Polynomial approximation in Bers spaces of non-Carathéodory domains

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1. Introduction

The reasonings in our recent work [5] conceals a considerably stronger result than that which we had stated as our main theorem. Specifically, the requirement on the domain D to be a **Carathéodory** domain can be replaced by a weaker assumption, namely that D possesses the so called **Farrell—Markuševič property** (to be defined later). We are grateful to J. Brennan for drawing our attention to this possibility.

We retain the notation of [5]. However, here D stands for an arbitrary bounded simply connected domain, not necessarily a Carathéodory domain. If $a(z)$ is a continuous positive function in D, we denote by $H^p(a:D)$ the class of all analytic functions $f(z)$ in D for which

$$\|f\|_a^p = \iint_D |f(z)|^p a(z)\,dx\,dy < \infty, \quad 0 < p < \infty.$$

Throughout this paper φ will denote a conformal map of D onto the open unit disc U and $\psi = \varphi^{-1}$ will be the inverse mapping. We denote by $\delta_D(z)$ the distance from z to ∂D and $\lambda_D(z)$ stands for the Poincaré metric of D. Here $\lambda_U(w) = (1-|w|^2)^{-1}$ and $\lambda_D(z) = \lambda_U(\varphi(z))\,|\varphi'(z)|$. The fact that $\lambda_D(z)$ is decreasing with D and Koebe's 1/4 theorem imply that

$$1/4 \leqq \lambda_D(z)\delta_D(z) \leqq 1, \quad z \in D. \tag{1.1}$$

We shall confine our attention to those weights $a(z)$ which behave like $\delta_\alpha^D(z)$, $a \in \mathbf{R}$. In view of (1.1) and the conformal invariance of $\lambda_D(z)$, it is more convenient to replace $a(z)$ by $\lambda_D^{-\alpha}(z)$. Specifically, we let

$$t_D = \operatorname{Sup}\{q \in \mathbf{R}\colon \mu_q(D) = \infty\},\ \mu_q(D) = \iint_D \lambda_D^{2-q}(z)\,dx\,dy.$$

Then $1 \leqq t_D \leqq 2$; and, moreover, $t_D = 1$ if ∂D is rectifiable. For these and further properties of t_D see [5].

We define the interval

$$I(t_D) = \begin{cases} [t_D, \infty), & \mu_{t_D}(D) < \infty \\ (t_D, \infty), & \mu_{t_D}(D) = \infty, \end{cases}$$

and note that $\{q \in R : \mu_q(D) < \infty\} = I(t_D)$. Of course, $I(1) = (1, \infty)$ and $I(2) = [2, \infty)$.

For $q \in I(t_D)$ and $0 < p < \infty$ we define $B_q^p(D)$ as the space $H^p(\lambda_D^{2-q} : D)$. $A_{q/p}^p(D) = B_q^p(D)$ is called the **Bers space**; it is a Frechét space of analytic functions, $f(z)$ D, "normed" by

$$\|f\|_{q,p} = \left\{\iint_D |f(z)|^p \lambda_D^{2-q}(z)\, dx\, dy\right\}^{1/p}.$$

Since D is bounded, the assumption $q \in I(t_D)$ implies that the polynomials belong to $B_q^p(D)$ for all $0 < p < \infty$.

The question of polynomial density in $B_q^p(D)$ has been considered by various authors (see [5] and [7] for more details). In 1934 Farrell and Markuševič proved independently that the polynomials are dense in $B_2^p(D) = H^p(1 : D)$ whenever D is a Carathéodory domain (see for example [7]). Recall that, a domain D in the complex plane is called a **Carathéodory domain** if it is simply connected, bounded, and its boundary coincides with the boundary of the infinite component of the complement of $\bar{D}$. Recently [5], we showed that for a Carathédory domain D, the polynomials are dense in $B_q^p(D)$ for $q \in I(t_D)$ and all $0 < p < \infty$. We proved this by perturbing the Farrell—Markuševič theorem to $q \geqq 2$ and to $q < 2$, $q \in I(t_D)$, by using a weak invertibility argument. The argument of the above proof, in effect conceals the possibility of further sharpening the results of [5]. In fact, the requirement on D to be a Carathédory domain can be replaced by a weaker assumption, namely that D has the Farrell—Markuševič property. We now make this notion precise.

Definition. *Let $p \in (0, \infty)$ be fixed. D is said to have the* p-**Farrell—Markuševič property** *or $D \in FM(p)$ if the polynomials are dense in $B_2^p(D)$.*

Clearly, $D \in FM(p)$ for all $p \in (0, \infty)$ whenever D is a Carathéodory domain. However, not only the Carathéodory domains have this property as the examples of [7, pp. 116, 158] and [1] show. We show (Proposition 2) that if $D \in FM(p_0)$ for some fixed $p_0 \in (0, \infty)$ then $D \in FM(p)$ for all $p \in (0, p_0]$. Using this and some facts similar to those exhibited in our previous work [5] we arrive at our main results (Propositions 3 and 4). The above mentioned three propositions when orchestrated yield the principal theorem of this paper (Theorem 1); namely, if $D \in FM(p)$ for all $p \geqq p_0$, where p_0 is some fixed number in $(0, \infty)$, then the polynomials are dense in $B_2^p(D)$ for $q \in I(t_D)$ and all $p \in (0, \infty)$. This, of course, extends our earlier work [5]. Proposition 1 of this paper is rather surprising and it is due to Brennan [3].

2. Auxiliary Facts

Lemma 1. *Let D be a bounded simply connected domain. Then*

$$\iint_D |\varphi'(z)|^p \, dx \, dy < \infty$$

whenever $0 \leqq p < 3$.

Proof. We may obviously assume that $p > 2$. Since ψ is a bounded schlicht function, it follows that $|\psi'(w)| \geqq M(1-|w|^2)$ for all $w \in U$ and some positive constant M. Therefore,

$$\iint_D |\varphi'(z)|^p \, dx \, dy = \iint_U |\psi'(w)|^{2-p} \, du \, dv \leqq M^{2-p} \iint_U (1-|w|^2)^{2-p} \, du \, dv.$$

The last integral is finite if $2-p > -1$ or if $p < 3$.

The assertions of this lemma are more than sufficient for our purposes. However, it is interesting to note that Brennan [3] has recently obtained a further extension of this lemma in the form:

Proposition 1. (Brennan). *Let D be a bounded simply connected domain. Then*

$$\iint_D |\varphi'|^p \, dx \, dy < \infty$$

whenever $0 \leqq p < 3 + \tau$. *Here* τ *is some positive constant which does not depend on* D.

The proof of this proposition is based on certain estimates for harmonic measures and on the following lemma which is fairly classical. The proof of this lemma appeared in Hedberg [6]. Because the proof in [6] is quite difficult we here provide a simpler proof which is similar to that of Lemma 1.

Lemma 2. *Let D be a bounded simply connected domain. Then there exists a positive constant K such that*

$$1 - |\varphi(z)|^2 \leqq K \sqrt{\delta_D(z)}$$

for every z in D.

Proof. As in Lemma 1, $|\psi'(w)| \geqq M(1-|w|^2)$ for all $w \in U$. Therefore, using (1.1), we have

$$\begin{aligned} 1 - |\varphi(z)|^2 &= \lambda_D^{-1}(z) |\varphi'(z)| = \lambda_D^{-1}(z) |\psi'(w)|^{-1} \\ &\leqq 4\delta_D(z) |\psi'(w)|^{-1} \leqq 4M^{-1} \delta_D(z) (1-|w|^2)^{-1}. \end{aligned}$$

Thus $1 - |\varphi(z)|^2 \leqq 2M^{-1/2} \delta_D^{1/2}(z)$ which concludes the proof.

Lemma 3. *The polynomials are dense in $B_q^p(U)$ for $q > 1$ and all $p \in (0, \infty)$.*

Proof. This is trivial, for the polynomials are dense $H^p(a\colon U)$ with $a(z) = a(|z|)$ (cf. [7]). In our case $a(z) = (1-|z|^2)^{q-2}$, $q > 1$.

Our main results are in part based on the following elementary fact (cf. [2, 5, 6] and [7, p. 136]) which, contrary to its parallel in [5], does not make use of the Carathéodory property.

Lemma 4. *Let D be a bounded simply connected domain and let $q\in I(t_D)$ and $p\in(0,\infty)$ be fixed. Then the polynomials are dense in $B_q^p(D)$ if and only if $\varphi^n(\varphi')^{q/p}$ is in the $B_q^p(D)$-closure of the polynomials for each $n=0,1,\dots$.*

Proof. The necessity is obvious since $\varphi^n(\varphi')^{q/p}$ is in $B_q^p(D)$ for $n=0,1,\dots$ and for all $q>1$. Indeed,

$$\|\varphi^n(\varphi')^{q/p}\|_{q,p}^p=\iint_D|\varphi|^{np}|\varphi'|^q\lambda^{2-q}\,dx\,dy$$
$$\leqq\iint_D|\varphi'|^q\lambda_D^{2-q}\,dx\,dy=\iint_U(1-|w|^2)^{q-2}\,du\,dv=\frac{\pi}{q-1}.$$

For the sufficiency let $f\in B_q^p(D)$ and $\varepsilon>0$. Then $Tf=f_0\psi(\psi')^{p/q}\in B_q^p(U)$. According to Lemma 3 there is a polynomial $Q(w)$ with

$$\|Tf-Q\|_{q,p,U}^p<\varepsilon/2.$$

By assumption, since Q is a polynomial, there is a polynomial $P(z)$ such that

$$\|Q(\varphi)(\varphi')^{q/p}-P\|_{q,p}^p<\varepsilon/2. \tag{2.1}$$

But

$$\|Tf-Q\|_{q,p,U}^p=\iint_U|f(\psi)(\psi')^{q/p}-Q|^p(1-|w|^2)^{q-2}\,du\,dv$$
$$=\iint_D|f(\varphi')^{-q/p}-Q(\varphi)|^p|\varphi'|^q\lambda_D^{2-q}\,dx\,dy$$
$$=\iint_D|f-Q(\varphi)(\varphi')^{q/p}|^p\lambda_D^{2-q}\,dx\,dy=\|f-Q(\varphi)(\varphi')^{q/p}\|_{q,p}^p.$$

Thus

$$\|f-Q(\varphi)(\varphi')^{q/p}\|_{q,p}^p<\varepsilon/2. \tag{2.2}$$

Hence, assuming $0<p\leqq 1$ (the case $1<p<\infty$ is of course similar), we have, by (2.1) and (2.2), $\|f-P\|_{q,p}^p<\varepsilon$. This concludes the proof of the lemma.

3. Main Results

We note first the elementary fact that $B_2^{p_0}(D)\subset B_2^p(D)$ for $0<p\leqq p_0<\infty$ and

$$\|f\|_{2,p}\leqq A(p,p_0)\|f\|_{2,p_0},\quad A(p,p_0)\equiv\mu_2(D)^{\frac{1}{p}-\frac{1}{p_0}}. \tag{3.1}$$

Now, Lemmas 3, 4 and (3.1) lead to the following interesting proposition:

Proposition 2. *If $D\in FM(p_0)$ for some fixed $p_0\in(0,\infty)$ then $D\in FM(p)$ for all $p\in(0,p_0]$.*

Proof. It is sufficient to show that $D\in FM(p)$ for $4/5p_0\leqq p\leqq p_0$. In this case $\varphi^n(\varphi')^{2/p}\in B_2^{p_0}(D)$ for all $n=0, 1, —$. Indeed,

$$\|\varphi^n(\varphi')^{2/p}\|_{2,p_0}^{p_0} = \iint_D |\varphi|^{np}|\varphi'|^{2\frac{p_0}{p}}\,dx\,dy \leqq \iint_D |\varphi'|^{2\frac{p_0}{p}}\,dx\,dy$$

where, according to Lemma 1, the last integral is finite because $0<2\frac{p_0}{p}\leqq 5/2<3$. Since $D\in FM(p_0)$, given $\varepsilon>0$ there is a polynomial P with

$$\|\varphi^n(\varphi')^{2/p}-P\|_{2,p_0} < \varepsilon/A(p, p_0), \quad n = 0, 1, \ldots.$$

Consequently, using (3.1),

$$\|\varphi^n(\varphi')^{2/p}-P\|_{2,p} < \varepsilon, \quad n = 0, 1, \ldots.$$

Therefore, according to Lemma 4, $D\in FM(p)$ for $4/5p_0\leqq p\leqq p_0$.

Corollary. *If, for some $p_0\in(0, \infty)$, $D\in FM(p)$ for all $p\geqq p_0$ then $D\in FM(p)$ for all $p\in(0, \infty)$.*

We are now in a position to use arguments along the same lines exhibited in [5]. Exactly as in Proposition 2 of [5] we can prove, using the present Lemma 4 and Proposition 2, the following more general proposition:

Proposition 3. *Let $D\in FM(p_0)$, $0<p_0<\infty$. Then the polynomials are dense in $B_q^p(D)$ for $q\geqq 2$ and all $p\in(0,p_0]$.*

The case $1<q<2$ is of course more complicated. However, a careful examination of the proof of Theorem 1 of [5] for the case $1\leqq t_D<2$ coupled with the present Lemma 4 and Proposition 2 yields the following sharper result:

Proposition 4. *Let $D\in FM(p_0+\varepsilon)$ for any $\varepsilon>0$, and assume that $1\leqq t_D<2$. Then the polynomials are dense in $B_q^p(D)$ for $q\in I(t_D)$ and all $p\in(0,p_0]$.*

The combination of the last three propositions leads to our principal result:

Theorem 1. *Let $D\in FM(p)$ for all $p\geqq p_0$, where p_0 is some fixed number in $(0, \infty)$. Then the polynomials are dense in $B_q^p(D)$ for $q\in I(t_D)$ and all $p\in(0, \infty)$.*

This theorem is applicable to many non-Carathéodory domains such as those described in [7, p. 116] and [1, p. 182]. A classical instance of these domains is the "**crescent domain**" i.e., a domain which is topologically equivalent to the domain bounded by two internally tangent circles (see also [1] for the extension of this definition). For example, if D is a crescent domain which is sufficiently thin at the multiple boundary point then $D\in FM(p)$ for all $p\geqq 1$ and hence for all $p\in(0, \infty)$. Thus the polynomials are dense in $B_q^p(D)$, for such domains D. for $q\in I(t_D)$ and all $p\in(0, \infty)$. Especially, if ∂D is rectifiable then $t_D=1$ (or $\lambda_D^{2-q}\in L^1(D)$ for all $q>1$), and we recover a theorem of Metzger (cf. [4, 5, 8]) for these domains.

A more bizarre situation occurs when ∂D consists almost entirely of cuts. There are examples of such domains D for which $D\in FM(p)$ for all $p\geqq 1$ (cf. [2, p. 138]). In this case $t_D=2$, and hence the polynomials are dense in $B_q^p(D)$ for $q\geqq 2$ and all $p\in(0, \infty)$.

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