Singular integral operators on nilpotent Lie groups

Roe Goodman*

0. Introduction

In this paper, we extend in two directions the theorem of Knapp and Stein [12] concerning the L_2-boundedness of principal-value singular convolution operators on nilpotent Lie groups: we consider an arbitrary nilpotent Lie group (which may not have any dilating automorphisms), and we replace the regular representation by an arbitrary unitary representation. For both of these aspects it is the singular behavior of the kernel at infinity which requires special attention.

We use our results [10] on comparison of nilpotent group structures (cf. [11], Ch. I). Some technical refinements in our treatment are the weak smoothness condition required of the kernel (a condition of "Dini type" on the modulus of continuity) and the explicit estimate for the norm of the operator defined by a singular kernel.

In Section 1, we briefly recall the basic facts about dilations, homogeneous functions, and nilpotent Lie group structures on $\mathbf{R}^n$, and show the equivalence of various "mean-value zero" conditions. In Section 2 we introduce a class of smooth singular kernels and state the main theorem. In Section 3 we prove the theorem, and in Section 4 we extend the theorem to certain non-unitary representations, obtaining a generalization of the "ergodic Hilbert transform" of Cotlar [5] and Calderón [2].

Other extensions and applications of the original Knapp—Stein boundedness theorem can be found, e.g. in [1], [4], [8], [9], [11], [13], [15].

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1. Dilations and nilpotent group structures on $\mathbf{R}^n$

Consider a one-parameter group $\{\delta_r\}_{r>0}$ of **dilations** on $\mathbf{R}^n$, acting by

$$\delta_r(x_1, \dots, x_n) = (r^{\lambda_1} x_1, \dots, r^{\lambda_n} x_n),$$

where $\lambda_i \geqq 1$. A function f on $\mathbf{R}^n \sim \{0\}$ will be called **homogeneous of degree** μ (relative to the dilations) if $f \circ \delta = r^\mu f$. Of particular interest is the case $\mu = -\Sigma \lambda_i$, since the measure $f(x)dx$ is then dilation-invariant (where dx denotes Lebesgue measure on $\mathbf{R}^n$ and f is also assumed to be locally integrable on $\mathbf{R}^n \sim \{0\}$). We shall call $Q = \Sigma \lambda_i$ the **homogeneous dimension** of $\mathbf{R}^n$, relative to the given group of dilations.

A **homogeneous norm** on $\mathbf{R}^n$ (relative to the dilations) is a continuous function $x \to |x|$, homogeneous of degree one, with $|x| \geqq 0$ and $x = 0$ only when $x = 0$. It is **symmetric** if $|x| = |-x|$, and **smooth** if it is C^1 on $\mathbf{R}^n \sim \{0\}$. For example,

$$|x| = \{\sum |x_i|^{\frac{p}{\lambda_i}}\}^{\frac{1}{p}} \tag{1.1}$$

is a symmetric, homogeneous norm for any $p > 0$, and is smooth if $p > \max\{\lambda_i\}$.

Relative to any choice of homogeneous norm, one has the following integration formula [12]:

$$\int_{\mathbf{R}^n} f(x) g(|x|)\, dx = M(f) \int_0^\infty g(r) r^{-1}\, dr. \tag{1.2}$$

Here we assume, e.g., that f is homogeneous of degree $-Q$ and continuous on $\mathbf{R}^n \sim \{0\}$, while $g \in L^1(\mathbf{R}^+, r^{-1}dr)$. (The existence of a constant $M(f)$ such that (1.2) holds is immediate from the uniqueness of the Haar measure $r^{-1}dr$ on the multiplicative group $\mathbf{R}^+$, since the left side of (1.2) is translation-invariant as a functional of g.)

Let $\mathscr{H}_{-Q}$ denote the space of continuous functions on $\mathbf{R}^n \sim \{0\}$ which are homogeneous of degree $-Q$. The linear functional $f \to M(f)$ on $\mathscr{H}_{-Q}$ is called the **mean-value**. It is independent of the choice of homogeneous norm appearing in (1.2). Indeed, if $|x|_0$ and $|x|_1$ are two homogeneous norms, then so is $|x|_t = |x|_0^{1-t} |x|_1^t$ for $0 \leqq t \leqq 1$. Denote by $M_t(f)$ the constant appearing in (1.2) for $|x| = |x|_t$. Take $g \in C_c^\infty(\mathbf{R}^+)$ with $\int_0^\infty g(r) r^{-1} dr = 1$. Then from (1.2) one sees that $M_t(f)$ is a smooth function of t, and

$$\frac{\partial}{\partial t} M_t(f) = \int_{\mathbf{R}^n} f(x) g'(|x|_t) |x|_t \log\left(\frac{|x|_1}{|x|_0}\right) dx.$$

But $h(x)=f(x)\log(|x|_1/|x|_0)$ is again homogeneous of degree $-Q$, so that

$$\frac{\partial}{\partial t}M_t(f)=M_t(h)\int_0^\infty g'(r)\,dr=0.$$

Thus $M_0(f)=M_1(f)$, as claimed.

A more explicit formula for $M(f)$ can be obtained using the $(n-1)$-form

$$\tau=\sum_{i=1}^{n}(-1)^{i-1}\lambda_i x_i\,d_i x,$$

where $d_i x=dx_1\ldots dx_{i-1}dx_{i+1}\ldots dx_n$. Take $|x|$ as given by (1.1) for some large p, and write $r(x)=|x|$. Then the function $r\in C^1(\mathbf{R}^n\sim\{0\})$, and $r^{-1}dr\wedge\tau=dx$. If f is C^1 and homogeneous of degree $-Q$, then the form $f\tau$ on $\mathbf{R}^n\sim\{0\}$ is closed, and hence $f\,dx=d[(\log r)f\tau]$. It follows from Stokes' theorem that

$$M(f)=\int_S f\tau,$$

where S is any smooth oriented hypersurface cobordant to $\{|x|=1\}$ in $\mathbf{R}^n\sim\{0\}$ (cf. [6], [7], [14]). We shall be primarily interested in functions f with the cancellation property $M(f)=0$, i.e.

$$\int_{a<|x|<b}f(x)\,dx=0$$

for all $0<a<b$.

For the sequel, we fix some choice of smooth, symmetric homogeneous norm, and note the following particular cases of (1.2):

$$\int_{a<|x|<b}|x|^{\lambda-Q}\,dx=\begin{cases}\dfrac{C_0(b^\lambda-a^\lambda)}{\lambda} & \text{if } \lambda\neq 0\\[2ex] C_0\log\left(\dfrac{b}{a}\right), & \text{if } \lambda=0,\end{cases}$$

where $C_0=M(f)$, with $f(x)=|x|^{-Q}$.

We now introduce a class of Lie group structures on $\mathbf{R}^n$ which can be viewed as "small perturbations" of the additive group structure, as measured by a homogeneous norm [7]. Define $H_\lambda=\{x\in\mathbf{R}^n\colon\ \delta_r x=r^\lambda x\}$, and set

$$V_\mu=\sum_{\lambda\geqq\mu}H_\lambda.$$

Definition. A Lie algebra structure $[x,y]$ on $\mathbf{R}^n$ is

(i) graded relative to the dilations if $[H_\lambda,H_\mu]\subseteqq H_{\lambda+\mu}$ for all $\lambda,\mu>0$;

(ii) filtered relative to the dilations if $[V_\lambda,V_\mu]\subseteqq V_{\lambda+\mu}$ for all $\lambda,\mu\geqq 0$.

Every graded structure is filtered, of course, but not conversely. If $\mathfrak{g}$ denotes $\mathbf{R}^n$ with a filtered Lie algebra structure, then $\mathfrak{g}$ is obviously a nilpotent Lie algebra, and every finite-dimensional nilpotent Lie algebra over $\mathbf{R}$ is of this form for some choice of $\{\lambda_i\}$ (cf. [11], Ch. I). The associated connected and simply-connected Lie group G will be identified with $\mathbf{R}^n$ again, with multiplication given by the Campbell—Hausdorff formula

$$xy = x+y+\frac{1}{2}[x, y]+\frac{1}{12}[x, [x, y]]+\frac{1}{12}[y, [y, x]]+\dots$$

(The right-hand side is a finite sum, due to the nilpotence.) One has $x^{-1}=-x$ and 0 for the identity element of G. Haar measure on G is given by Lebesgue measure on $\mathbf{R}^n$. We shall refer to such a Lie group structure on $\mathbf{R}^n$ as "graded" or "filtered" relative to the dilations when the Lie algebra $\mathfrak{g}$ has these respective properties. (Graded structures are characterized by the property that $\delta_r \in \operatorname{Aut} G$ for all $r>0$.)

2. Singular kernels and unitary representations

Let G be $\mathbf{R}^n$, equipped with a filtered Lie group structure. Suppose K is a continuous function on $\mathbf{R}^n \sim \{0\}$, homogeneous of degree $-Q$, with mean value $M(K)=0$. Assume that K satisfies the smoothness condition

$$|K(x)-K(y)| \leqq B\omega(|x-y|) \tag{2.1}$$

if $|x|=|y|=1$. Here ω is assumed to be a continuous, concave, non-decreasing function on $[0, \infty)$ with $\omega(0)=0$ and $\omega(t)\geqq Ct$ for some $C>0$. Define

$$|K|_\omega = \sup_{\substack{|x|=|y|=1 \\ x\neq y}} \left\{\frac{|K(x)-K(y)|}{\omega(|x-y|)}\right\}$$

$$\|K\|_\infty = \sup_{|x|=1} |K(x)|,$$

and set

$$\|K\|_\omega = |K|_\omega+\|K\|_\infty .$$

If $0<\varepsilon\leqq 1\leqq R$, we define truncated kernels $K_\varepsilon=\chi_{[\varepsilon,1]}K$ and $K^R=\chi_{[1,R]}K$, where $\chi_{[A,B]}(y)=1$ if $A\leqq|y|\leqq B$ and is zero otherwise.

Theorem 2.1. *Suppose $\omega^{1/2}$ satisfies the Dini condition:*

$$\int_0^1 \omega(t)^{\frac{1}{2}} t^{-1}\, dt < \infty \tag{2.2}$$

Then for any continuous unitary representation π of G, $\lim_{R\to\infty} \pi(K^R)$ exists in the strong operator topology, and defines a bounded operator $\pi(K^\infty)$ with

$$\|\pi(K^\infty)\| \leqq C\|K\|_\omega$$

(C depending only on ω and G).

If G is graded relative to the dilations, then $\lim_{\varepsilon\to 0}\pi(K_\varepsilon)$ *also exists in the strong operator topology, and defines a bounded operator* $\pi(K_0)$ *of norm* $\leqq C\|K\|_\omega$ *(C depending only on* ω *and G).*

Remark 1. When K is C^∞ on $\mathbf{R}^n\sim\{0\}$, π is the regular representation of G on $L^2(G)$, and G is graded, this theorem was first proved by Knapp—Stein ([12]; they comment that the smoothness assumptions on K can be considerably weakened)

Remark 2. When G is the additive group of $\mathbf{R}^n$, then (2.2) can be weakened to

$$\int_0^\infty \omega(t)t^{-1}\,dt<\infty. \tag{2.3}$$

([16], Ch. II § 4.2). Of course, for $\omega(t)=t^\alpha$, $0<\alpha\leqq 1$, corresponding to (non-isotropic) Hölder continuity, this makes no difference.

3. Proof of Theorem

The proof of Theorem 2.1 involves two parts:

(1) Establishing a uniform bound for the operators $\pi(K_\varepsilon)$ and $\pi(K^R)$, depending only on $\|K\|_\omega$ and the structure of G;

(2) Proving the existence of $\lim_{\varepsilon\to 0}\pi(K_\varepsilon)v$ and $\lim_{R\to\infty}\pi(K^R)w$ for v and w in suitable dense subspaces of the Hilbert space $\mathscr{H}(\pi)$ on which π acts.

For both parts the following lemma is the basic tool:

Lemma 3.1. *Let* $a=\min\{1/\lambda_i\}$, *where* $\{\lambda_i\}$ *are the exponents of the dilations, and set* $b=a^2$. *There are constants* C_0, $C, M>0$ *depending only on the group G, such that*

$$\int_{A<|x|<B}|K(xy)-K(x)|\,dx\leqq C_0\|K\|_\omega\int_I\omega(t)\frac{dt}{t}, \tag{3.1}$$

where $I=[C|y|^bB^{-b},\ C|y|^bA^{-b}]$. *Here one assumes that* $A\geqq M|y|$ *if G is graded or that* $A\geqq M(|y|+1)$ *when G is only assumed to be filtered.*

Proof. We must convert the smoothness condition (2.1) into an estimate involving translations on G. First consider the special case where G is the additive group of $\mathbf{R}^n$. Since the norm is C^1 and homogeneous of degree 1,

$$\big||x+h|-1\big|\leqq C|h| \tag{3.2}$$

when $|x|=1$ and $|h|\leqq\varepsilon$, for some constants C, ε. Introduce the notation $y=\delta_{|y|}y'$, and set $x+h=y$. Then when $|x|=1$ and $|h|\leqq\varepsilon$,

$$\begin{aligned}|K(x+h)-K(x)|&\leqq |y|^{-Q}\big|K(y')-K(x)\big|+\big|1-|y|^{-Q}\big|\,|K(x)|\\&\leqq C\|K\|_\omega[\omega(|y'-x|)+|h|].\end{aligned}$$

But $|y'-x|=|y|^{-1}|y-\delta_{|y|}x|\leqq C[|y-x|+|x-\delta_{|y|}x]$, and one has $|x-\delta_r x|\leqq C|1-r|^a$ for r near 1 and $|x|=1$. Using (3.2), we thus have an estimate

$$|K(x+h)-K(x)| \leqq C_0\|K\|_\omega \omega(C|h|^a)$$

for all $|x|=1$ and $|h|\leqq\varepsilon$, where C_0, C, ε are independent of K. (Recall that we are assuming that $\omega(t)\geqq Ct$.) By the homogenity of K, this estimate is equivalent to

$$(3.3) \qquad |K(x+y)-K(x)| \leqq C_0\|K\|_\omega |x|^{-Q}\omega(C|y|^a|x|^{-a})$$

for all $x\neq 0$ and $|y|\leqq\varepsilon|x|$.

Now we want to replace $x+y$ by xy in (3.3). For this we recall the following comparison between the additive and nilpotent group structures:

$$(3.4) \qquad |xy-x| \leqq C|x|^{1-a}|y|^a,$$

which holds for $|y|\leqq|x|$ when G is graded, and for $|y|+1\leqq|x|$ when G is filtered (cf. [10] and [11], Ch. I). Using (3.4) in (3.3) thus gives the estimate

$$(3.5) \qquad |K(xy)-K(x)| \leqq C_0\|K\|_\omega |x|^{-Q}\omega(C|y|^b|x|^{-b})$$

($b=a^2$). Integrating (3.5) using formula (1.2), we obtain (3.1), proving the lemma.

Returning to the proof of the theorem, for part (1) we use the Cotlar—Knapp—Stein method of "almost orthogonal" decomposition of the operators $\pi(K_\varepsilon)$ and $\pi(K^R)$ ([4], Ch. VI). Namely, let $\{\varphi_j\}_{j\in\mathbf{Z}}$ be the partition of unity on $\mathbf{R}^n\sim\{0\}$ defined by

$$\varphi_j(x)=\begin{cases}1, & \text{if } \ 2^j\leqq|x|<2^{j+1}\\ 0, & \text{otherwise.}\end{cases}$$

Set $f_j=\varphi_j K$. Then one has the following L_1 estimates (For $f, g\in L_1, f*g$ denotes the convolution relative to the group G, and $f^*(x)=\overline{f(-x)}$.):

Lemma 3.2. *There are constants $C>0$ and $r<1$ (independent of K), such that the following holds: For all $j\in\mathbf{Z}$,*

$$(3.6) \qquad \|f_j\|_{L_1} \leqq C\|K\|_\omega.$$

If $i,j\geqq 0$, then

$$(3.7) \qquad \|f_i*f_j^*\|_{L_1} \leqq C\|K\|_\omega^2\omega(r^{|i-j|}).$$

If G is graded, then (3.7) also holds when $i,j\leqq 0$.

Proof. Since $|K(x)|\leqq\|K\|_\omega|x|^{-Q}$, estimate (3.6) follows from (1.3). For the rest of the lemma, note that $\|f_i*f_j^*\|_{L_1}=\|f_j*f_i^*\|_{L_1}$, so one may assume $i\leqq j$.

Suppose first that $i,j\geqq 0$. Using the mean-value zero condition, write

$$f_i*f_j^*(x)=\int f_i(y)\varphi_j(x^{-1}y)[K^*(y^{-1}x)-K^*(x)]\,dy$$
$$+\int f_i(y)K^*(x)[\varphi_j(x^{-1}y)-\varphi_j(x^{-1})]\,dy.$$

Thus, $\|f_i * f_j^*\|_{L_1} \leqq I_1 + I_2$, where

$$I_1 = \iint \varphi_i(y)\varphi_j(x)\,|K(y)|\,|K(x)-K(xy^{-1})|\,dx\,dy$$

and

$$I_2 = \iint \varphi_i(y)\,|\varphi_j(xy)-\varphi_j(x)|\,|K(x)|\,|K(y)|\,dx\,dy.$$

We can estimate I_1 using Lemma 3.1, provided $j \geqq i+m$, where m is independent of K. (For the remaining finite set of values of $j-i$, we use the estimate $\|f_i * f_j^*\|_{L_1} \leqq \|f_i\|_{L_1}\|f_j\|_{L_1}$ and apply (3.6).) Thus by the monotonicity of ω and Lemma 3.1,

$$I_1 \leqq C_0\|K\|_\omega^2 \int \varphi_i(y)\,|y|^{-Q}\omega(C|y|^b 2^{-bj})\,dy \leqq C_1\|K\|_\omega^2\,\omega(r^{j-i}),$$

where $r<1$ is independent of K.

To estimate I_2, use the Lipschitz condition

$$\big||xy|-|x|\big| \leqq C|y|, \tag{3.8}$$

which holds for $|x| \geqq M(|y|+1)$ ([10], Theorem 3) when M is sufficiently large. This shows that the integrand in I_2 has support where $2^i \leqq |y| \leqq 2^{i+1}$ and either $2^j - C2^i \leqq |x| \leqq 2^j$ or else $2^{j+1} \leqq |x| \leqq 2^{j+1}+C2^i$. Since it is bounded by $\|K\|_\omega^2 |x|^{-Q}|y|^{-Q}$, one obtains from formula (1.3) the estimate

$$I_2 \leqq C\|K\|_\omega^2 \log\left(\frac{1+t}{1-t}\right) \leqq Ct\|K\|_\omega^2,$$

where $t = C2^{i-j}$, providing $j \gg i$. This implies (3.7), since $\omega(t) \geqq Ct$.

When G is graded, (3.8) is valid for $|x| \geqq M|y|$, and the argument just given also applies to the case $i, j \leqq 0$, proving Lemma 3.2.

To complete the proof of part (1), write

$$K^R = \sum_{j=0}^J f_j + g,$$

where $2^J \leqq R < 2^{J+1}$. Since $\|\pi(f)\| \leqq \|f\|_{L_1}$, it follows from Lemma 3.2 and Lemma 11 of Cotlar—Knapp—Stein [12] that

$$\Big\|\sum_{j=0}^J \pi(f_j)\Big\| \leqq B\|K\|_\omega,$$

where

$$B = C\sum_{k=0}^\infty \omega(r^k)^{\frac{1}{2}} < \infty.$$

(By the concavity and monotonicity of ω, this series is majorized by $\int_0^1 \omega(t)^{\frac{1}{2}} t^{-1}dt$.) Clearly $\|g\|_{L_1} \leqq \|f_{J+1}\|_{L_1} \leqq C\|K\|_\omega$, so we obtain the desired bound for $\|\pi(K^R)\|$. When G is graded, the same argument applies to $\pi(K_\varepsilon)$.

For part (2) of the proof, recall that if $v \in \mathscr{H}(\pi)$ is a C^1-vector for π, then $\|\pi(x)v - v\| \leqq C\|x\|$. Since $\|x\| \leqq C|x|$ when $|x| \leqq 1$, we can use the mean value zero property of K to estimate

$$\|\pi(K_\varepsilon)v-\pi(K_\delta)v\| \leqq \int_{\varepsilon<|x|<\delta} |K(x)|\,\|\pi(x)v-v\|\,dx$$
$$\leqq C\|K\|_\infty \int_{\varepsilon<|x|<\delta} |x|^{-Q+1}\,dx \leqq C\|K\|_\infty(\delta-\varepsilon),$$

if $0<\varepsilon<\delta\leqq 1$. Thus $\lim_{\varepsilon\to 0}\pi(K_\varepsilon)v$ exists for v in the dense subspace of C^1 vectors.

To remove the cutoff at infinity on K, we use a regularisation technique naturally suggested by part (**1**). Assume $\psi\in L_1(G)$ has compact support, and that

$$\int_G \psi(x)\,dx = 0. \tag{3.9}$$

Lemma 3.3. *Let* $0<A<B$, *and set* $\varphi=\chi_{[A,B]}$. *Then* $\lim_{A,B\to\infty}\|(\varphi K)*\psi\|_{L_1}=0$.

Proof. Using (3.9), we can write

$$(\varphi K)*\psi(x) = \int [K(xy^{-1})-K(x)]\varphi(xy^{-1})\psi(y)\,dy+\int [\varphi(xy^{-1})-\varphi(x)]K(x)\psi(y)\,dy.$$

Hence $\|(\varphi K)*\psi\|_{L_1}\leqq I_1+I_2$, where

$$I_1 = \int |\psi(y)|\left\{\int_{A<|x|<B} |K(xy)-K(x)|\,dx\right\}dy$$

and

$$I_2 = \iint |\varphi(xy^{-1})-\varphi(x)|\,|K(x)|\,|\psi(y)|\,dx\,dy.$$

Since ψ is assumed to have compact support, we can use Lemma 3.1 when A, B are sufficiently large, to obtain the estimate

$$I_1 \leqq C_0\|K\|_\omega\|\psi\|_{L_1}\int_E \omega(t)t^{-1}\,dt,$$

where $E=[CB^{-b}, CA^{-b}]$, with a constant C depending on Supp(ψ). Since ω is bounded on $[0, 1]$, condition (2.2) implies that $\int_0^1 \omega(t)t^{-1}dt<\infty$, so that $I_1\to 0$ as $A, B\to\infty$.

For I_2, the integrand is zero unless either $B<|x|<B+C$ or $A-C<|x|<A$, where C depends on Supp(ψ), by (3.8). Thus, just as in the proof of Lemma 3.2, we have

$$I_2 \leqq C_0\|K\|_\infty\|\psi\|_{L_1}\log\left[\frac{1+\left(\frac{C}{B}\right)}{1-\left(\frac{C}{A}\right)}\right],$$

so that $I_2\to 0$ as $A, B\to\infty$, proving the lemma.

To complete the proof of part (**2**), we note that if ψ, φ are as in Lemma 3.3, then

$$\|\pi(K^A)\pi(\psi)v-\pi(K^B)\pi(\psi)v\| \leqq \|(\varphi K)*\psi\|_{L_1}\|v\|.$$

Hence $\lim_{R\to\infty} \pi(K^R)\pi(\psi)v$ exists for all $v\in\mathscr{H}$. Let $\mathscr{H}_1$ be the subspace spanned by all vectors of the form $\pi(\psi)v$, for $v\in\mathscr{H}$ and ψ a compactly supported L_1 function on G satisfying (3.9). Then $\mathscr{H}_0\equiv\mathscr{H}_1^{\perp}$ consists of all $w\in\mathscr{H}$ such that

$$\int \psi(x)\big(\pi(x)v, w\big)\,dx = 0$$

for all such ψ, and all $v\in\mathscr{H}$. But this implies that $x\to\big(\pi(x)v, w\big)$ is constant on G, and hence $\mathscr{H}_0=\mathscr{H}^G$, the space of G-fixed vectors. But $\pi(K^R)w=0$ if $w\in\mathscr{H}^G$, by the mean-value zero condition. Hence $\lim_{R\to\infty} \pi(K^R)u$ exists for u in the dense subspace $\mathscr{H}_0+\mathscr{H}_1$. Together with the uniform bounds from part (1), this completes the proof of Theorem 2.1.

4. Non-unitary representations

Various parts of Theorem 2.1 and the proof just given extend to certain non-unitary representations of G. We can summarize the situation as follows:

A. If π is any Banach-space representation of G, then $\lim_{\varepsilon\to 0} \pi(K_\varepsilon)v$ exists when v is a C^1 vector for π (cf. Section 3).

B. If π is a uniformly-bounded representation of G on a Banach space $\mathscr{H}$ ($\sup_g \|\pi(g)\|<\infty$), then $\lim_{R\to\infty} \pi(K^R)v$ exists for $v\in\mathscr{H}^G+\mathscr{H}_1$, where $\mathscr{H}^G$ consists of the G-fixed vectors in $\mathscr{H}$, and $\mathscr{H}_1^{\perp}=(\mathscr{H}^*)^G$ consists of the G-fixed vectors in $\mathscr{H}^*$. (Cf. Section 3.) Thus if we assume that

$$(\mathscr{H}^G)^{\perp}\cap(\mathscr{H}^*)^G = 0, \tag{4.1}$$

then $\lim_{R\to\infty} \pi(K^R)v$ exists for v in a dense subspace of $\mathscr{H}$. For example, if G acts ergodically as measure-preserving transformations on a measure space $\mathscr{M}$, and π is the induced action on $\mathscr{H}=L^p(\mathscr{M})$, $1<p<\infty$, then (4.1) is satisfied. (If $\mathscr{M}$ has finite measure, $(\mathscr{H}^*)^G=\mathscr{H}^G=$constants, and $(\mathscr{H}^G)^{\perp}=$functions of integral zero. If $\mathscr{M}$ has infinite measure, $(\mathscr{H}^*)^G=0$.)

C. If π is the regular representation of G on $L^p(G)$, $1<p<\infty$, then the operators $\pi(K^R)$ are uniformly bounded as $R\to\infty$. The same is true for $\pi(K_\varepsilon)$ as $\varepsilon\to 0$ if G is assumed to be graded. (From Lemma 3.1 we have the inequality

$$\int_{|x|\geqq C|y|} |K(xy)-K(x)|\,dx \leqq C_0\|K\|_{\omega}\int_0^C \omega(t)t^{-1}\,dt, \tag{4.2}$$

valid for all y in the graded case, and $|y|\geqq M$ in the filtered case. Using the Coifman—Weiss theory of integral operators on spaces of "homogeneous type", the L^2 boundedness together with (4.2) implies L^p boundedness; cf. [4] Ch. III and [13].)

D. If π is a uniformly-bounded representation of G on an L^p space, $1<p<\infty$, then the uniform L^p estimates in **C** can be "transferred" to give uniform bounds for $\pi(K^R)$, and also for $\pi(K_\varepsilon)$ when G is graded [3]. The property of G which is used is the **Folner condition:** For every compact set $C\subset G$ and every $\varepsilon>0$, there is a neighborhood V of 0 such that

$$\text{(4.3)} \qquad \frac{\text{meas}\,(V\cdot C)}{\text{meas}\,(V)} \leqq 1+\varepsilon.$$

For V we can take the ball $B_r=\{|x|\leqq r\}$ with sufficiently large r. Indeed, by (3.8) we have $B_r\cdot C\subseteqq B_{r+a}$, for some $a>0$ depending on C but independent of r, and by (1.3)

$$\frac{\text{meas}\,(B_{r+a})}{\text{meas}\,(B_r)} = \left[1+\left(\frac{a}{r}\right)\right]^Q.$$

Combining the results **A—D**, we finally obtain the following generalization of the "ergodic Hilbert transform" studied by Cotlar [5] and Calderón [2]:

Theorem 4.1. *Let the filtered nilpotent group G act ergodically as measure-preserving transformations on a σ-finite measure space $\mathcal{M}$. Let K be homogeneous of degree $-Q$, with mean-value zero, satisfying the smoothness condition* (2.1), *and assume $\omega^{1/2}$ satisfies the Dini condition* (2.2). *If $1<p<\infty$ and $f\in L^p(\mathcal{M})$, then*

$$T^\infty f(m) = \lim_{R\to\infty}\int_{1\leqq|x|\leqq R} K(x)f(x\cdot m)\,dx$$

exists in $L^p(\mathcal{M})$, and $\|T^\infty\|_{L_p}\leqq C\|K\|_\omega$, where C depends only on p, ω, and G. If G is also graded, then

$$T_0 f(m) = \lim_{\varepsilon\to 0}\int_{\varepsilon\leqq|x|\leqq 1} K(x)f(x\cdot m)\,dx$$

also exists in $L^p(\mathcal{M})$, and $\|T_0\|_{L_p}\leqq C\|K\|_\omega$, where C depends only on p, ω, and G.

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Roe Goodman
Department of Mathematics
Rutgers, The State University
New Brunswick, N. J. 08 903
U.S.A.