Examples of $\mathscr{L}_1$ spaces*

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In this note we present a class of new examples of simple but interesting $\mathscr{L}_1$ spaces. Let us first recall the definition of $\mathscr{L}_1$ spaces. A Banach space X is said to be an $\mathscr{L}_{1,\lambda}$ space for some $\lambda \geqq 1$ if for every finite-dimensional subspace B of X there is a finite-dimensional subspace C of X containing B so that $d(C, l_1^n) \leqq \lambda$ where $n = \dim C$ ($d(U, V)$ denotes the Banach—Mazur distance between U and V. See [6] for details and also for the basic facts concerning $\mathscr{L}_{p,\lambda}$ spaces, $1 \leqq p \leqq \infty$). A Banach space is said to be an $\mathscr{L}_1$ space if it is an $\mathscr{L}_{1,\lambda}$ space for some $\lambda < \infty$. It is known (cf. [6]) that X is an $\mathscr{L}_{1,1+\varepsilon}$ space for every $\varepsilon > 0$ if and only if X is isometric to the space $L_1(\mu)$ for some measure μ. Consequently, there are up to isomorphism only two examples of separable infinite-dimensional spaces which are $\mathscr{L}_{1,1+\varepsilon}$ spaces for every $\varepsilon > 0$, namely l_1 and $L_1(0, 1)$. (Up to isometry there are countably many such spaces, according to the number of atoms of μ.) It is also known that there are $\mathscr{L}_1$ spaces which are not isomorphic to $L_1(\mu)$ spaces. In [4] a sequence of mutually non-isomorphic separable infinite-dimensional $\mathscr{L}_1$ spaces was constructed. It was not known however, till now whether there exist uncountably many different spaces of this type, or even if there are for a given $\lambda < \infty$, infinitely many mutually non-isomorphic separable and infinite-dimensional $\mathscr{L}_{1,\lambda}$ spaces. The examples presented here solve these problems. They also provide the first examples of separable $\mathscr{L}_1$ spaces which on the one hand do not embed in l_1 and on the other hand do not contain isomorphic copies of $L_1(0, 1)$.

Our construction here was motivated by a paper of McCartney and O'Brien [7]. In this paper the authors produced an example of a separable space which has the Radon—Nikodym property (R. N. property in short, see [2] for a detailed discussion of this property) but which does not embed into a separable conjugate space.

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We noticed that a modification (and simplification) of the construction in [7] yields $\mathcal{L}_1$ spaces. The $\mathcal{L}_1$ spaces we obtain also have the R. N. property without being subspaces of separable conjugate spaces (such examples were constructed independently of [7] and at about the same time also by Bourgain and Delbaen [1]. The examples in [1] are $\mathcal{L}_\infty$ spaces).

The basic building blocks of our examples are the following spaces. Let $0<\alpha<1$ and let T be a quotient map from l_1 onto $L_1(0, 1)$. Let X_α be the graph of $\alpha^{-1}T$; i.e., the subspace $\{(\alpha x, Tx),\ x\in l_1\}$ of $(l_1\oplus L_1(0, 1))_1$. The space X_α depends of course also on the special choice of T. We did not indicate T explicitly in the notation of X_α since the special form of T will be of no importance in the sequel. Moreover, from the isomorphic point of view X_α does not really depend on T. It was proved in [5] that there is an absolute constant K so that if T_1 and T_2 are both quotient maps from l_1 onto $L_1(0, 1)$ then there is an automorphism τ of l_1 with $\|\tau\|, \|\tau^{-1}\|\leqq K$ and $T_1=T_2\tau$. The map $\varrho\colon X_\alpha(T_1)\to X_\alpha(T_2)$ defined by

$$\varrho(\alpha x, T_1 x) = (\alpha\tau x, T_2\tau x)$$

is thus an isomorphism with $\|\varrho\|, \|\varrho^{-1}\|\leqq K$.

We exhibit next some simple properties of the spaces X_α.

Proposition 1. a) *There is a constant λ (independent of α and T) so that each X_α is an $\mathcal{L}_{1,\lambda}$ space.*

b) *$d(X_\alpha, l_1)\leqq(1+\alpha)/\alpha$ for every $\alpha>0$.*

c) *For every subspace Z of l_1 and every $\alpha>0$, $d(X_\alpha, Z)\geqq 1/2\alpha(1+\alpha)$.*

Proof. a) The annihilator $X_\alpha^\perp$ of X_α in $(l_\infty\oplus L_\infty(0, 1))_\infty$ consists of all the vectors of the form $(-\alpha^{-1}T^*y^*, y^*)$ with $y^*\in L_\infty(0, 1)$. Since T^* is an isometry it follows that $X_\alpha^\perp$ is isometric to $L_\infty(0, 1)$. Since $L_\infty(0, 1)$ is an injective space (i.e., a P_1 space) there is a projection of norm 1 from $(l_\infty\oplus L_\infty(0, 1))_\infty$ onto $X_\alpha^\perp$. Consequently, there is a projection of norm $\leqq 2$ from $(l_\infty\oplus L_\infty(0,1))_\infty^*$ (which is an $L_1(\mu)$ space for some μ) onto $X_\alpha^{\perp\perp}$ (which is isometric to X_α^{**}). The desired result follows now from [6, Theorem II.5.7.]. It can be easily checked from the proof of that theorem that one can take as λ any constant larger than 10. (If one takes as T the "most natural" quotient map; i.e., the operator which maps the unit vectors e_{2^n+i}, $0\leqq i<2^n$, $n=0, 1, 2, \dots$ of l_1 to the vectors $2^n\chi_{[i2^{-n}, (i+1)2^{-n}]}$ of $L_1(0, 1)$ then a simple direct argument shows that X_α is an $\mathcal{L}_{1,2+\varepsilon}$ space for every $\varepsilon>0$.)

Assertion b) follows by considering the isomorphism $x\to(\alpha x, Tx)$ from l_1 onto X_α. In order to verify assertion c) we note first that if $\{u_n\}_{n=1}^\infty$ is a sequence in the unit ball of $l_1=c_0^*$ so that $\|u_n-u_m\|\geqq 2\gamma$ for some $\gamma>0$ and every $n\neq m$, then every w^* limit point u of $\{u_n\}_{n=1}^\infty$ satisfies $\|u\|\leqq 1-\gamma$. Moreover, for every $\varepsilon>0$, there is a sequence $\{n_k\}_{k=1}^\infty$ of integers so that $u_{n_k}=u+y_k+w_k$ with the $\{y_k\}_{k=1}^\infty$

having mutually disjoint supports and $\sum_{k=1}^{\infty}\|w_k\|<\varepsilon$. In particular

$$\left\|\sum_{k=1}^{\infty}\lambda_k(u_{n_{2k}}-u_{n_{2k+1}})\right\| \geqq (2\gamma-\varepsilon)\sum_{k=1}^{\infty}|\lambda_k|$$

for every choice of scalars $\{\lambda_k\}_{k=1}^{\infty}$.

Let now $r_n(t)=\text{sign}\sin 2^n\pi t$, $n=1, 2, \dots$ be the Rademacher functions on $[0, 1]$. Let $x_n\in l_1$ be such that $Tx_n=r_n$ with $\|x_n\|\to 1$ as $n\to\infty$ and consider the vectors $v_n=(\alpha x_n, r_n)\in X_\alpha$. Then clearly $\|v_n\|=1+\alpha+o(1)$ and $\|v_n-v_m\|\geqq 1$ for every $n\neq m$. Since (by Khintchine's inequality) the sequence $\{r_n\}_{n=1}^{\infty}$ in $L_1(0, 1)$ is equivalent to the unit vector basis in l_2 it follows that if $\{n_k\}_{k=1}^{\infty}$ and ϱ are such that $\|\sum_{k=1}^{\infty}\lambda_k(v_{n_{2k}}-v_{n_{2k+1}})\|\geqq\varrho\sum_{k=1}^{\infty}|\lambda_k|$ for every choice of scalars $\{\lambda_k\}_{k=1}^{\infty}$, then necessarily $\varrho\leqq 2\alpha$. Assertion c) is an immediate consequence of this fact and the preceding observation. ∎

As an easy consequence of Proposition 1 we get

Theorem 1. *Let* $1\geqq\alpha_1>\alpha_2>\dots$ *be a sequence decreasing to* 0 *and let* $Y=Y(\{\alpha_i\}_{i=1}^{\infty})=(\sum_{i=1}^{\infty}\oplus X_{\alpha_i})_1$. *Then*

(i) *Y is an $\mathscr{L}_1$ space.*

(ii) *Y has the Radon—Nikodym property.*

(iii) *Y has the Schur property (i.e., a sequence in Y tends w to* 0 *only if it tends in norm to* 0*).*

(iv) *Y is not isomorphic to a subspace of l_1.*

(v) *Y is not isomorphic to a subspace of a separable conjugate space.*

Proof. Part (i) follows from Proposition 1a. Parts (ii) and (iii) follow from Proposition 1b and the easy and well-known fact that if $\{Z_n\}_{n=1}^{\infty}$ all have the R. N. property (resp. the Schur property) then the same is true for $(\sum_{n=1}^{\infty}\oplus Z_n)_1$. Part (iv) follows from Proposition 1c. Finally Part (v) is a consequence of (1), and (iv) in view of a result of Lewis and Stegall [3] which asserts that an $\mathscr{L}_1$ space which embeds in a separable conjugate space already embeds in l_1. ∎

We are going to prove next that by taking different sequences $\{\alpha_i\}_{i=1}^{\infty}$ in Theorem 1 we can obtain $2^{\aleph_0}$ many different isomorphism types among the spaces $Y(\{\alpha_i\}_{i=1}^{\infty})$. This is essentially a consequence of the fact that if $0<\beta<\alpha$ with β much smaller than α, then it is impossible to embed X_α in X_β in such a way that there is a projection from X_β onto the image of X_α whose norm is substantially smaller than α^{-1}. A precise statement of this fact in a somewhat stronger form is the content of the following proposition.

Proposition 2. *Let* $0<\beta<\alpha\leqq 1$, *let S be an operator of norm* $\leqq\beta$ *from* l_1 *into itself with* $\ker S=\{0\}$ *and let* $T\colon l_1\to L_1(0, 1)$ *be a quotient map. Let Z be the subspace* $\{(Sx, Tx);\ x\in l_1\}$ *of* $(l_1\oplus L_1(0, 1))_1$. *Then for every pair of operators* $U\colon X_\alpha\to Z$, $V\colon Z\to X_\alpha$ *such that* $VU=$*identity of* X_α, *we have* $\|U\|\,\|V\|\geqq\alpha/(20\beta+50\alpha^2)$.

(Note that the same operator T is used in the definition of Z and X_α. This is done just for notational convenience and is of no significance in the proof.)

Proof. We assume, as we clearly may, that $\|V\|=1$. Let $\{r_n\}_{n=1}^\infty$ be the Rademacher functions in $L_1(0, 1)$ and let $\{x_n\}_{n=1}^\infty$ be elements of norm $\leqq 2$ in l_1 so that $Tx_n=r_n$ for every n. Let $\{u_n\}_{n+1}^\infty$ be defined by the relation

$$U(\alpha x_n, Tx_n) = (Su_n, Tu_n).$$

As in the proof of Proposition 1c, we can find a sequence of integers $\{n_k\}_{k=1}^\infty$ and a constant σ so that if $v_k=u_{n_{2k}}-u_{n_{2k+1}}$ then $\|Sv_k\|\leqq 2\sigma$ for every k and $\left\|\sum_{k=1}^\infty \lambda_k Sv_k\right\| \geqq \sigma \sum_{k=1}^\infty |\lambda_k|$ for every choice of scalars $\{\lambda_k\}_{k=1}^\infty$. Putting $y_k=x_{n_{2k}}-x_{n_{2k+1}}$, we have

$$(1) \qquad (\alpha y_k, Ty_k) = V(Sv_k, Tv_k).$$

Note that $\|Ty_k\|=1$ for every k (for future reference, note also that $\|Ty_k-Ty_h\|\geqq 1$ for every $k\neq h$). Also we have that $\left\|\sum_{k=1}^n Ty_k\right\|=0(\sqrt{n})$. Hence

$$n\sigma \leqq \left\|\sum_{k=1}^n Sv_k\right\| \leqq \left\|\left(\sum_{k=1}^n Sv_k, \sum_{k=1}^n Tv_k\right)\right\| \leqq$$
$$\leqq \|U\|\left(\alpha \sum_{k=1}^n \|y_k\| + \left\|\sum_{k=1}^n Ty_k\right\|\right) \leqq \|U\|\left(4n\alpha+0(\sqrt{n})\right)$$

consequently, $\sigma\leqq 4\|U\|\alpha$, i.e.

$$(2) \qquad \|Sv_k\| \leqq 8\|U\|\alpha \quad k = 1, 2, \dots .$$

The sequence $\{Tv_k\}_{k=1}^\infty$, as any bounded sequence in $L_1(0, 1)$, can be represented (after passing to a subsequence if necessary) as

$$Tv_k = f_k + h_k \quad k = 1, 2, \dots$$

where $\{f_k\}_{k=1}^\infty$ is equi-integrable and even weakly convergent in $L_1(0, 1)$, the $\{h_k\}_{k=1}^\infty$ have disjoint supports and $|h_k|\wedge|f_k|=0$ for every k. By passing to a further subsequence, if necessary, we may assume that the sequence $\|h_k\|$ is almost constant (up to a factor 2, say). Now it is well-known that $L_1(0, 1)$ has the Banach—Saks property; that is, every weakly convergent sequence in $L_1(0, 1)$ has a subsequence whose Cesaro averages are norm convergent. Thus by passing to a suitable subsequence of the v_k's, we may assume that

$$\left\|\sum_{k=1}^n (-1)^k f_k\right\| = o(n).$$

By repeating the argument used to prove (2) (using $\sum_{k=1}^n (-1)^k v_k$ instead of $\sum_{k=1}^n v_k$) we get

$$(3) \qquad \|h_k\| \leqq 8\|U\|\alpha \quad k = 1, 2, \dots .$$

Since

$$\|f_k\| \leqq \|Tv_k\| \leqq \|U\| \, \|(\alpha y_k, Ty_k)\| \leqq \|U\|(1+4\alpha) < 5\|U\|$$

and since T is a quotient map there are $\{z_k\}_{k=1}^\infty$ in l_1 so that $\|z_k\|\leqq 5\|U\|$ and $Tz_k=f_k$.

Let now $w_k \in l_1$ be such that

$$V(Sz_k, f_k) = V(Sz_k, Tz_k) = (\alpha w_k, Tw_k). \tag{4}$$

If the sequence $\{w_k\}_{k=1}^{\infty}$ is not a Cauchy sequence then by passing to a subsequence we may assume without loss of generality that there is a constant $\gamma > 0$ so that $\|w_{2k+1} - w_{2k}\| \leqq 2\gamma$ for every k and $\left\|\sum_{k=1}^{\infty} \lambda_k (w_{2k+1} - w_{2k})\right\| \geqq \gamma \sum_{k=1}^{\infty} |\lambda_k|$ for every choice of scalars $\{\lambda_k\}_{k=1}^{\infty}$. By repeating the argument used to prove (2) and (3) (noting that $\|V\| = 1$ and $\|Sz_k\| \leqq 5\beta \|U\|$) we get that

$$\|w_{2k+1} - w_{2k}\| \leqq 20\beta \|U\|/\alpha \quad k = 1, 2, \dots . \tag{5}$$

Clearly (5) is also valid for large k if $\{w_k\}_{k=1}^{\infty}$ is a Cauchy sequence. We have for every k

$$\begin{aligned} \|T(y_{2k+1} - y_{2k} - w_{2k+1} + w_{2k})\| &\geqq \|T(y_{2k+1} - y_{2k})\| - 20\beta \|U\|/\alpha \\ &\geqq 1 - 20\beta \|U\|/\alpha. \end{aligned} \tag{6}$$

On the other hand, since $\|V\| = 1$ we get by (1) and (2), (3) that

$$\begin{aligned} &\|T(y_{2k+1} - y_{2k} - w_{2k+1} + w_k)\| \\ &\leqq \|S(v_{2k+1} - v_{2k} - z_{2k+1} + z_{2k})\| + \|T(v_{2k+1} - v_{2k} - z_{2k+1} + z_k)\| \\ &\leqq \|Sv_{2k+1}\| + \|Sv_{2k}\| + \|Sz_{2k+1}\| + \|Sz_{2k}\| + \|h_{2k+1}\| + \|h_{2k}\| \\ &\leqq 32\alpha \|U\| + 10\beta \|U\| \leqq 42\alpha \|U\|. \end{aligned} \tag{7}$$

By combining (6) and (7) we get

$$\|U\| \geqq \alpha/(20\beta + 50\alpha^2). \quad \square$$

Theorem 2. *Let* $\alpha_n = (1/2)^{2^n}$, $n = 1, 2, \dots$ *and let* $\{m_k\}_{k=1}^{\infty}$ *and* $\{n_k\}_{k=1}^{\infty}$ *be two increasing sequences of integers. Then* $(\sum_{k=1}^{\infty} \oplus X_{\alpha_{n_k}})_1$ *and* $(\sum_{k=1}^{\infty} \oplus X_{\alpha_{m_k}})_1$ *are isomorphic if and only if the sequences are eventually equal, i.e. if there are integers* $k_0 \geqq 1$ *and* i_0 *so that* $n_k = m_{k+i_0}$ *for every* $k \geqq k_0$. *In particular, there are* $2^{\aleph_0}$ *many isomorphism types among the spaces of the form* $(\sum_{k=1}^{\infty} \oplus X_{\alpha_{n_k}})_1$.

Proof. The "if" assertion is obvious. In order to prove the "only if" assertion it is enough to prove that if N_0 is a subset of the integers so that $n_0 \notin N_0$ then for every

$$U\colon X_{\alpha_{n_0}} \to (\textstyle\sum_{n \in N_0} \oplus X_{\alpha_n})_1, \quad V\colon (\textstyle\sum_{n \in N_0} \oplus X_{\alpha_n})_1 \to X_{\alpha_{n_0}} \tag{$*$}$$

such that $VU =$ identity of $X_{\alpha_{n_0}}$ we have $\|U\| \|V\| \geqq K/\alpha_{n_0}$ where K is an absolute constant.

Let us decompose N_0 into a union $N_0' \cup N_0''$ where $N_0' = \{n \in N_0, n < n_0\}$, $N_0'' = \{n \in N_0, n > n_0\}$. It follows from Proposition 1b that $d(l_1, (\sum_{n \in N_0'} \oplus X_{\alpha_n})_1) \leqq 1/\alpha_{n_0 - 1}$ and hence since $(\sum_{n \in N_0''} \oplus X_{\alpha_n})_1 = Z$ contains a subspace isometric to l_1 onto which there is a projection of norm 1 we deduce that $d(Z, (\sum_{n \in N_0} \oplus X_{\alpha_n})_1) \leqq 4/\alpha_{n_0 - 1}$.

Each X_{α_n} can be represented as $\{(\alpha_n x, T_n x);\ x \in l_1(n)\}$ where $l_1(n)$ is isometric to l_1 and where T_n is a quotient map from $l_1(n)$ onto a space $L_1(0, 1)(n)$ which is isometric to $L_1(0, 1)$. The space $(\sum_{n\in N_0''}\oplus l_1(n))_1$ is isometric to l_1, the space $(\sum_{n\in N_0''}\oplus L_1(0, 1)(n))_1$ is isometric to $L_1(0, 1)$ and the map $T\colon (\sum_{n\in N_0''}\oplus l_1(n))_1 \to (\sum_{n\infty N_0''}\oplus L_1(0, 1)(n))_1$ defined by $T|_{l_1(n)}=T_n$ is a quotient map. Let S be the operator from $(\sum_{n\in N_0''}\oplus l_1(n))_1$ into itself defined by $S|_{l_1(n)}=\alpha_n\cdot$identity. It is clear that

$$Z = \{(Sy, Ty);\ y\in(\textstyle\sum_{n\in N_0''}\oplus l_1(n))_1\}$$

and that $\|S\| \leqq \alpha_{n_0+1}$. Hence, by Proposition 2 for every U and V as in $(*)$ we get

$$\|U\|\,\|V\| \geqq \alpha_{n_0-1}\alpha_{n_0}/4(50\alpha_{n_0}^2+20\alpha_{n_0+1}).$$

The desired result follows now from our choice of the sequence $\{\alpha_n\}_{n=1}^{\infty}$. □

To conclude this paper, let us recall the following result from [4]. If X is a separable $\mathscr{L}_1$ space and if $U\colon l_1\to X$ is a quotient map then ker U is an $\mathscr{L}_1$ space which determines X uniquely (i.e. if $U_1\colon l_1\to X_1$, $U_2\colon l_1\to X_2$ are quotient maps and X_1 and X_2 are $\mathscr{L}_1$ spaces then $X_1\approx X_2$ if and only if $\ker U_1\approx\ker U_2$). Hence from Theorem 2 we can deduce that there are $2^{\aleph_0}$ many mutually non-isomorphic $\mathscr{L}_1$ subspaces of l_1.

Remark. Recently Bourgain, Rosenthal, and Schechtman have constructed uncountably many separable $\mathscr{L}_p$ spaces for $1<p<\infty$, $p\neq 2$. Their work also gives other new information about the structure of L_p; e.g., their examples provide for $2<p<\infty$ the first examples of subspaces of $L_p(0, 1)$ which do not contain isomorphic copies of $L_p(0, 1)$ and yet do not embed into $(l_2\oplus l_2\oplus\ldots)_p$.

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