conclusions (1.5) and (1.7) hold and in a related theorem, Theorem 2, an estimate for the Riesz mass of u outside a narrowing angular neighbourhood of the curve $\arg z = \theta(r)$ will be given.

2. Statement of results

Given numbers k and δ, with $k > 1$ and $0 \leqq \delta \leqq \pi$, let

$$S(r, k) = \{z : k^{-1}r < |z| \leqq kr\}, \tag{2.1}$$

$$S(\zeta, k, \delta) = \{z : k^{-1}r < |z| \leqq kr \text{ and } |\arg z\zeta^{-1}| \geqq \delta\}. \tag{2.2}$$

Let ϱ be a positive number less than one and let $u(z)$ be a function subharmonic in the plane and of order ϱ satisfying

(A) there is a finite constant K such that

$$\int_{r_1}^{r_2}\{m(r,u)-\cos\pi\varrho M(r,u)\}\frac{dr}{r^{\varrho+1}}\leqq K,\quad 1<r_1<r_2<\infty, \tag{2.3}$$

and

(B) there are numbers α and β, with $0<\alpha<\beta<\infty$, such that for all large r

$$\alpha r^{\varrho}\leqq\mu\{z:|z|<r\}\leqq\beta r^{\varrho}. \tag{2.4}$$

Condition (A) is implicit in the work of Kjellberg [7] and has been much exploited subsequently, notably by Anderson [2] and Essén [4].

We will prove

Theorem 1. *Let $u(z)$ be a function subharmonic in the plane and of order ϱ, where $0<\varrho<1$, which satisfies conditions* (A) *and* (B) *above and let*

$$k=\left(\frac{2\beta}{\alpha}\right)^{1/\varrho}. \tag{2.5}$$

Then there is a curve $C: z=re^{i\varphi(r)}$, where $\varphi(r)$ is a continuous function satisfying

$$|\varphi(R_2)-\varphi(R_1)|=o\left|\log\frac{R_2}{R_1}\right|^{1/2}\quad as\quad \min(R_1,R_2)\to\infty, \tag{2.6}$$

and a continuous function $\Delta(t)$ satisfying $\pi\geqq\Delta(t)\geqq 0$, $\Delta(t)\to 0$ as $t\to\infty$ and

$$\int^{\infty}\frac{\Delta(t)^2}{t}\,dt<\infty, \tag{2.7}$$

for which the following is true. If ζ is a point of C, then

$$\mu[S(\zeta,k,\Delta(|\zeta|)]=o\{\mu[S(|\zeta|,k)]\}, \tag{2.8}$$

where μ is the Borel measure occurring in the representation (1.1).

We will also prove the related theorem:

Theorem 2. *Let $u(z)$ be a function subharmonic in the plane and of order ϱ, where $0<\varrho<1$, satisfying conditions* (A) *and* (B) *above. Let k be as given by* (2.5) *and let C be the curve, μ the measure, of Theorem* 1. *Then there is a function $\varepsilon(t)$ satisfying $\pi\geqq\varepsilon(t)\geqq 0$ and $\varepsilon(t)\to 0$ as $t\to\infty$, and a function $\nu(t)$ satisfying $1\geqq\nu(t)\geqq 0$, $\nu(t)\to 0$ as $t\to\infty$ and*

$$\int^{\infty}\frac{\nu(t)}{t}<\infty \tag{2.9}$$

for which the following is true. If ζ is any point on C, then

$$\mu[S(\zeta,k,\varepsilon(|\zeta|)]\leqq\nu(|\zeta|)\mu[S(|\zeta|,k)]. \tag{2.10}$$

The case $\varrho=\frac{1}{2}$ of Theorem 1 has much in common with a particular case of a theorem appearing in [3]. In [3] David Drasin is concerned with the behaviour of the curves on which a function possesses asymptotic values and, in the case of an entire function of order $\frac{1}{2}$ with a single asymptotic value, he shows that the curve is close (in a certain sense) to a curve: $z=re^{i\theta(r)}$, where $\theta(r)$ is a continuous, piecewise linear function of $\log r$ satisfying

$$\int^{\infty} r\theta'(r)^2\,dr < \infty.$$

Such a curve is of precisely the same kind as that occuring in Theorem 1 and it follows from the asymptotic behaviour of the entire function that

$$\theta(r)-\varphi(r) = o(1) \quad \text{as} \quad r\to\infty.$$

It seems not unlikely that the two curves are in fact interchangeable.

3. Preliminaries to the proofs of Theorems 1 and 2

Let $u(z)$ be a subharmonic function satisfying the hypotheses of Theorems 1 and 2. We assume that $u(0)=0$, which may be done without loss of generality. We define

$$\mu^*(t) = \mu[\{z: |z| \leqq t\}] \tag{3.1}$$

and introduce a new function $U(z)$ given by

$$U(z) = \int_0^{\infty} \left|1+\frac{z}{t}\right| d\mu^*(t). \tag{3.2}$$

$U(z)$ is a well-known auxilliary function, the properties of which are summarized on page 204 of [5]. Among these properties is the following relation:

$$m(r,u)+M(r,u) \geqq m(r,U)+M(r,U). \tag{3.3}$$

We will require the following lemmas, the first of which is a consequence of (2.4) and the properties of U, the second being an important result of Kjellberg [7].

Lemma 1. *Let $U(z)$ be the subharmonic function* (3.2). *Then*

$$\alpha\pi \operatorname{cosec} \pi\varrho \leqq \varliminf_{r\to\infty} \frac{M(r,U)}{r^{\varrho}} \leqq \varlimsup_{r\to\infty} \frac{M(r,U)}{r^{\varrho}} \leqq \beta\pi \operatorname{cosec} \pi\varrho. \tag{3.4}$$

Lemma 2 [7]. *Let $U(z)$ be the subharmonic function* (3.2). *Then there exist constants B_1 and B_2, depending only on ϱ, such that*

$$\int_{r_1}^{r_2} \{m(r,U)-\cos\pi\varrho M(r,U)\}\frac{dr}{r^{\varrho+1}} \geqq B_1\frac{M(r_1,U)}{r_1^{\varrho}} - B_2\frac{M(r_2,U)}{r_2^{\varrho}} \tag{3.5}$$

for all r_1, r_2 satisfying $0<r_1<r_2<\infty$.

4. A basic lemma

A useful preliminary result is:

Lemma 3. *Let $U(z)$ be the subharmonic funcion* (3.2). *Then*

$$\int^{\infty} \{m(r, u) - m(r, U)\} \frac{dr}{r^{\varrho+1}} < \infty. \tag{4.1}$$

Let $\sigma = \max(0, -\cos \pi\varrho)$ and let

$$P(r) = m(r, u) - \cos \pi\varrho M(r, u) - m(r, U) + \cos \pi\varrho M(r, U).$$

Then, from (2.3), (3.4) and (3.5), there is a finite constant K_1 such that

$$\int_{r_1}^{r_2} P(r) \frac{dr}{r^{\varrho+1}} \leqq K_1 \tag{4.2}$$

for all large values of r_1 and r_2, with $r_2 \geqq r_1$.

From (3.3) we see that

$$\begin{aligned} P(r) &= m(r, u) - m(r, U) + \cos \pi\varrho \{M(r, U) - M(r, u)\} \\ &\geqq m(r, u) - m(r, U) - \sigma\{M(r, U) - M(r, u)\} \\ &\geqq (1-\sigma)\big(m(r, u) - m(r, U)\big). \end{aligned} \tag{4.3}$$

From (4.2) and (4.3) it follows that

$$(1-\sigma) \int_{r_1}^{r_2} \{m(r, u) - m(r, U)\} \frac{dr}{r^{\varrho+1}} \leqq K_1$$

for all large values of r_1 and r_2 with $r_2 \geqq r_1$. Since $\sigma < 1$ and $m(r, u) \geqq m(r, U)$, (4.1) follows.

5. Some definitions

We begin by making the observation that the set S on which $m(r, U) = -\infty$ has zero measure. This follows from Lemma 2. Also, for r outside S, $m(r, u) \geqq m(r, U) > -\infty$.

Let r be a positive number outside S and let

$$E(r) = \{z : |z| = r \text{ and } u(z) < m(r, u) + 1\}.$$

$E(r)$ is a non-empty open set, since $u(z)$ is upper semi-continuous. Let $E_1(r), \ldots, E_n(r)$, say, be the largest components of $E(r)$ and let $\zeta_1(r), \ldots, \zeta_n(r)$ be their centres. We choose that one of the $\zeta_i(r)$, $i = 1, 2, \ldots, n$, which is such that, with "arg" denoting principal argument with range $(-\pi, \pi]$ (notation with which we will continue),

$$\arg \zeta_i(r) = \max_{j=1}^{n} \{\arg \zeta_j(r)\},$$

and write $\zeta(r)=\zeta_i(r)$. $\zeta(r)$ is thus uniquely chosen for each r outside S and

$$u(\zeta(r)) < m(r, u)+1. \tag{5.1}$$

Let ε be any positive number less than one and let k be as given by (2.5). We define $\delta(r, \varepsilon)$ to be the upper bound of all positive numbers t for which

$$\mu[S(\zeta(r), k, t)] \geqq \varepsilon' \mu[S(r, k)], \tag{5.2}$$

where

$$\varepsilon' = \varepsilon \left(\frac{\alpha}{4\beta}\right)^2. \tag{5.3}$$

In the case where there is no positive number t for which (5.2) holds we define $\delta(r, \varepsilon)$ to be zero.

Let $\tau(r, \varepsilon)$ be given by

$$\mu[S(\zeta(r), k, \delta(r, \varepsilon))] = \tau(r, \varepsilon)\mu[S(r, k)], \tag{5.4}$$

so that $1 \geqq \tau(r, \varepsilon) \geqq \varepsilon'$ and let $\eta(r, \varepsilon)$ be given by

$$\mu[S(\zeta(r), k, \tfrac{1}{2}\varepsilon)] = \eta(r, \varepsilon)\mu[S(r, k)], \tag{5.5}$$

$1 \geqq \eta(r, \varepsilon) \geqq 0$.

6. The functions $\delta(r, \varepsilon)$ and $\eta(r, \varepsilon)$

We have

Lemma 4. *Let $\delta(r, \varepsilon)$, $\eta(r, \varepsilon)$ be as defined in the previous section. Then*

$$\int^{\infty} \frac{\delta(r, \varepsilon)^2}{r} dr < \infty \tag{6.1}$$

and

$$\int^{\infty} \frac{\eta(r, \varepsilon)}{r} dr < \infty. \tag{6.2}$$

Let r be a positive number outside S and define two positive real valued functions $\mu_1^*(t)$, $\mu_2^*(t)$ for $k^{-1}r < t \leqq kr$ by

$$\mu_1^*(t) = \mu[\{z : k^{-1}r < |z| \leqq t \text{ and } |\arg z\zeta(r)^{-1}| \geqq \delta(r, \varepsilon)\}], \tag{6.3}$$

$$\mu_2^*(t) = \mu[\{z : k^{-1}r < |z| \leqq t \text{ and } |\arg z\zeta(r)^{-1}| < \delta(r, \varepsilon)\}]. \tag{6.4}$$

We have, writing $\zeta=\zeta(r)$ and recalling that $|\zeta(r)|=r$,

$$u(\zeta) = \int_{|z|<\infty} \log\left|1-\frac{\zeta}{z}\right| d\mu e_z$$

$$\geqq \int_0^{k^{-1}r} \log\left|1-\frac{r}{t}\right| d\mu^*(t)+\int_{kr}^{\infty} \log\left|1-\frac{r}{t}\right| d\mu^*(t)$$

$$+\int_{k^{-1}r}^{kr} \log\left|1-\frac{r}{t}\right| d\mu_2^*(t)+\int_{k^{-1}r}^{kr} \log\left|1-\frac{re^{i\delta(r,\varepsilon)}}{t}\right| d\mu_1^*(t)$$

$$= m(r, U)+\int_{k^{-1}r}^{kr} \left\{\log\left|1-\frac{re^{i\delta(r,\varepsilon)}}{t}\right|-\log\left|1-\frac{r}{t}\right|\right\} d\mu_1^*(t).$$

Hence

(6.5) $$u(\zeta)-m(r, U) \geqq \int_{k^{-1}r}^{kr} \log\left|\frac{t-re^{i\delta(r,\varepsilon)}}{t-r}\right| d\mu_1^*(t).$$

Now,

(6.6) $$|t-re^{i\delta(r,\varepsilon)}|^2|t-r|^{-2} = 1+\frac{4rt}{(t-r)^2}\sin^2\frac{\delta(r,\varepsilon)}{2} \geqq 1+k_2\delta(r,\varepsilon)^2,$$

where $k_2=4k\pi^{-2}(k-1)^{-2}$. Substituting (6.6) into (6.5) and setting $k_3=\frac{1}{2}k_2(1+k_2\pi^2)^{-1}$, we obtain

(6.7) $$\begin{aligned} u(\zeta)-m(r, U) &\geqq \tfrac{1}{2}\log\left(1+k_2\delta(r,\varepsilon)^2\right)\mu_1^*(kr) \\ &\geqq k_3\delta(r,\varepsilon)^2\mu_1^*(kr) \\ &= k_3\delta(r,\varepsilon)^2\mu[S(\zeta, k, \delta(r,\varepsilon))] \\ &\geqq \varepsilon' k_3\delta(r,\varepsilon)^2\mu[S(r,k)] \\ &\geqq \varepsilon'\beta k_3\delta(r,\varepsilon)^2 r^{\varrho} \end{aligned}$$

since, from (2.4) and (2.5), $\mu[S(r,k)]\geqq\alpha(kr)^{\varrho}-\beta(k^{-1}r)^{\varrho}=r^{\varrho}(2\beta-\frac{1}{2}\alpha)>\beta r^{\varrho}$.

From (6.7) and the definition of $\zeta(r)$ we obtain, for large r outside S,

$$m(r, u)-m(r, U)+1 \geqq \varepsilon'\beta k_3\delta(r,\varepsilon)^2 r^{\varrho},$$

and from this together with Lemma 3 we obtain (6.1).

In a precisely similar way, but with $\frac{1}{2}\varepsilon$ replacing $\delta(r,\varepsilon)$ in (6.3) and (6.4), we deduce that

(6.8) $$m(r, u)-m(r, U)+1 \geqq \tfrac{1}{4}\varepsilon^2\beta k_3\eta(r,\varepsilon)r^{\varrho}.$$

(6.2) follows from (6.8) and Lemma 3.

7. Further definitions

Let r_0 be a large positive number and define intervals

(7.1) $$I_n = [k^{n/4} r_0, k^{(n+1)/4} r_0], \qquad n = 0, 1, 2, \ldots.$$

Within each of these intervals we select a point $s_n = s_n(\varepsilon)$ at which both

(7.2) $$\delta(s_n, \varepsilon)^2 < \frac{9}{\log k} \int_{I_n} \frac{\delta(r, \varepsilon)^2}{r} dr,$$

(7.3) $$\eta(s_n, \varepsilon) < \frac{9}{\log k} \int_{I_n} \frac{\eta(r, \varepsilon)}{r} dr.$$

Such a point must exist. For each of (7.2), (7.3) is false in a subset of I_n of logarithmic measure at most $\frac{1}{9} \log k$ and consequently the subset on which one or both of (7.2) and (7.3) is false has logarithmic measure at most $\frac{2}{9} \log k$, which is less than $\frac{1}{4} \log k$, the logarithmic measure of I_n.

Let $\xi_n(\varepsilon) = \zeta(s_{2n})$, let $\theta(s_{2n}, \varepsilon) = \arg \xi_n(\varepsilon)$, $n = 0, 1, 2, \ldots$, and let $\Gamma(\varepsilon)$ be the curve defined by

(7.4) $$\Gamma(\varepsilon): z = re^{i\{\theta(s_{2n}, \varepsilon) + k_n(r - s_n)\}}, \qquad s_{2n} \leqq r \leqq s_{2n+2},$$

where k_n is a constant given by

$$k_n = (s_{2n+2} - s_{2n})^{-1} \arg \frac{\xi_{n+1}(\varepsilon)}{\xi_n(\varepsilon)}.$$

Then for $r \geqq s_0$ we may represent $\Gamma(\varepsilon)$ by

(7.5) $$\Gamma(\varepsilon) = re^{i\Psi(r, \varepsilon)},$$

where $\Psi(r, \varepsilon)$ is continuous for $r \geqq s_0$.

We define

$$D(r, \varepsilon) = 2\{\delta(s_{2n}, \varepsilon) + 2\delta(s_{2n+1}, \varepsilon) + \delta(s_{2n+2}, \varepsilon) + (s_{2n+2})^{-1}\} \quad \text{for} \quad s_{2n} \leqq r \leqq s_{2n+2},$$
(7.6)

(7.7) $$E(r, \varepsilon) = \left(\frac{2\beta}{\alpha}\right)^{7/4} \{\eta(s_{2n+2}, \varepsilon) + \eta(s_{2n}, \varepsilon)\} \quad \text{for} \quad s_{2n} \leqq r \leqq s_{2n+2},$$

for $n = 0, 1, 2, \ldots$.

The remainder of the proof proceeds as follows. We will show that $\Psi(r, \varepsilon)$ satisfies the condition (2.6), also that $D(r, \varepsilon)$ tends to zero as r tends to infinity and that (2.7) holds for $D(r, \varepsilon)$. We will show that, for any point ζ on $\Gamma(\varepsilon)$, $\mu[S(\zeta, k, D(|\zeta|, \varepsilon))] \leqq \varepsilon \mu[S(|\zeta|, k)]$ and $\mu[S(\zeta, k, \varepsilon)] \leqq E(|\zeta|, \varepsilon) \mu[S(|\zeta|, k)]$, and that

$E(r, \varepsilon)$ tends to zero as r tends to infinity and that (2.9) holds for $E(r, \varepsilon)$. Having done this the remainder of the proof is straightforward, though lengthy. It will be shown that the functions $\Delta(r)$, $\nu(r)$ may be constructed form the functions $D(r, 1/n)$, $E(r, 1/n)$, $n=1, 2, 3, \ldots$, and also that the curve C may be obtained by piecing together segments of the curves $\Gamma(1/n)$, $n=1, 2, 3, \ldots$.

8. Concerning $\Gamma(\varepsilon)$, $D(r, \varepsilon)$, $E(r, \varepsilon)$

(1) $\Gamma(\varepsilon)$

Lemma 5. *Let ξ_n, ξ_{n+1} be successive vertices of $\Gamma(\varepsilon)$. Then*

$$|\arg \xi_n \xi_{n+1}^{-1}| \leqq \tfrac{1}{2} D(|\xi_n|, \varepsilon). \tag{8.1}$$

Lemma 5 follows from (7.6) and two applications of the following result:

Lemma 6. *With s_n as defined in section 7,*

$$|\arg e^{i(\theta(s_n, \varepsilon) - \theta(s_{n+1}, \varepsilon))}| \leqq \delta(s_n, \varepsilon) + \delta(s_{n+1}, \varepsilon), \tag{8.2}$$

for $n=0, 1, 2, 3, \ldots$.

From the definition of s_n, $s_{n+1} \leqq k^{1/2} s_n$. Also, if (8.2) is false we have, for all small positive numbers t,

(8.3)
$$\begin{aligned}
&\mu[\{z : ks_n \geqq |z| > k^{-1} s_{n+1}\}] \\
&\quad \leqq \mu[S(s_n e^{i\theta(s_n, \varepsilon)}, k, \delta(s_n, \varepsilon) + t)] + \mu[S(s_{n+1} e^{i\theta(s_{n+1}, \varepsilon)}, k, \delta(s_{n+1}, \varepsilon) + t)] \\
&\quad \leqq \varepsilon' \{\beta(ks_n)^\varrho - \alpha(k^{-1}s_n)^\varrho\} + \varepsilon' \{\beta(ks_{n+1})^\varrho - \alpha(k^{-1}s_{n+1})^\varrho\} \\
&\quad \leqq \varepsilon' s_n^\varrho \{\beta k^{3\varrho/2} + \beta k^\varrho - \alpha k^{-\varrho/2} - \alpha k^{-\varrho}\} \\
&\quad \leqq 2\beta k^{2\varrho} \varepsilon' s_n^\varrho \\
&\quad < \tfrac{1}{2} \beta s_n^\varrho,
\end{aligned}$$

from (2.5), (5.3) and the fact that $\varepsilon < 1$.

However

$$\begin{aligned}
\mu[\{z : ks_n \geqq |z| > k^{-1} s_{n+1}\}] &\geqq \alpha(ks_n)^\varrho - \beta(k^{-1}s_{n+1})^\varrho \\
&\geqq s_n^\varrho (\alpha k^\varrho - \beta k^{-\varrho/2}) \\
&\geqq s_n^\varrho (\alpha k^\varrho - \beta) \\
&= \beta s_n^\varrho,
\end{aligned} \tag{8.4}$$

from (2.5), which contradicts (8.3). Lemma 6 is thus proved.

(2) $D(r, \varepsilon)$

We will prove

Lemma 7. *Let $D(r,\varepsilon)$ be as given by (7.6). Then*

(i) *if ζ is a point on $\Gamma(\varepsilon)$ then for large $|\zeta|$*

$$\mu[S(\zeta, k, D(|\zeta|, \varepsilon))] \leqq \varepsilon\mu[S(|\zeta|, k)];$$

(ii) $\int^{\infty} \frac{D(r,\varepsilon)^2}{r}\, dr < \infty$;

(iii) *$D(r,\varepsilon)$ tends to zero as r tends to infinity.*

Let ζ be a point on $\Gamma(\varepsilon)$ and let q be the integer for which

$$s_{2q} = |\xi_q| \leqq |\zeta| < |\xi_{q+1}| = s_{2q+2},$$

where the points ξ_n are the vertices of $\Gamma(\varepsilon)$, $n=0, 1, 2, \ldots$.

Now from Lemma 5 and the fact that $D(|\zeta|, \varepsilon)$ is constant for $s_{2q} \leqq |\zeta| < s_{2q+2}$ we have, with $t=(s_{2q+2})^{-1}$,

$$\begin{aligned}
\mu[S(\zeta, k, D(|\zeta|, \varepsilon))] &\leqq \mu[S(\xi_q, k, \tfrac{1}{2}D(|\zeta|, \varepsilon)) \cup S(\xi_{q+1}, k, \tfrac{1}{2}D(|\zeta|, \varepsilon))] \\
&\leqq \mu[S(\xi_q, k, \delta(|\xi_q|, \varepsilon)+t) \cup S(\xi_{q+1}, k, \delta(|\xi_{q+1}|, \varepsilon)+t)] \\
&\leqq \varepsilon'\{\beta(ks_{2q+2})^{\varrho} - \alpha(k^{-1}s_{2q+2}) + \beta(ks_{2q})^{\varrho} - \alpha(k^{-1}s_{2q})^{\varrho}\} \\
&\leqq 2\varepsilon' s_{2q+2}^{\varrho}(\beta k^{\varrho} - \alpha k^{-\varrho}) \\
&\leqq 2\varepsilon' s_{2q}^{\varrho} k^{7\varrho/4} \\
&\leqq 2\varepsilon'\beta\left(\frac{2\beta}{\alpha}\right)^2 s_{2q}^{\varrho} \\
&< \varepsilon\beta s_{2q}^{\varrho}, \\
&\leqq \varepsilon\mu[S(|\zeta|, k)],
\end{aligned}$$

from (5.3) and (8.4), which proves (i).

Turning to the proof of (ii) we have, from (7.2) and (7.6),

$$\int_{s_0}^{\infty} \frac{D(r,\varepsilon)^2}{r}\, dr \leqq 432 \sum_0^{\infty} \int_{I_n} \frac{\delta(r,\varepsilon)^2}{r}\, dr + 2\int_{s_0}^{\infty} r^{-3}\, dr < \infty.$$

Finally, to prove (iii) it is sufficient to show that $\delta(s_n, \varepsilon)$ tends to zero as n tends to infinity, and this follows from (6.1) and (7.2).

(3) $E(r, \varepsilon)$

The analogue of Lemma 7 for $E(r, \varepsilon)$ is

Lemma 8. *Let* $E(r, \varepsilon)$ *be as given by* (7.7). *Then*

(i) *if* ζ *is a point on* $\Gamma(\varepsilon)$, *then*

$$\mu[S(\zeta, k, \varepsilon)] \leqq E(|\zeta|, \varepsilon)\mu[S(|\zeta|, k)];$$

(ii) $\int^{\infty} \frac{E(r, \varepsilon)}{r} dr < \infty$;

(iii) $E(r, \varepsilon)$ *tends to zero as* r *tends to infinity.*

The proofs of parts (ii) and (iii) follow as do those of parts (ii) and (iii) of Lemma 7.

Turning to the proof of (i), let ζ be a point on $\Gamma(\varepsilon)$ with $|\zeta|$ large so that $D(|\zeta|, \varepsilon) < \varepsilon$ and $E(|\zeta|, \varepsilon) < \varepsilon$, and let q be the integer such that

$$s_{2q} = |\xi_q| \leqq |\zeta| < |\xi_{q+1}| = s_{2q+2}.$$

From Lemma 5 and the fact that $D(|\zeta|, \varepsilon) < \varepsilon$,

$$\begin{aligned}\mu[S(\zeta, k, \varepsilon)] &\leqq \mu[S(\xi_q, k, \tfrac{1}{2}\varepsilon) \cup S(\xi_{q+1}, k, \tfrac{1}{2}\varepsilon)] \\ &\leqq \eta(s_{2q+2}, \varepsilon)\{\beta(ks_{2q+2})^{\varrho} - \alpha(k^{-1}s_{2q+2})^{\varrho}\} \\ &\quad + \eta(s_{2q}, \varepsilon)\{\beta(ks_{2q})^{\varrho} - \alpha(k^{-1}s_{2q})^{\varrho}\} \\ &\leqq \beta s_{2q}^{\varrho}\left(\frac{2\beta}{\alpha}\right)^{7/4}\{\eta(s_{2q+2}, \varepsilon) + \eta(s_{2q}, \varepsilon)\} \\ &= E(|\zeta|, \varepsilon)\beta s_{2q}^{\varrho}\end{aligned} \tag{8.5}$$

from (7.7). (8.4) and (8.5) together prove (i).

9. Concerning $\Psi(r, \varepsilon)$

The third result required before constructing the functions of Theorems 1 and 2 has to do with the slow angular variation of $\Gamma(\varepsilon)$.

Lemma 9. *Let* $\Gamma(\varepsilon)$ *be the curve* (7.5),

$$\Gamma(\varepsilon): z = re^{i\Psi(r, \varepsilon)}.$$

Then

$$|\Psi(R_2, \varepsilon) - \Psi(R_1, \varepsilon)| = o\left|\log \frac{R_2}{R_1}\right|^{1/2}$$

as $\min(R_1, R_2)$ *tends to infinity.*

From Lemma 5 and the definition of Ψ, if ζ is a point on $\Gamma(\varepsilon)$ with

$$s_{2q} = |\xi_q| \leqq |\zeta| < |\xi_{q+1}| = s_{2q+2},$$

then, from (7.4),

$$|\Psi'(|\zeta|,\varepsilon)| \leqq \frac{D(|\zeta|,\varepsilon)}{2(s_{2q+2}-s_{2q})} \leqq \frac{D(|\zeta|,\varepsilon)}{2|\zeta|(1-k^{-1/4})}, \tag{9.1}$$

so

$$\int^{\infty} t\Psi'(t,\varepsilon)^2\,dt \leqq \frac{1}{4}(1-k^{-1/4})^{-2}\int^{\infty}\frac{D(t,\varepsilon)^2}{t}\,dt < \infty.$$

Consequently, for any large numbers R_1 and R_2, with $R_2 \geqq R_1$,

$$\begin{aligned}|\Psi(R_2,\varepsilon)-\Psi(R_1,\varepsilon)| &\leqq \int_{R_1}^{R_2}|\Psi'(t,\varepsilon)|\,dt\\ &\leqq \left\{\int_{R_1}^{R_2} t\Psi'(t,\varepsilon)^2\,dt\right\}^{1/2}\left\{\int_{R_1}^{R_2}\frac{dt}{t}\right\}^{1/2}\\ &= o\left|\log\frac{R_2}{R_1}\right|^{1/2}\end{aligned}$$

as $\min(R_1, R_2)$ tends to infinity.

10. The functions $\Delta(r)$, $\nu(r)$, $\varepsilon(r)$ and $\varphi(r)$

In this section the functions Δ, ν, ε and φ occuring in Theorems 1 and 2 are defined inductively. Let

$$D_1(R,n) = \max\left\{\int_{k^{1/2}R}^{\infty}\frac{D\left(r,\frac{1}{n+1}\right)^2}{r}\,dr,\ \int_{k^{1/2}R}^{\infty}\frac{D\left(r,\frac{1}{n+2}\right)^2}{r}\,dr\right\}, \tag{10.1}$$

$$D_2(R,n) = \max\left\{D\left(R,\frac{1}{n}\right),\ D\left(R,\frac{1}{n+1}\right)\right\}, \tag{10.2}$$

$$E_1(R,n) = \max\left\{\int_{k^{1/2}R}^{\infty}\frac{E\left(r,\frac{1}{n+1}\right)}{r}\,dr,\ \int_{k^{1/2}R}^{\infty}\frac{E\left(r,\frac{1}{n+2}\right)}{r}\,dr\right\}, \tag{10.3}$$

$$E_2(R,n) = \max\left\{E\left(R,\frac{1}{n}\right),\ E\left(R,\frac{1}{n+1}\right)\right\}, \tag{10.4}$$

$$\Psi_1(t_1,t_2,n) = \left|\Psi\left(t_1,\frac{1}{n}\right)-\Psi\left(t_2,\frac{1}{n}\right)\right|. \tag{10.5}$$

Let R_1 be the smallest number for which Lemmas 7 and 8 hold for all $|\zeta| \geqq R_1$, with $\varepsilon=\frac{1}{4}$. Let R_2 be the smallest number greater than or equal to kR_1 which is such that Lemmas 7 and 8 hold for all $|\zeta|\geqq R_2$, with $\varepsilon=\frac{1}{5}$, and such that

$$D_1(R_2,4)\leqq 2^{-4},\quad E_1(R_2,4)\leqq 2^{-4},$$

$$D_2(R,4)\leqq\frac{1}{5}\quad\text{for all}\quad r\geqq R_2,\quad E_2(R,4)\leqq\frac{1}{5}\quad\text{for all}\quad R\geqq R_2,$$

$$\Psi_1(t_1,t_2,4)\leqq\frac{1}{4}\left|\log\frac{t_2}{t_1}\right|^{1/2}\quad\text{for all}\quad t_1,t_2\geqq R_2,$$

$$\Psi_1(t_1,t_2,5)\leqq\frac{1}{4}\left|\log\frac{t_2}{t_1}\right|^{1/2}\quad\text{for all}\quad t_1,t_2\geqq R_2.$$

For $R_2\geqq r>k^{1/2}R_1$, define

$$\Delta(r)=D\left(r,\frac{1}{4}\right),\quad \nu(r)=E\left(r,\frac{1}{4}\right),\quad \varphi(r)=\Psi\left(r,\frac{1}{4}\right),\quad \varepsilon(r)=\frac{1}{4}.$$

For $k^{1/2}R_2\geqq r>R_2$, define

$$\Delta(r)=2\left\{D\left(r,\frac{1}{4}\right)+D\left(r,\frac{1}{5}\right)\right\},\quad \nu(r)=E\left(r,\frac{1}{4}\right)+E\left(r,\frac{1}{5}\right),\quad \varepsilon(r)=1$$

and define $\varphi(r)$ for $k^{1/2}R_2\geqq r>R_2$ by

$$\varphi(r)=\Psi\left(r,\frac{1}{4}\right)+\frac{r-R_2}{R_2(k^{1/2}-1)}\left\{\Psi\left(r,\frac{1}{5}\right)-\Psi\left(r,\frac{1}{4}\right)-2c_1\pi\right\},$$

where c_1 is the integer for which

$$\left|\Psi\left(R_2,\frac{1}{5}\right)-\Psi\left(R_2,\frac{1}{4}\right)-2c_1\pi\right|\leqq\pi.$$

We proceed to define $\Delta(r)$, $\nu(r)$, $\varepsilon(r)$ and $\varphi(r)$ by induction. Assuming that $R_1,\dots,R_n$ and $c_1,\dots,c_{n-1}$ have been defined, let R_{n+1} be the smallest number no less than kR_n such that Lemmas 7 and 8 hold for $|\zeta|\geqq R_{n+1}$, with $\varepsilon=1/(n+4)$, and such that

$$D_1(R_{n+1},n+3)\leqq 2^{-n-3},\quad E_1(R_{n+1},n+3)\leqq 2^{-n-3},$$

$$D_2(R,n+3)\leqq\frac{1}{n+4}\quad\text{for}\quad R\geqq R_{n+1},\quad E_2(R,n+3)\leqq\frac{1}{n+4}\quad\text{for}\quad R\geqq R_{n+1},$$

$$\Psi_1(t_1,t_2,n+3)\leqq\frac{1}{n+3}\left|\log\frac{t_2}{t_1}\right|^{1/2}\quad\text{for}\quad t_1,t_2\geqq R_{n+1},$$

$$\Psi_1(t_1,t_2,n+4)\leqq\frac{1}{n+3}\left|\log\frac{t_2}{t_1}\right|^{1/2}\quad\text{for}\quad t_1,t_2\geqq R_{n+1}.$$

For $R_{n+1} \geqq r > k^{1/2} R_n$, define

$$\Delta(r) = D\left(r, \frac{1}{n+3}\right), \quad \nu(r) = E\left(r, \frac{1}{n+3}\right),$$

$$\varphi(r) = \Psi\left(r, \frac{1}{n+3}\right) - 2c_{n-1}\pi, \quad \varepsilon(r) = \frac{1}{n+3}.$$

For $k^{1/2} R_{n+1} \geqq r > R_{n+1}$, define

$$\Delta(r) = 2\left\{D\left(r, \frac{1}{n+3}\right) + D\left(r, \frac{1}{n+4}\right)\right\},$$

$$\nu(r) = E\left(r, \frac{1}{n+3}\right) + E\left(r, \frac{1}{n+3}\right), \quad \varepsilon(r) = \frac{4}{n+3},$$

and define $\varphi(r)$ for $k^{1/2} R_{n+1} \geqq r > R_{n+1}$ by

$$\varphi(r) = \Psi\left(r, \frac{1}{n+3}\right) - 2c_{n-1}\pi$$
$$+ \frac{r - R_{n+1}}{R_{n+1}(k^{1/2} - 1)}\left\{\Psi\left(r, \frac{1}{n+4}\right) - \Psi\left(r, \frac{1}{n+3}\right) + 2\pi(c_{n-1} - c_n)\right\},$$

where c_n is the integer for which

$$\left|\Psi\left(R_{n+1}, \frac{1}{n+4}\right) - \Psi\left(R_{n+1}, \frac{1}{n+3}\right) + 2\pi(c_{n-1} - c_n)\right| \leqq \pi.$$

This completes the definitions of Δ, ν, ε and φ. The proof of Theorems 1 and 2 will be completed in the following three sections, in which each of the functions is considered in turn.

11. Concerning $\Delta(r)$

It is evident from the definition of Δ that $\Delta(r)$ tends to zero as r tends to infinity. Also,

$$\int_{k^{1/2}R_1}^{\infty} \frac{\Delta(r)^2}{r}\, dr \leqq 4 \sum_1^\infty \int_{k^{1/2}R_n}^{R_{n+1}} \frac{D\left(r, \frac{1}{n+3}\right)^2}{r}\, dr$$
$$+ 4 \sum_1^\infty \int_{R_{n+1}}^{k^{1/2}R_{n+1}} \frac{\left\{D\left(r, \frac{1}{n+3}\right) + D\left(r, \frac{1}{n+4}\right)\right\}^2}{r}\, dr$$

$$\leqq 8\sum_1^\infty \int_{k^{1/2}R_1}^\infty \frac{\left\{D\left(r, \frac{1}{n+3}\right)^2 + D\left(r, \frac{1}{n+4}\right)^2\right\}}{r}\, dr$$

$$\leqq 8\int_{k^{1/2}R_1}^\infty \frac{\left\{D\left(r, \frac{1}{4}\right)^2 + D\left(r, \frac{1}{5}\right)^2\right\}}{r}\, dr + 8\sum_2^\infty 2^{-n-1}$$

$$< \infty,$$

which establishes (2.7).

Let ζ be any point on C, with $R_{n+1} \geqq |\zeta| > k^{1/2}R_n$. Then, from Lemma 7 part (i),

$$\mu[S(\zeta, k, \Delta(|\zeta|))] \leqq \frac{1}{n+3}\mu[S(|\zeta|, k)].$$

Let ζ be any point on C with $k^{1/2}R_{n+1} \geqq |\zeta| > R_{n+1}$. From the definition of $\varphi(r)$, C lies between $\Gamma(1/(n+3))$ and $\Gamma(1/(n+4))$. Also, taking into account Lemma 7 part (i), we have

$$\left|\arg\exp i\left\{\Psi\left(r, \frac{1}{n+3}\right) - \Psi\left(r, \frac{1}{n+4}\right)\right\}\right| \leqq D\left(r, \frac{1}{n+3}\right) + D\left(r, \frac{1}{n+4}\right),$$

for $k^{1/2}R_{n+1} \geqq r > R_{n+1}$. Hence

$$\mu[S(\zeta, k, \Delta(|\zeta|))] \leqq \mu\left[S\left(|\zeta|e^{i\Psi(|\zeta|, 1/(n+3))}, k, D\left(|\zeta|, \frac{1}{n+3}\right)\right)\right]$$

$$+\mu\left[S\left(|\zeta|e^{i\Psi(|\zeta|, 1/(n+4))}, k, D\left(|\zeta|, \frac{1}{n+4}\right)\right)\right]$$

$$\leqq \frac{2}{n+3}\mu[S(|\zeta|, k)].$$

Hence, since n tends to infinity as $|\zeta|$ tends to infinity,

$$\mu[S(\zeta, k, \Delta(|\zeta|))] = o\{\mu[S(|\zeta|, k)]\}$$

as $|\zeta|$ tends to infinity. That part of Theorem 1 concerning $\Delta(r)$ is thus proved.

12. Concerning $\nu(r)$ and $\varepsilon(r)$

From the definition of $\nu(r)$ it is apparent that $\nu(r)$ tends to zero as r tends to infinity. Moreover,

$$\begin{aligned}\int_{k^{1/2}R_1}^{\infty} \frac{\nu(r)}{r}\,dr &\leqq \sum_1^\infty \int_{k^{1/2}R_n}^{k^{1/2}R_{n+1}} \frac{\nu(r)}{r}\,dr \\ &\leqq \sum_1^\infty \int_{k^{1/2}R_n}^{\infty} \left\{E\left(r, \frac{1}{n+3}\right) + E\left(r, \frac{1}{n+4}\right)\right\}\frac{dr}{r} \\ &\leqq \int_{k^{1/2}R_1}^{\infty} \left\{E\left(r, \frac{1}{4}\right) + E\left(r, \frac{1}{5}\right)\right\}\frac{dr}{r} + \sum_2^\infty 2^{-n-1} \\ &< \infty,\end{aligned}$$

which establishes (2.9).

Let ζ be any point on C, with $R_{n+1} \geqq |\zeta| > k^{1/2}R_n$. Then, from Lemma 8 part (i),

$$\mu[S(\zeta, k, \varepsilon(|\zeta|))] = \mu\left[S\left(\zeta, k, \frac{1}{n+3}\right)\right] \leqq \nu(|\zeta|)\mu[S(|\zeta|, k)].$$

Let ζ be a point on C, with $k^{1/2}R_{n+1} \geqq |\zeta| > R_{n+1}$. From the definition of $\varphi(r)$, C lies between $\Gamma(1/(n+3))$ and $\Gamma(1/(n+4))$, and

$$(12.1)\qquad \left|\arg\exp i\left\{\Psi\left(r, \frac{1}{n+3}\right) - \Psi\left(r, \frac{1}{n+4}\right)\right\}\right| \leqq D\left(r, \frac{1}{n+3}\right) + D\left(r, \frac{1}{n+4}\right) < \frac{2}{n+4}.$$

Hence

$$\begin{aligned}\mu[S(\zeta, k, \varepsilon(|\zeta|))] &= \mu\left[S\left(\zeta, k, \frac{4}{n+3}\right)\right] \\ &\leqq \mu\left[S\left(|\zeta|e^{i\Psi(|\zeta|, 1/(n+3))}, k, \frac{1}{n+3}\right)\right] + \\ &\quad + \mu\left[S\left(|\zeta|e^{i\Psi(|\zeta|, 1/(n+4))}, k, \frac{1}{n+4}\right)\right] \\ &\leqq \nu(|\zeta|)\mu[S(|\zeta|, k)].\end{aligned}$$

Since $\varepsilon(r)$ obviously tends to zero as r tends to infinity, Theorem 2 is proved.

13. The function $\varphi(r)$

The proof of Theorem 1 will be complete when (2.6) has been proved.

For $R_{n+1} \geqq r > k^{1/2} R_n$,

(13.1)
$$|\varphi'(r)| = \left|\Psi'\left(r, \frac{1}{n+3}\right)\right| \leqq \frac{D\left(r, \frac{1}{n+3}\right)}{2r(1-k^{-1/4})}.$$

Also, for $k^{1/2} R_{n+1} \geqq r > R_{n+1}$,

(13.2)
$$|\varphi'(r)| \leqq \Psi'\left(r, \frac{1}{n+3}\right) + \frac{1}{R_{n+1}(k^{1/2}-1)} \left|\Psi\left(r, \frac{1}{n+4}\right) - \Psi\left(r, \frac{1}{n+3}\right) + 2\pi(c_{n-1}-c_n)\right| + \frac{r-R_{n+1}}{R_{n+1}(k^{1/2}-1)} \left|\Psi'\left(r, \frac{1}{n+4}\right) - \Psi'\left(r, \frac{1}{n+3}\right)\right|.$$

From the definition of the numbers c_i, and from (12.1) and the definition of $\Delta(r)$,

$$\left|\Psi\left(r, \frac{1}{n+4}\right) - \Psi\left(r, \frac{1}{n+3}\right) + 2\pi(c_{n-1}-c_n)\right| = \left|\arg\exp i\left\{\Psi\left(r, \frac{1}{n+4}\right) - \Psi\left(r, \frac{1}{n+3}\right)\right\}\right| \leqq D\left(r, \frac{1}{n+4}\right) + D\left(r, \frac{1}{n+3}\right) = \frac{1}{2}\Delta(r).$$

Hence, taking account of (9.1), we obtain

(13.3)
$$|\varphi'(r)| \leqq A_1 \frac{\Delta(r)}{r}$$

for some constant A_1 depending only on k. Hence, combining (13.1) and (13.3), we see that there is a constant A_2 depending only on k such that

(13.4)
$$|\varphi'(r)| \leqq A_2 \frac{\Delta(r)}{r}$$

for all $r \geqq k^{1/2} R_1$.

If T_1 and T_2 are two numbers satisfying $T_2 \geqq T_1 \geqq k^{1/2} R_1$,

$$\begin{aligned} |\varphi(T_2) - \varphi(T_1)| &\leqq \int_{T_1}^{T_2} |\varphi'(r)|\, dr \\ &\leqq A_2 \int_{T_1}^{T_2} \frac{\Delta(r)}{r}\, dr \\ &\leqq A_2 \left\{ \int_{T_1}^{T_2} \frac{\Delta(r)^2}{r}\, dr \right\}^{1/2} \left\{ \log \frac{T_2}{T_1} \right\}^{1/2} \\ &= o\left\{ \log \frac{T_2}{T_1} \right\}^{1/2} \end{aligned}$$

as $T_1 = \min(T_1, T_2)$ tends to infinity, since

$$\int^{\infty} \frac{\Delta(r)^2}{r}\, dr < \infty.$$

Theorems 1 and 2 are thus completely proved.

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