Regularity of spherical means

J. Peyrière and P. Sjölin

1. Introduction

Let $\mathbf{R}^n$ denote n-dimensional Euclidean space and let $|x|$ denote the norm of an element $x\in\mathbf{R}^n$. For $\beta\in\mathbf{R}$ and $f\in L^1_{\text{loc}}(\mathbf{R}^n)$ we set

$$F_{\beta,x}(t) = |t|^\beta \int_{S^{n-1}} f(x-ty)\,d\sigma(y), \quad x\in\mathbf{R}^n, \quad t\in\mathbf{R}, \tag{1}$$

where σ denotes the surface measure on $S^{n-1}=\{x\in\mathbf{R}^n;\ |x|=1\}$. It follows from Fubini's theorem that for every $x\in\mathbf{R}^n$, $F_{\beta,x}(t)$ is well-defined for almost all $t\in\mathbf{R}$. We also set $F_x(t)=F_{0,x}(t)$, $t\geqq 0$, and $F_x(t)=0$ for $t<0$.

E. M. Stein [2] has studied the maximal operator M defined by

$$Mf(x) = \sup_{t\geqq 0} |F_x(t)|, \quad x\in\mathbf{R}^n, \quad f\in\mathcal{S}(\mathbf{R}^n),$$

where $\mathcal{S}(\mathbf{R}^n)$ denotes the Schwartz class, and has proved that $\|Mf\|_{L^p(\mathbf{R}^n)}\leqq C_p\|f\|_{L^p(\mathbf{R}^n)}$ if $n\geqq 3$ and $p>n/(n-1)$.

The purpose of this paper is to study the regularity of the function $F_{\beta,x}$ in terms of Besov (=Lipschitz) spaces. We let the Besov spaces $B_p^{\alpha,q}=B_p^{\alpha,q}(\mathbf{R})$ be defined as in P. Brenner, V. Thomée and L. B. Wahlbin [1]. These spaces are known to coincide with the Lipschitz spaces $\Lambda_\alpha^{p,q}$ studied by M. H. Taibleson [3].

If f is a complex-valued function on $\mathbf{R}^n$ we write $\check{f}(x)=\overline{f(-x)}$. In Sections 2—4 we obtain the following results.

Theorem 1. *Assume $n\geqq 2, \alpha>0$ and $-1<\beta<(n-2)/2$. Then there exists a constant C such that*

$$\int_{\mathbf{R}^n} \|F_{\beta,x}\|^2_{B_2^{\alpha,2}(\mathbf{R})}\,dx \leqq C\int_{\mathbf{R}^n} |f*\check{f}(x)|\,|x|^{2(\beta-\alpha)-n+1}(1+|x|^{2\alpha})\,dx$$

for every continuous function f with compact support in $\mathbf{R}^n$.

Corollary 1. *Assume $n\geqq 2$ and $1\leqq p<q\leqq 2$. If $-1<\beta<\inf((n-2)/2, n(1/p-1/2)-1/2)$, $0<\alpha<\beta+1/2-n(1/q-1/2)$ and $f\in L^p(\mathbf{R}^n)\cap L^q(\mathbf{R}^n)$, then $F_{\beta,x}\in B_2^{\alpha,2}(\mathbf{R})$ for almost every $x\in\mathbf{R}^n$.*

Corollary 2. *Assume* $n\geqq 3$, $n/(n-1)<q\leqq 2$ *and* $f\in L^q_{\mathrm{loc}}(\mathbf{R}^n)$. *If* $\alpha<n(1-1/q)-1$, *then for almost every* $x\in\mathbf{R}^n$ *the function* F_x *coincides almost everywhere on* $]0,\infty[$ *with a function which belongs locally to* $\Lambda_\alpha(]0,\infty[)$.

In Section 5 we use a different method to prove the following theorem.

Theorem 2. *If* $n\geqq 3$, $n/(n-1)<p<\infty$, $\alpha<1/p$ *and* $f\in L^p(\mathbf{R}^n)$, *then* $F_x\in B_p^{\alpha,1}(\mathbf{R})$ *for almost every* $x\in\mathbf{R}^n$.

We remark that it is easy to see that Theorem 2 holds also if the function F_x is replaced by $F_{0,x}$.

2. Some lemmas

To prove Theorem 1 we use the following characterization of the Besov space $B_2^{\alpha,2}(\mathbf{R})$. If g is a function in $L^2(\mathbf{R})$ let u denote its Poisson integral. Then g lies in $B_2^{\alpha,2}(\mathbf{R})$ if and only if for some (or every) integer $k>\alpha$ the quantity

$$\|g\|_{L^2(\mathbf{R})}+\left[\int_0^{+\infty} y^{2(k-\alpha)-1}\left(\int_{\mathbf{R}} |(\partial/\partial y)^k u(x,y)|^2\,dx\right)dy\right]^{1/2}$$

is finite. This defines equivalent norms on $B_2^{\alpha,2}(\mathbf{R})$.

Let us consider $u_0(x,y)=\pi^{-1}y/(x^2+y^2)$ the Poisson kernel of the upper half plane. We set $u_k=(\partial/\partial y)^k u_0$.

Since u_{2k} is homogeneous of degree $-(2k+1)$ with respect to (x,y) and is an odd function of y, there exists a constant C_k such that

$$|u_{2k}(x,y)|\leqq C_k y/(x^2+y^2)^{k+1}\quad (x\in\mathbf{R},\ y\in\mathbf{R}^+).$$

Lemma 1. *We have* $u_k(.,y)*u_k(.,y)=u_{2k}(.,2y)$.

Proof. This lemma is an easy consequence of the formula

$$\mathscr{F}\big(u(.,y)\big)(\xi)=e^{-|\xi|y}.$$

Let us denote by σ_n the rotation invariant probability measure on S^n, by s_n the area of S^n and by χ the characteristic function of the set $\{(x,y,z)\in\mathbf{R}^3;\ \big||x|-|y|\big|<|z|<|x|+|y|\}$. When $\chi(x,y,z)$ equals 1 we set

$$\Delta(x,y,z)=\tfrac{1}{4}\left([(x+y)^2-z^2][z^2-(x-y)^2]\right)^{1/2}$$

(it is the area of the triangle with sides of length $|x|$, $|y|$, $|z|$).

Lemma 2. *Let n be a positive integer, r and s two non-zero real numbers. Let us denote by μ_n the image measure of $\sigma_n\times\sigma_n$ by the mapping $(y,z)\mapsto ry+sz$ from $S^n\times S^n$ to $\mathbf{R}^{n+1}$. We have*

$$d\mu_n(x)=\frac{2^{n-2}s_{n-1}}{s_n^2}\frac{[\Delta(|x|,r,s)]^{n-2}}{(|rs|\cdot|x|)^{n-1}}\chi(|x|,r,s)\,dx.$$

Proof. We may suppose $0<r\leqq s$. Let t be a non-negative number. Let us compute $\omega(t)=\mu_n(\{x;\,|x|\leqq t\})$. We have

$$\omega(t)=\int_{S^n}\Big(\int_{S^n\cap\{y;\,|ry+sz|\leqq t\}}d\sigma_n(y)\Big)d\sigma_n(z).$$

The inner integral does not depend on z so we get

$$\omega(t)=\int_{S^n\cap\{y;\,|ry+sz|\leqq t\}}d\sigma_n(y),$$

where z is any point in S^n. Clearly $\omega(t)$ is zero if t is less than $s-r$ and is 1 if t is greater than $s+r$. If t is between $s-r$ and $s+r$ we denote by φ_0 the number such that $0<\varphi_0<\pi$ and $t^2=r^2+s^2-2rs\cos\varphi_0$. We then have

$$\omega(t)=\frac{s_{n-1}}{s_n}\int_0^{\varphi_0}(\sin\varphi)^{n-1}\,d\varphi\quad\text{and}\quad rs\sin\varphi_0=2\Delta(r,s,t)$$

so

$$\omega'(t)=2^{n-2}\frac{s_{n-1}}{s_n}\frac{t[\Delta(r,s,t)]^{n-2}}{(rs)^{n-1}}\chi(r,s,t).$$

The result follows because μ_n is rotation invariant.

Lemma 3. *Let v and w be two real numbers such that $v<0$, $w>-1$, $2(v+w)<-1$. Let us set*

$$\lambda(s)=\begin{cases}\int_1^{+\infty}(t^2-s^2)^v(t^2-1)^w\,dt & \text{when}\quad |s|<1,\\ \int_0^1(s^2-t^2)^v(1-t^2)^w\,dt & \text{when}\quad |s|>1.\end{cases}$$

Then we have, when s tends to 1,

$$\lambda(s)=\begin{cases}O(1) & \text{if}\quad v+w>-1,\\ O(\mathrm{Log}\,(1/|1-s|)) & \text{if}\quad v+w=-1,\\ O(|1-s|^{v+w+1}) & \text{if}\quad v+w<-1.\end{cases}$$

Proof. First let us study the case when s tends to 1^+. If s is less than 2 we have

$$\lambda(s)\leqq C\int_0^1(s-t)^v(1-t)^w\,dt=C\int_0^1(s-1+t)^v t^w\,dt$$
$$=C(s-1)^{v+w+1}\int_0^{1/(s-1)}(1+t)^v t^w\,dt$$

and we conclude easily.

Let us now study the case when s tends to 1^-. We have

$$\lambda(s)\leqq\int_1^2(t^2-s^2)^v(t^2-1)^w\,dt+\int_2^{+\infty}(t^2-1)^{v+w}\,dt$$
$$\leqq C\Big(1+\int_1^2(t-s)^v(t-1)^w\,dt\Big)\leqq C\Big(1+\int_0^1(1-s+t)^v t^w\,dt\Big)$$

and we conclude as above.

3. Proof of Theorem 1

Let us compute first $\int_{\mathbf{R}^n} \|F_{\beta,x}\|^2_{L^2(\mathbf{R}^n)}\,dx$.

We have

$$|F_{\beta,x}(t)|^2 = |t|^{2\beta} \iint_{S^{n-1}\times S^{n-1}} f(x-ty_1)\overline{f(x-ty_2)}\,d\sigma(y_1)\,d\sigma(y_2),$$

thus

$$\int_{\mathbf{R}^n} |F_{\beta,x}(t)|^2\,dx = |t|^{2\beta} \iint_{S^{n-1}\times S^{n-1}} f*\check{f}(-ty_1+ty_2)\,d\sigma(y_1)\,d\sigma(y_2).$$

Using Lemma 2 we get

$$\int_{\mathbf{R}^n} |F_{\beta,x}(t)|^2\,dx = C|t|^{2\beta} \int_{\mathbf{R}^n} f*\check{f}(x)\,\frac{[\Delta(|x|,t,t)]^{n-3}}{(t^2|x|)^{n-2}}\,\chi(|x|,t,t)\,dx,$$

therefore

$$\int_{\mathbf{R}^n} \|F_{\beta,x}\|^2_{L^2(\mathbf{R})}\,dx = C\int_{\mathbf{R}^n} f*\check{f}(x)\Big[\int_{|x|/2}^{+\infty} \frac{t^{2(\beta-n+2)}}{|x|}(4t^2-|x|^2)^{(n-3)/2}\,dt\Big]\,dx$$

$$= C\Big[\int_{1/2}^{+\infty} t^{2(\beta-n+2)}(4t^2-1)^{(n-3)/2}\,dt\Big]\int_{\mathbf{R}^n} f*\check{f}(x)\,|x|^{2\beta-n+1}\,dx$$

(the first integral converges since we have $\beta<(n-2)/2$).

Now let us estimate

$$\int_{\mathbf{R}^n}\Big[\int_0^{+\infty} h^{2(k-\alpha)-1}\Big(\int_{\mathbf{R}} |u_k(.,h)*F_{\beta,x}(t)|^2\,dt\Big)\,dh\Big]\,dx,$$

where k is the first integer greater than α. We have

$$F_{\beta,x}*u_k(.,h)(\tau) = \iint_{\mathbf{R}\times S^{n-1}} |t|^{\beta} f(x-ty)\,u_k(\tau-t,h)\,dt\,d\sigma(y)$$

so

$$|F_{\beta,x}*u_k(.,h)(\tau)|^2 = \iiiint_{\mathbf{R}^2\times S^{n-1}\times S^{n-1}} |t_1t_2|^{\beta} f(x-t_1y_1)\overline{f(x-t_2y_2)}$$
$$\times u_k(\tau-t_1,h)\,u_k(\tau-t_2,h)\,dt_1\,dt_2\,d\sigma(y_1)\,d\sigma(y_2).$$

Using Lemma 1 we get

$$\int_{\mathbf{R}} |F_{\beta,x}*u_k(.,h)(\tau)|^2\,d\tau = \iiiint_{\mathbf{R}^2\times S^{n-1}\times S^{n-1}} |t_1t_2|^{\beta} f(x-t_1y_1)\overline{f(x-t_2y_2)}$$
$$\times u_{2k}(t_1-t_2,2h)\,dt_1\,dt_2\,d\sigma(y_1)\,d\sigma(y_2).$$

If we set $A(\beta,h)=\iint_{\mathbf{R}^n\times\mathbf{R}} |F_{\beta,x}*u_k(.,h)(\tau)|^2\,dx\,d\tau$ and $\varphi(x)=f*\check{f}(x)$ we get

$$A(\beta,h) = \iiiint_{\mathbf{R}^2\times S^{n-1}\times S^{n-1}} |t_1t_2|^{\beta}\varphi(t_2y_2-t_1y_2)\,u_{2k}(t_1-t_2,2h)\,dt_1\,dt_2\,d\sigma(y_1)\,d\sigma(y_2).$$

By Lemma 2 we obtain

$$A(\beta,h)$$
$$= C\iint_{\mathbf{R}^2} |t_1t_2|^{\beta} u_{2k}(t_1-t_2,2h)\Big[\int_{\mathbf{R}^n}\varphi(x)\frac{[\Delta(|x|,t_1,t_2)]^{n-3}}{(|t_1t_2|\cdot|x|)^{n-2}}\,\chi(|x|,t_1,t_2)\,dx\Big]\,dt_1\,dt_2.$$

Using homogeneity properties we get

$$A(\beta, h) = C\int_{\mathbf{R}^n} \varphi(x)\,|x|^{2(\beta-k)+1-n} K(2h/|x|)\,dx,$$

where

$$K(\tau) = \iint_{\mathbf{R}^2} |t_1 t_2|^{\beta-n+2} u_{2k}(t_1-t_2, \tau)\big(\Delta(1, t_1, t_2)\big)^{n-3}\chi(1, t_1, t_2)\,dt_1\,dt_2.$$

We shall show later that we have $|K(\tau)| \leqq C/(1+\tau^{2k+1})$. This being granted we have

$$\int_0^{+\infty} h^{2(k-\alpha)-1} A(\beta, h)\,dh \leqq C\int_{\mathbf{R}^n} |\varphi(x)|\,|x|^{2(\beta-k)-n+1}\left[\int_0^{+\infty} \frac{h^{2(k-\alpha)-1}}{1+\big((2h/|x|)^{2k+1}\big)}\,dh\right] dx$$

$$\leqq C\left[\int_0^{\infty} \frac{\tau^{2(k-\alpha)-1}}{1+\tau^{2k+1}}\,d\tau\right]\left[\int_{\mathbf{R}^n} |\varphi(x)|\,|x|^{2(\beta-\alpha)-n+1}\,dx\right].$$

The first integral converges and collecting both estimates we get

$$\int_{\mathbf{R}^n} \|F_{\beta,x}\|^2_{B_2^{\alpha,2}(\mathbf{R})}\,dx \leqq C\int_{\mathbf{R}^n} |f*\check{f}(x)|\,|x|^{2(\beta-\alpha)-n+1}(1+|x|^{2\alpha})\,dx.$$

We now have to prove the estimate of $K(\tau)$. By change of variables we get

$$K(\tau) = C\iint_{\{(s,t)\in\mathbf{R}^2;\,(t^2-1)(1-s^2)>0\}} |t^2-s^2|^{\beta-n+2} u_{2k}(s, \tau)\,|(t^2-1)(1-s^2)|^{(n-3)/2}\,ds\,dt.$$

Let us set

$$L(s) = \begin{cases} (1-s^2)^{(n-3)/2}\int_1^{+\infty} (t^2-s^2)^{\beta-n+2}(t^2-1)^{(n-3)/2}\,dt & \text{if } |s| < 1, \\ (s^2-1)^{(n-3)/2}\int_0^1 (s^2-t^2)^{\beta-n+2}(1-t^2)^{(n-3)/2}\,dt & \text{if } |s| > 1. \end{cases}$$

Both integrals converge because $\beta < (n-2)/2$ and $n \geqq 2$.

L is a C^∞ function on $]-1, 1[$ and by Lemma 3 it is integrable in a neighbourhood of -1 and of 1 (because $\beta > -1$). In addition, when $|s|$ tends to infinity, we have $L(s) = O(s^{2\beta-n+1})$ so L is integrable.

We have $K(\tau) = C\int_{-\infty}^{+\infty} u_{2k}(s, \tau)L(s)\,ds$, so when s tends to zero, $K(\tau)$ tends, save for a multiplicative factor, to the $2k^{\text{th}}$ derivative of L at the origin.

In addition $|K(\tau)| \leqq C\int_{-\infty}^{+\infty} \tau(\tau^2+s^2)^{-k-1}|L(s)|\,ds$, thus $K(\tau) = O(|\tau|^{-2k-1})$, when $|\tau|$ tends to infinity. And the proof is complete.

4. Proof of the Corollaries

Proof of the first Corollary.

The hypothesis $f \in L^p(\mathbf{R}^n) \cap L^q(\mathbf{R}^n)$ implies $f*\check{f} \in L^r(\mathbf{R}^n) \cap L^s(\mathbf{R}^n)$ where r and s are defined by $1/r = 2/p - 1$, $1/s = 2/q - 1$.

More precisely $\|f*\check{f}\|_{L^r(\mathbf{R}^n)} \leqq \|f\|^2_{L^p(\mathbf{R}^n)}$ and $\|f*\check{f}\|_{L^s(\mathbf{R}^n)} \leqq \|f\|^2_{L^q(\mathbf{R}^n)}$.

Let us denote by r' and s' the conjugate exponents. We have

$$\int_{\mathbf{R}^n} \|F_{\beta,x}\|^2_{B_2^{\alpha,2}(\mathbf{R})}\,dx \leqq C\|f\|_p^2\Big(\int_{|x|>1}|x|^{(2\beta-n+1)r'}\,dx\Big)^{1/r'}$$
$$+C\|f\|_q^2\Big(\int_{|x|<1}|x|^{(2\beta-2\alpha-n+1)s'}\,dx\Big)^{1/s'}.$$

The first integral converges if $\beta<n(1/p-1/2)-1/2$, the second if $\alpha<\beta+1/2-n(1/q-1/2)$. So we get a result if

$$-1<\beta<\inf\big((n-2)/2,\ n(1/p-1/2)-1/2\big)$$

and

$$0<\alpha<\beta+1/2-n(1/q-1/2).$$

Proof of Corollary 2.

Let f be a function in $L^q(\mathbf{R}^n)$, let us multiply f by the characteristic function of a ball and use Corollary 1 with $p=1$: if α lies in $]1/2,(n-1)/2-n(1/q-1/2)[$ one can choose a suitable β. We conclude using the inclusion $B_2^{\alpha,2}(\mathbf{R})\subset \Lambda_{\alpha-1/2}(\mathbf{R})$.

5. Proof of Theorem 2

We shall first prove the following inequality.

Lemma 4. *Let Q denote a cube in $\mathbf{R}^n$ with diameter equal to 1. Then*

$$\int_Q \|F_x\|^p_{B_p^{\alpha,1}}\,dx \leqq C_{p,\alpha}\int_{\mathbf{R}^n}|f(x)|^p\,dx,\quad f\in\mathscr{S}(\mathbf{R}^n), \tag{2}$$

if $n\geqq 3$, $n/(n-1)<p<\infty$ and $\alpha<1/p$.

We need the following notation. Choose $\psi\in C_0^\infty(\mathbf{R}^n)$ such that $\operatorname{supp}(\psi)\subset\{\xi: 1/2<|\xi|<2\}$ and

$$\textstyle\sum_{\nu=-\infty}^{\infty}\psi(2^{-\nu}\xi)=1,\quad \xi\neq 0.$$

Set $\psi_\nu(\xi)=\psi(2^{-\nu}\xi)$, $\nu\in\mathbf{Z}$, and let φ and φ_ν, $\nu=1,2,\ldots$, be defined by $\hat\varphi=\psi$ and $\hat\varphi_\nu=\psi_\nu$. Here the Fourier transform $\hat\varphi$ is defined by

$$\hat\varphi(\xi)=\int_{\mathbf{R}} e^{-i\xi t}\varphi(t)\,dt.$$

It follows that $\varphi_\nu(t)=2^\nu\varphi(2^\nu t)$, $\nu=1,2,\ldots$. We also define φ_0 by setting $\hat\varphi_0=1-\sum_1^\infty\psi_\nu$. Then the norm in the Besov space $B_p^{\alpha,q}$ is given by

$$\|f\|_{B_p^{\alpha,q}}=\big(\textstyle\sum_{\nu=0}^{\infty}2^{\nu\alpha q}\|\varphi_\nu*f\|_p^q\big)^{1/q},\quad 1\leqq p,q\leqq\infty,\quad \alpha>0.$$

Here $\|\ \|_p$ denotes the norm in $L^p(\mathbf{R})$ and we make the usual modification for $q=\infty$.

We observe that it suffices to prove the lemma and the theorem with $B_p^{\alpha,1}$ replaced by $B_p^{\alpha,p}$. This follows from a well-known application of Hölder's inequality:

$$\|F_x\|_{B_p^{\alpha,1}} = \sum_0^\infty 2^{\nu(\alpha-r)} 2^{\nu r} \|\varphi_\nu * F_x\|_p \leqq \big(\sum_0^\infty 2^{\nu(\alpha-r)p'}\big)^{1/p'} \big(\sum_0^\infty 2^{\nu rp} \|\varphi_\nu * F_x\|_p^p\big)^{1/p}$$
$$\leqq C_{p,\alpha} \|F_x\|_{B_p^{r,p}}, \quad \text{where} \quad \alpha < r < 1/p \quad \text{and} \quad 1/p + 1/p' = 1.$$

Proof of Lemma 4. Let p and α satisfy the conditions in the lemma. We shall prove the inequalities

$$\int_Q \|\varphi_0 * F_x\|_p^p \, dx \leqq C_p \int_{\mathbf{R}^n} |f(x)|^p \, dx \tag{3}$$

and

$$\int_{\mathbf{R}^n} \big(\sum_{\nu=1}^\infty 2^{\nu\alpha p} \|\varphi_\nu * F_x\|_p^p\big) \, dx \leqq C_{p,\alpha} \int_{\mathbf{R}^n} |f(x)|^p \, dx \tag{4}$$

for $f \in \mathcal{S}(\mathbf{R}^n)$. It is clear that the lemma is a consequence of (3) and (4). We have

$$\varphi_\nu * F_x(u) = \int_{\mathbf{R}} \varphi_\nu(u-t) F_x(t) \, dt = \int_{S^{n-1}} \Big(\int_0^\infty \varphi_\nu(u-t) f(x-ty') \, dt\Big) d\sigma(y') \tag{5}$$
$$= \int_{\mathbf{R}^n} \varphi_\nu(u-|y|) f(x-y) |y|^{-n+1} \, dy = \varphi_{\nu,u} * f(x), \quad x \in \mathbf{R}^n, \quad u \in \mathbf{R}, \quad \nu = 0, 1, 2, \ldots,$$

where $\varphi_{\nu,u}(y) = \varphi_\nu(u-|y|)|y|^{-n+1}$.

We first prove (3). Since

$\|\varphi_0 * F_x\|_p \leqq \|\varphi_0\|_1 \|F_x\|_p$ it is sufficient to prove that

$$\int_Q \|F_x\|_p^p \, dx \leqq C_p \int_{\mathbf{R}^n} |f(x)|^p \, dx. \tag{6}$$

Using the Minkowski inequality we obtain

$$\Big(\int_Q \|F_x\|_p^p dx\Big)^{1/p} = \Big(\iint_{Q \times \mathbf{R}^+} \Big|\int_{S^{n-1}} f(x-ty') \, d\sigma(y')\Big|^p dx \, dt\Big)^{1/p}$$
$$\leqq \int_{S^{n-1}} \Big(\iint_{Q \times \mathbf{R}^+} |f(x-ty')|^p \, dx \, dt\Big)^{1/p} d\sigma(y').$$

Let δ denote the diameter of Q. We then have

$$\iint_{Q \times \mathbf{R}^+} |f(x-ty')|^p \, dx \, dt = \sum_{k=0}^\infty \int_{3k\delta}^{3(k+1)\delta} \Big(\int_Q |f(x-ty')|^p \, dx\Big) dt$$
$$= \int_0^{3\delta} \Big(\sum_{k=0}^\infty \int_Q \big|f(x-(t+3k\delta)y')\big|^p dx\Big) dt \leqq 3\delta \int_{\mathbf{R}^n} |f(x)|^p \, dx$$

for every $y' \in S^{n-1}$. (6) follows from this estimate and hence (3) is proved. We now prove (4) and first observe that

$$\|\varphi_{\nu,u}\|_{L^1(\mathbf{R}^n)} = \int_{\mathbf{R}^n} \big|\varphi_\nu(u-|y|)\big| \, |y|^{-n+1} dy = C \int_{\mathbf{R}} |\varphi_\nu(t)| \, dt = C, \quad \nu = 1, 2, \ldots \tag{7}$$

For the Fourier transform $\hat{\varphi}_{\nu,u}$ of $\varphi_{\nu,u}$ we then obtain the estimate

$$\|\hat{\varphi}_{\nu,u}\|_{L^\infty(\mathbf{R}^n)} \leqq C, \quad \nu = 1, 2, \ldots; \quad u \in \mathbf{R}. \tag{8}$$

We shall also prove that

$$\|\hat{\varphi}_{\nu,u}\|_{L^\infty(\mathbf{R}^n)} \leqq C(2^\nu|u|)^{-(n-1)/2}, \quad 2^\nu|u| \geqq 1. \tag{9}$$

For $u<0$ this follows from the inequality

$$\|\hat{\varphi}_{\nu,u}\|_{L^\infty(\mathbf{R}^n)} \leqq \|\varphi_{\nu,u}\|_{L^1(\mathbf{R}^n)} = C\int_0^\infty 2^\nu|\varphi(2^\nu(u-t))|\,dt$$
$$= C\int_{-\infty}^{2^\nu u}|\varphi(t)|\,dt \leqq C\int_{2^\nu|u|}^\infty (1+t)^{-N}\,dt \leqq C(2^\nu|u|)^{-N'},$$

where N and N' are large positive numbers. For $u>0$ we set $a=2^\nu u$ and assume $a\geqq 1$.

Performing a change of variable $x=uy$ we have

$$\hat{\varphi}_{\nu,u}(\xi) = \int_{\mathbf{R}^n} e^{-i\xi\cdot x}\varphi_\nu(u-|x|)\,|x|^{-n+1}\,dx = \int_{\mathbf{R}^n} e^{-iu\xi\cdot y}a\varphi(a(1-|y|))\,|y|^{-n+1}\,dy.$$

Hence

$$\|\hat{\varphi}_{\nu,u}\|_{L^\infty(\mathbf{R}^n)} = \|J\|_{L^\infty(\mathbf{R}^n)}, \quad \text{where} \quad J(\xi) = \int_{\mathbf{R}^n} e^{-i\xi\cdot y}a\varphi(a(1-|y|))\,|y|^{-n+1}\,dy.$$

Assuming $|\xi|\geqq a/2$ and invoking the well-known estimate

$$\hat{\sigma}(\xi) = \int_{S^{n-1}} e^{-i\xi\cdot y}\,d\sigma(y) = O(|\xi|^{-(n-1)/2}), \quad |\xi| \to \infty,$$

we obtain

$$|J(\xi)| = \left|\int_0^\infty a\varphi(a(1-t))\hat{\sigma}(t\xi)\,dt\right|$$
$$\leqq \int_{1/2}^{3/2} a|\varphi(a(1-t))|\,|\hat{\sigma}(t\xi)|\,dt + C\int_{|v|\geqq 1/2} a|\varphi(av)|\,dv$$
$$\leqq C\int_{1/2}^{3/2} a|\varphi(a(1-t))|(t|\xi|)^{-(n-1)/2}dt + \int_{|v|\geqq a/2}|\varphi(v)|\,dv \leqq Ca^{-(n-1)/2}.$$

For $|\xi|<a/2$ we use the fact that $\hat{\varphi}(t)$ vanishes for $|t|\leqq 1/2$ and get

$$|J(\xi)| \leqq \left|\int_{S^{n-1}}\left(\int_{-\infty}^\infty e^{-it\xi\cdot y'}a\varphi(a(1-t))\,dt\right)d\sigma(y')\right| + C\int_{-\infty}^0 a|\varphi(a(1-t))|\,dt$$
$$\leqq \int_{S^{n-1}}|\hat{\varphi}(\xi\cdot y'/a)|\,d\sigma(y') + C\int_1^\infty a|\varphi(av)|\,dv \leqq C\int_a^\infty |\varphi(t)|\,dt \leqq Ca^{-N},$$

where N is a large positive number. Thus (9) is proved.

We let $\|\;\|_{M_p}$ denote the norm in the space $M_p(\mathbf{R}^n)$ of Fourier multipliers on $L^p(\mathbf{R}^n)$, $1\leqq p\leqq\infty$. It follows from (7) that

$$\|\hat{\varphi}_{\nu,u}\|_{M_\infty} \leqq C,$$

and from (8) and (9) we conclude that

$$\|\hat{\varphi}_{\nu,u}\|_{M_2} \leqq C(1+2^\nu|u|)^{-(n-1)/2}.$$

Interpolation between $p=2$ and $p=\infty$ yields

$$\|\hat{\varphi}_{\nu,u}\|_{M_p} \leqq C(1+2^\nu|u|)^{-(n-1)/p}, \quad 2\leqq p\leqq\infty. \tag{10}$$

By duality we also obtain

$$\|\hat{\varphi}_{\nu,u}\|_{M_p} \leqq C(1+2^\nu|u|)^{-(n-1)(p-1)/p}, \quad 1 \leqq p \leqq 2. \tag{11}$$

We now use (10) and (11) to prove (4). Denoting the left hand side of (4) by B we have

$$B = \sum_{\nu=1}^{\infty} 2^{\nu\alpha p} \int_{\mathbf{R}} \Big(\int_{\mathbf{R}^n} |\varphi_{\nu,u} * f(x)|^p \, dx \Big) du$$
$$\leqq \sum_{\nu=1}^{\infty} 2^{\nu\alpha p} \Big(\int_{\mathbf{R}} \|\hat{\varphi}_{\nu,u}\|_{M_p}^p \, du \Big) \Big(\int_{\mathbf{R}^n} |f(x)|^p \, dx \Big).$$

We denote the first integral on the above right hand side by $I_{p,\nu}$. We shall prove that

$$I_{p,\nu} \leqq C_p 2^{-\nu}, \tag{12}$$

from which (4) follows, since $\alpha < 1/p$. For $p \geqq 2$, (10) yields

$$I_{p,\nu} \leqq C \int_0^\infty (1+2^\nu u)^{-n+1} \, du = C 2^{-\nu} \int_0^\infty (1+u)^{-n+1} \, du = C 2^{-\nu},$$

where we used the assumption $n \geqq 3$. For $n/(n-1) < p < 2$ we have $(n-1)(p-1) > 1$ and from (11) it follows that

$$I_{p,\nu} \leqq C \int_0^\infty (1+2^\nu u)^{-(n-1)(p-1)} \, du = C 2^{-\nu} \int_0^\infty (1+u)^{-(n-1)(p-1)} \, du = C_p 2^{-\nu}.$$

We conclude that (4) holds and the proof of the lemma is complete.

Proof of Theorem 2. It is sufficient to prove that if Q is a cube with diameter 1 and $f \in L^p(\mathbf{R}^n)$ then

$$\int_Q \|F_x\|_{B_p^{\alpha,p}}^p \, dx \leqq C_{p,\alpha} \int_{\mathbf{R}^n} |f(x)|^p \, dx. \tag{13}$$

This can be proved by use of the fact that (5) holds for almost every $x \in \mathbf{R}^n$ if $f \in L^p(\mathbf{R}^n)$, but one can also argue as follows. We may assume that f is non-negative and let $(f_k)_1^\infty$ denote a non-decreasing sequence of step functions tending to f almost everywhere. It follows from the proof of Lemma 4 that (13) holds with f replaced by f_k and F_x by the corresponding function $F_{x,k}$. Fatou's lemma yields

$$\int_\varepsilon^N \varliminf_{k\to\infty} \big(F_x(t) - F_{x,k}(t)\big) \, dt \leqq \varliminf_{k\to\infty} \int_\varepsilon^N \big(F_x(t) - F_{x,k}(t)\big) \, dt$$
$$\leqq \varepsilon^{-n+1} \varliminf_{k\to\infty} \int_\varepsilon^N \big(F_x(t) - F_{x,k}(t)\big) t^{n-1} \, dt$$
$$= \varepsilon^{-n+1} \varliminf_{k\to\infty} \int_{\varepsilon<|y|<N} \big(f(x-y) - f_k(x-y)\big) \, dy = 0$$

for $0 < \varepsilon < N$. We conclude that for every $x \in \mathbf{R}^n$, $F_x(t) = \lim_{k\to\infty} F_{x,k}(t)$ for almost every t. We have

$$\int_Q \|F_{x,k}\|_{B_p^{\alpha,p}}^p \, dx \leqq C_{p,\alpha} \int_{\mathbf{R}^n} |f_k(x)|^p \, dx \tag{14}$$

and hence also

$$\int_Q \|F_{x,k}\|_p^p \, dx \leqq C_p \int_{\mathbf{R}^n} |f_k(x)|^p \, dx. \tag{15}$$

Letting k tend to infinity in (15) we obtain

$$\int_Q \|F_x\|_p^p \, dx \leqq C_p \int_{\mathbf{R}^n} |f(x)|^p \, dx$$

and it follows that $F_x \in L^p(\mathbf{R})$ and $\lim_{k\to\infty} \|F_x - F_{x,k}\|_p = 0$ for almost every x. As a consequence we also have $\lim_{k\to\infty} \|\varphi_\nu * F_{x,k}\|_p = \|\varphi_\nu * F_x\|_p$, $\nu = 0, 1, 2, \ldots,$ for almost every x. An application of Lebesgue's theorem on dominated convergence yields

$$\lim_{k\to\infty} \int_Q \|\varphi_\nu * F_{x,k}\|_p^p \, dx = \int_Q \|\varphi_\nu * F_x\|_p^p \, dx$$

and letting k tend to infinity in (14) we obtain (13). The proof of the theorem is complete.

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J. Peyrière
Université de Paris VII
U. E. R. de Mathématiques
2, Place Jussieu
F—75 221 Paris Cedex 05
France

P. Sjölin
University of Stockholm
Dept. of Mathematics
Box 6701
S—113 85 Stockholm
Sweden