Bounding subsets of some metric vector spaces

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1. Introduction

A subset of a complex topological vector space E is said to be *bounding* if it is mapped onto a bounded set by every holomorphic (i.e. continuous and Gâteaux-analytic) function on E.

Seán Dineen, Bengt Josefson and Martin Schottenloher have studied the bounding sets in many locally convex vector spaces. The first author proved in [1] that if E is a separable or reflexive Banach space, then the closed bounding subsets of E are compact. He proved in [2] that the non-compact set $D=\{e_j\colon j\in\mathbf{N}\}$ of unit vectors $e_j\in l_\infty$, $e_j=(\delta_{ji})_{i\in\mathbf{N}}$, is bounding in l_∞, the space of all bounded complex functions on $\mathbf{N}$. The second author studied in detail the bounding subsets of l_∞ and extended in [3] Dineen's result in [2] by proving that a subset of c_0, the Banach space of all complex sequences tending to zero, is bounding as a subset of l_∞ if and only if it is bounded. Schottenloher [5, Proposition 1] generalized Dineen's result in [1] to separable quasicomplete locally convex spaces: *in such a space any closed bounding set is sequentially compact.* Also Kamil Rusek [4] has studied bounding subsets of separable Banach spaces.

In this paper we study bounding subsets of some spaces which are not locally convex. We shall prove that if a separable complete metric vector space E, with translation invariant metric, has the bounded approximation property and the closed balls are polynomially convex, then a subset of E is bounding if and only if it is relatively compact. We shall also prove that the metric vector space l_p, $0<p<1$, of complex sequences with metric d defined by $d(x,y)=\sum_{j=1}^{\infty}|x_j-y_j|^p$, $x\in l_p$, $y\in l_p$, is an example of such a space.

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2. The main result

A metric vector space E is said to have the *bounded approximation property* if there is an equicontinuous family $(\varphi_t)_{t\in T}$ of linear mappings $\varphi_t\colon E\to\varphi_t(E)$ of finite rank such that, for every $\varepsilon>0$ and every compact set K in E, there is a $t\in T$ with $d(\varphi_t(x), x)\leqq\varepsilon$ for all $x\in K$.

Theorem. *Let E be a complete separable complex metric vector space with a translation invariant metric d which has the bounded approximation property and is such that the closed balls in E are polynomially convex. Then a subset of E is bounding if and only if it is relatively compact.*

Proof. A relatively compact set is obviously bounding since any continuous function on a compact set is bounded.

Assume that $A\subset E$ is not relatively compact. Then A is not precompact (since its completion is equal to its closure in E), so there exists a sequence (x_j) of elements of A such that $d(x_j, x_k)\geqq 5R>0$ when $j\neq k$ for some suitable number R. The proof will depend on whether or not (x_j) stays close to a finite-dimensional subspace. In fact we have two cases to study.

Let $\varrho_t=I-\varphi_t$ where $(\varphi_t)_{t\in T}$ is the given family of linear mappings, and let $B(x, r)$ denote the closed ball of center x and radius r.

Case 1. For some $t\in T$ we have $\varrho_t(x_j)\in B(0, 2R)$ for every j.

In this case the sequence of points (y_j), $y_j=\varphi_t(x_j)$, satisfies

$$d(x_j, x_k)\leqq d(x_j, y_j)+d(y_j, y_k)+d(y_k, x_k),\quad j\neq k,$$

i.e.

$$d(y_j, y_k)\geqq d(x_j, x_k)-d(x_j, y_j)-d(x_k, y_k)\geqq 5R-2R-2R=R,\quad j\neq k.$$

Hence $\{y_j\}$ is a subset of $E_t=\varphi_t(E)$ which is not precompact. Since E_t is of finite dimension we know that there exists a linear form ξ on E_t such that $(\xi(y_j))$ is unbounded. The linear form $q=\xi\circ\varphi_t$ is a holomorphic function on E with $(q(x_j))$ unbounded. Hence $\{x_j\}$ is not bounding in this case. Note that here the family of linear mappings need not be equicontinuous.

Case 2. For every $t\in T$ there is a j such that $\varrho_t(x_j)\notin B(0, 2R)$.

By the triangle inequality for the translation invariant metric d we have

$$B(0, R)+B(0, R)\subset B(0, 2R).$$

Choose $\delta>0$ such that $\varrho_t(B(0, \delta))\subset B(0, R)$ for every $t\in T$; this is possible in view of the equicontinuity of the family (ϱ_t). Define

$$V_n=\bigcup_{j=1}^{n} B(z_j, \delta),$$

where $\{z_j\}$ is a denumerable dense set in E; hence (V_n) is a covering of E. Then for every $t \in T$,

$$\varrho_t(V_n) \subset \bigcup_{j=1}^{n} B(\varrho_t(z_j), R).$$

Let now n be fixed. Then for some $t \in T$ we have $\varrho_t(z_j) \in B(0, R), j=1, \ldots, n$. This is because $\varphi_t(z_j) = z_j - \varrho_t(z_j)$ can be brought arbitrarily close to z_j, $j=1, \ldots, n$. Hence for this t we have

$$\varrho_t(V_n) \subset B(0, R) + B(0, R) \subset B(0, 2R).$$

But on the other hand we have by assumption in this case $\varrho_t(x_j) \notin B(0, 2R)$ for some j. By assumption $B(0, 2R)$ is polynomially convex, so there is a polynomial q with $|q(\varrho_t(x_j))| > 1$ and $|q| < 1$ on $B(0, 2R)$. The polynomial $Q = q \circ \varrho_t$ satisfies $|Q(x_j)| > 1$ and $|Q| < 1$ on V_n.

We can now define inductively a covering (W_m) of E and a sequence (a_m) with $a_m \in W_{m+1} \setminus W_m$, $m = 1, 2, \ldots,$ as follows: $W_m = V_{n(m)}$, $a_m = x_{j(m)}$, where $n(m)$ and $j(m)$ are found by the procedure just described. If $n = n(m)$ has already been found we let $j(m) = j$ be the index j found above and then we put $n(m+1)$ as the smallest integer such that $a_m = x_j \in V_{n(m+1)} = W_{m+1}$. This is possible since (V_n) is a covering of E.

We now have $a_m \in W_{m+1} \setminus W_m$ and polynomials Q_m, $m = 1, 2, \ldots,$ with

$$|Q_m| < 1 \quad \text{on} \quad W_m \quad \text{and} \quad |Q_m(a_m)| > 1.$$

One can find $\alpha_m \in \mathbf{N}$ and constants c_m such that the polynomial $f_m = c_m Q_m^{\alpha_m}$ satisfies

$$|f_m| < 2^{-m} \quad \text{on} \quad W_m \quad \text{and} \quad |f_m(a_m)| \geqq m + 1 + \textstyle\sum_{k=1}^{m-1} |f_k(a_m)|.$$

Since (W_m°) is an increasing open covering of E, the function $f = \sum_{m=1}^{\infty} f_m$ is holomorphic on E and

$$\begin{aligned} |f(a_m)| &= |f_m(a_m) + \textstyle\sum_1^{m-1} f_k(a_m) + \sum_{m+1}^{\infty} f_k(a_m)| \\ &\geqq |f_m(a_m)| - \textstyle\sum_1^{m-1} |f_k(a_m)| - \sum_{m+1}^{\infty} \|f_k\|_{W_k} \geqq m. \end{aligned}$$

Hence $\{a_m\} = \{x_{j(m)}\} \subset A$ is not bounding in this case.

Remark. The proof in Schottenloher [5] depends on a similar construction with a covering (V_n) consisting of convex sets. In our notation

$$V_n = \{z_1, \ldots, z_n\} + B(0, \delta)$$

is the vector sum of a finite set and a polynomially convex set. If $B(0, \delta)$ is convex we can use

$$V_n' = \operatorname{cvx} \{z_1, \ldots, z_n\} + B(0, \delta),$$

where "cvx" denotes convex hull; as the sum of two convex sets V_n' is then convex. However, the vector sum of a convex set and a polynomially convex set need not be polynomially convex. This difficulty explains the difference between our proof and that in the locally convex case and also the role played by the mappings ϱ_t.

3. Sequence spaces with polynomially convex balls

We shall now see that there exist spaces which are not locally convex and still satisfy the hypotheses of the theorem.

The space l_p of all complex sequences (x_j) such that $\sum |x_j|^p < +\infty$ is a metric vector space for $0<p\leqq 1$ if we define the distance by

$$d(x, y) = \sum |x_j - y_j|^p, \quad x, y \in l_p.$$

If $0<p<1$ it is not locally convex. The space l_p clearly has the bounded approximation property, in fact, even the projective approximation property: we put $T=\mathbf{N}$ and let

$$\varphi_t(x) = \pi_t(x) = \pi_t\left(\sum_0^\infty x_j e_j\right) = \sum_0^t x_j e_j.$$

The crucial property of the balls we formulate as a proposition.

Proposition. *The closed unit ball B of l_p, $0<p<1$, is polynomially convex. More precisely, for every point $b\in l_p$, $0<p<1$, with $b\notin B$ one can find a continuous monomial P on l_p with*

$$\sup_{x\in B} |P(x)| < 1 \leqq P(b).$$

Proof. The construction of P is a finite-dimensional problem. This is because for every $b\notin B$, there exists an $n\in\mathbf{N}$ such that $\pi_n(b)\notin\pi_n(B)=B\cap E_n$. We can assume that all $b_j\geqq 0, j=1, \ldots, n$, by multiplying b_j by complex numbers of modulus one, and indeed that $b_j>0$ by ignoring coordinates for which $b_i=0$.

Consider the mapping $\varphi\colon \mathbf{R}^n\to\mathbf{R}^n$ defined by $\varphi(t_1, \ldots, t_n)=(e^{t_1}, \ldots, e^{t_n})$, where $t=(t_1, \ldots, t_n)\in\mathbf{R}^n$. Let $B^*=\varphi^{-1}(\pi_n(B))=\{t\in\mathbf{R}^n;\ \sum_{j=1}^n e^{pt_j}\leqq 1\}$ and $b^*=\varphi^{-1}(b)=(\log b_1, \ldots, \log b_n)$. The set B^* is closed and convex in $\mathbf{R}^n$. This is because $t\to\sum_1^n e^{pt_j}$ is a continuous convex function on $\mathbf{R}^n$. Since $b^*\notin B^*$, by the Hahn—Banach theorem there exists a continuous linear functional v on $\mathbf{R}^n$ such that

$$\sup_{t\in B^*} v(t) < v(b^*).$$

Here v takes the form $v(t)=\sum_{j=1}^n m_j t_j$, $t\in\mathbf{R}^n$, where necessarily $m_j\geqq 0$, $j=1, \ldots, n$, since B^* contains with t every point s with $s_j\leqq t_j$. But the hyperplane $\{t;\ \sum_{j=1}^n m_j t_j = v(b^*)\}$ can be moved so that all m_j become rational numbers,

$j=1, \ldots, n$. Thus there exists a positive integer m such that $mm_j=k_j$ are all non-negative integers. The monomial $P(x)=x_1^{k_1}\ldots x_n^{k_n}$, $x\in\mathbf{C}^n$, satisfies

$$P(b)=\prod_{j=1}^n b_j^{k_j}=\exp\left(\sum_{j=1}^n k_j b_j^*\right)=\exp\left(mv(b^*)\right),$$

and

$$\sup_{x\in B}|P(x)|=\sup_{t\in B^*}\exp\left(mv(t)\right)<P(b),$$

so $P/P(b)$ has the desired properties.

Remark. The method of proof shows that the proposition holds for every metric vector space with a Schauder basis (e_j) such that all the sets

$$\left\{z\in\mathbf{C}^n;\ d\left(\sum_1^n \exp(z_j)\, e_j, 0\right)<r\right\}$$

are convex (in particular $d(tx, 0)=d(x, 0)$ for $t\in\mathbf{C}$, $|t|=1$). As an example we take the space

$$l^{\{p_j\}}=\left\{(x_j);\ x_j\in\mathbf{C},\ d(x, 0)=\sum_{j=1}^{\infty}|x_j|^{p_j}<+\infty\right\}$$

where $0<p_j<1$.

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