On extreme operators on finite-dimensional Banach spaces whose unit balls are polytopes

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1. Introduction

The space of all bounded linear operators from a Banach space Y to a Banach space Z is denoted $L(Y, Z)$. X_r denotes the ball $\{x: \|x\| \leqq r\}$ in the space X, and its dual space is written X^*. The set of extreme points of a convex set C is written $\partial_e C$, and the convex hull of a set S, conv (S). Thus $\partial_e L(Y, Z)_1$ denotes the set of extreme operators in the unit ball of $L(Y, Z)$.

l_1^m denotes $\mathbf{R}^m$ with the norm $\|(x_1, \ldots, x_m)\| = \sum_{i=1}^m |x_i|$ and l_∞^m is the dual of l_1^m. The l_1-sum of two spaces X and Y is written $X \oplus_1 Y$ and their l_∞-sum $X \oplus_\infty Y$.

We shall assume that all spaces are real and finite-dimensional.

In [8] J. Lindenstrauss and M. A. Perles studied the set of extreme operators $\partial_e L(X, X)_1$. Their two main theorems are.

Theorem 1.1. *If X is a finite-dimensional Banach space, then the following statements are equivalent:*

(1) $T \in \partial_e L(X, X)_1,\ x \in \partial_e X_1 \Rightarrow Tx \in \partial_e X_1$.

(2) $T_1, T_2 \in \partial_e L(X, X)_1 \Rightarrow T_1 \circ T_2 \in \partial_e L(X, X)_1$.

(3) $\{T_i\}_{i=1}^m \subseteq \partial_e L(X, X)_1 \Rightarrow \|T_1 \circ \ldots \circ T_m\| = 1$ *for all* m.

Theorem 1.2. *Assume* $\dim X \leqq 4$. *If X has properties* (1) *to* (3) *of Theorem* 1.1 *then*

either

(i) *X is an inner product space*

or

(ii) *X_1 is a polytope such that $X_1 = \mathrm{conv}\,(K \cup -K)$ for every maximal proper face K of X_1.*

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In [4] D. Larman showed that Theorem 1.2 is true when $\dim X \leqq 6$. Lindenstrauss and Perles [8] conjectured that Theorem 1.2 is true for all real finite-dimensional spaces.

Every inner product space has properties (1) to (3) of Theorem 1.1 and Lindenstrauss and Perles showed that this is also the case if X satisfies (ii) of Theorem 1.2 and $\dim X \leqq 4$. However, they showed that $X = \{(x_1, \ldots, x_6) \in l_\infty^6 : \sum x_i = 0\}$ satisfies (ii) of Theorem 1.2 but not (1) of Theorem 1.1.

We call X a CL-space if $X_1 = \operatorname{conv}(K \cup -K)$ for every maximal proper face K of X_1. The object of this paper is to prove the following theorem.

Theorem 1.3. *Assume that X is a real finite-dimensional* CL*-space. Then X has properties* (1) *to* (3) *of Theorem* 1.1 *if and only if either X is an l_1-sum of an l_1^m-space and finitely many copies of l_∞^3 or X is an l_∞-sum of an l_∞^m-space and finitely many copies of l_1^3.*

X is said to have the 3.2 intersection property (3.2.I.P.) if whenever $x_1, x_2, x_3 \in X$ are such that $\|x_i - x_j\| \leqq 2$ for all i and j, then there exists $x \in X$ such that $\|x - x_i\| \leqq 1$ for all i. If X has the 3.2.I.P., then X is a CL-space [5]. The CL-spaces appearing in Theorem 1.3 are simple examples of spaces with the 3.2.I.P.

Since the structure of finite-dimensional spaces with the 3.2.I.P. is well known [3], the proof of Theorem 1.3 in case X has the 3.2.I.P. is simple. This is done in Sections 2, 3 and 4. The more difficult part of the proof is to show that no CL-space without the 3.2.I.P. has properties (1) to (3) of Theorem 1.1. This is done in Sections 5 and 6. A main result here is Theorem 5.4 which characterizes CL-spaces without the 3.2.I.P.

2. Sufficient conditions

In this section we shall prove the "if" part of Theorem 1.3. It is an easy corollary of the following theorem.

Theorem 2.1. *Assume that X has the* 3.2.I.P. *and that* $\dim X < \infty$. *Let* $Y = l_1^m \oplus_1 l_\infty^3 \oplus_1 \ldots \oplus_1 l_\infty^3$ *(k copies of l_∞^3). Then $L(Y, X)$ satisfies: If $x \in \partial_e Y_1$ and $T \in \partial_e L(Y, X)_1$, then $Tx \in \partial_e X_1$.*

Corollary 2.2. *If X equals* $l_1^m \oplus_1 l_\infty^3 \oplus_1 \ldots \oplus_1 l_\infty^3$ *or* $l_\infty^m \oplus_\infty l_1^3 \oplus_\infty \ldots \oplus_\infty l_1^3$, *then $L(X, X)$ satisfies* (1) *to* (3) *of Theorem* 1.1.

Since $L(Y, X) = X \oplus_\infty \ldots \oplus_\infty X \oplus_\infty L(l_\infty^3, X) \oplus_\infty \ldots \oplus_\infty L(l_\infty^3, X)$, the theorem follows from Propositions 2.4 and 2.5 below.

We shall need the following characterization of CL-spaces.

Theorem 2.3. *If* $\dim X < \infty$, *then the following statements are equivalent:*
(1) *X is a* CL*-space.*
(2) *X^* is a* CL*-space.*
(3) $e \in \partial_e X_1, f \in \partial_e X_1^* \Rightarrow f(e) = \pm 1$.
(4) *If* $e \in \partial_e X_1$ *and* $x \in X$ *are such that* $\|x\| = 1$ *and* $e \notin \text{face}(x)$, *then* $\|x - e\| = 2$.

The proof can be found in [6] and [7]. Since the proof of (3)⇒(4) is not explicitly given in [6], and this implication is important in Section 5, we shall give the proof here.

Assume $e \in \partial_e X_1$, $\|x\| = 1$ and $e \notin \text{face}(x)$. Let $F = \{f \in X_1^* \colon f(e) = 1\}$. By (2) we get, $X_1^* = \text{conv}(F \cup -F)$. Let $a = \inf\{f(x) \colon f \in F\}$ and $b = \sup\{f(x) \colon f \in F\}$, and define $y = 2^{-1}(a+b)e$. We have $1 = \|x\| = \|y\| + \|x - y\|$. Since $e \notin \text{face}(x)$, we get $a + b \leqq 0$, such that $\|x\| = \sup\{-f(x) \colon f \in F\}$. But then $2 = \|x\| + \|e\| = \sup\{(e-x)(f) \colon f \in F\} = \|e - x\|$.

Proposition 2.4. *If X and Y are finite dimensional, then the following statements are equivalent:*
(1) *$L(X, Y)$ is a* CL*-space.*
(2) *X and Y are* CL*-spaces and* $e \in \partial_e X_1$, $T \in \partial_e L(X, Y)_1 \Rightarrow Te \in \partial_e Y_1$.
(3) *Y is a* CL*-space and* $e \in \partial_e X_1$, $T \in \partial_e L(X, Y)_1 \Rightarrow Te \in \partial_e Y_1$.

Proof. (2)⇒(3) is trivial.

(3)⇒(1). Let F be a maximal proper face of the unit ball of $L(X, Y)$. Then, by Theorem 5.1 in [6], there exist $e \in \partial_e X_1$ and a maximal proper face G of Y_1 such that

$$F = \{T \colon \|T\| \leqq 1 \text{ and } Te \in G\}.$$

Since $Y_1 = \text{conv}(G \cup -G)$, we get from (3) that $Te \in G \cup -G$ for every $T \in \partial_e L(X, Y)_1$. Thus $L(X, Y)_1 = \text{conv}(F \cup -F)$.

(1)⇒(2). Let $e \in \partial_e X_1$ and let G be a maximal proper face of Y_1. Define F by

$$F = \{T \colon \|T\| \leqq 1 \text{ and } Te \in G\}.$$

F is a proper face of the unit ball of $L(X, Y)$. Hence $F \subseteq K$ for some maximal proper face K of $L(X, Y)_1$. By Theorem 5.1 in [6], we have

$$K = \{T \colon \|T\| \leqq 1 \text{ and } Tx \in H\}$$

for some $x \in \partial_e X_1$ and some maximal proper face H of Y_1.

We want to show that, by changing sign of x and H if necessary, $e = x$ and $G = H$. Let $f \in X_1^*$ such that $f(e) = 1$. Choose $y \in G$ and define T by $Tz = f(z)y$. Then $T \in F \subseteq K$. Hence $Tx = f(x)y \in H$. Thus $|f(x)| = 1$ and $y \in H \cup -H$. We may assume $G = H$ and $f \in X_1^*$ and $f(e) = 1 \Rightarrow f(x) = 1$. Now choose $y \in \partial_e H$ and let $h \in \partial_e Y_1^*$ such that $h = 1$ on H. Let $e_n = e - n^{-1}x$ and let $z_n = \|e_n\|^{-1}e_n$. By

Lemma 3.1 in [8], there exist $U_n\in\partial_e L(X, Y)_1$, such that $U_n z_n=y$. Let $g_n=h\cdot U_n\in X_1^*$. By (1), $U_n\in K\cup -K$ for all n. Hence $g_n(x)=\pm 1$ for all n. Since X_1^* is compact, we may assume that $g_n\to g$ in norm. We also have $z_n\to e$. From $g_n(z_n)=1$, we get $g(e)=1$. Hence $g(x)=1$. But then $g_n(x)=1$ for large n. This implies that

$$\begin{aligned} g_n(e) &\leqq \|e\| \\ &\leqq \|n^{-1}x\|+\|e-n^{-1}x\| \\ &= g_n(n^{-1}x)+g_n(e-n^{-1}x) \\ &= g_n(e) \end{aligned}$$

for large n. Thus $\|e\|=\|n^{-1}x\|+\|e-n^{-1}x\|$ for large n, such that $x=e\in\partial_e X_1$. Hence $F=K$ and $\partial_e L(X, Y)_1\subseteqq F\cup -F$.

Let $T\in\partial_e L(X, Y)_1$. Then $Te\in G\cup -G$ for every maximal proper face G of Y_1. Hence $Te\in\partial_e Y_1$.

Next let G be a maximal proper face of Y_1, let $e\in\partial_e X_1$ and let $y\in\partial_e Y_1$. By Lemma 3.1 in [8], there exists $T\in\partial_e L(X, Y)_1$ such that $Te=y$. But by the argument above, $Te=y\in G\cup -G$. Hence $\partial_e Y_1\subseteqq G\cup -G$, and Y is a CL-space.

That also X is a CL-space follows from $L(X, Y)=L(Y^*, X^*)$, Theorem 2.3 and the argument above.

In [3] it was proved that if X has the 3.2.I.P. and $2\leqq\dim X<\infty$, then X contains proper subspaces Y and Z such that $X=Y\oplus_1 Z$ or $X=Y\oplus_\infty Z$ and Y and Z also have the 3.2.I.P. Note that by Proposition 2.4, if $L(l_\infty^3, X)$ is a CL-space, then X is a CL-space.

Proposition 2.5. *Assume X has the* 3.2.I.P. *and* $\dim X<\infty$. *Then $L(l_\infty^3, X)$ is a* CL-*space.*

Proof. The statement is trivially true if $\dim X$ is 1 or 2. Assume that we have proved that the statement is true when $\dim X\leqq n$.

Suppose $\dim X=n+1$. By Theorem 7.3 in [3], there exist proper subspaces Y and Z of X with the 3.2.I.P. and such that $X=Y\oplus_1 Z$ or $X=Y\oplus_\infty Z$. Then the proposition is true for Y and Z.

Case 1. $X=Y\oplus_\infty Z$. Then $L(l_\infty^3, X)=L(l_\infty^3, Y)\oplus_\infty L(l_\infty^3, Z)$. Hence $L(l_\infty^3, X)$ is a CL-space.

Case 2. $X=Y\oplus_1 Z$. We want to show that (3) in Proposition 2.4 is satisfied. Let $T\in\partial_e L(l_\infty^3, X)_1$ and let $x_1=(1, 1, 1)$, $x_2=(1, -1, 1)$, $x_3=(1, -1, -1)$ and $x_4=(1, 1, -1)$. Then $Tx_1+Tx_3=Tx_2+Tx_4$. We want to show that $Tx_i\in\partial_e X_1$.

It is easy to see that we may assume $\|Tx_i\|=1$ for $i=1, 2, 3$. Suppose $\|Tx_4\|<1$. If $Tx_1\notin\partial_e X_1$, then choose $y\in X$, $y\neq 0$, such that $\|Tx_1\pm y\|\leqq 1$ and

$\|Tx_4\|+\|y\|\leqq 1$. Define S by $Sx_1=Sx_4=y$ and $Sx_2=Sx_3=0$. Then $\|T\pm S\|\leqq 1$. Since T is extreme, we have got a contradiction. Hence Tx_1, and similarly Tx_3, are extreme points in X_1. But then

$$\|Tx_1+Tx_3\| = \|Tx_2+Tx_4\|$$

equals 0 or 2. This is impossible since $\|Tx_4\|\neq\|Tx_2\|$. Hence we have $\|Tx_i\|=1$ for all i.

Write

$$H^4(X) = \{(u_1, \dots, u_4)\colon u_i\in X \text{ and } \textstyle\sum u_i = 0\}$$

equipped with the norm $\|(u_1, \dots, u_4)\|=\sum\|u_i\|$. We have

$$(Tx_1, -Tx_2, Tx_3, -Tx_4)\in H^4(X)_4.$$

From Lemma 4.1 [5], we get that there exist $u_1, \dots, u_6$, $v_{kj}\in X$, $\lambda_i\geqq 0$ and $\alpha_j\geqq 0$ such that $1=\sum\lambda_i+\sum\alpha_j$, $(v_{1j}, -v_{2j}, v_{3j}, -v_{4j})\in\partial_e H(X)_4$ and

$$\begin{aligned}
&(Tx_1, \ -Tx_2, \ Tx_3, \ -Tx_4)\\
&= 2\lambda_1(u_1, \ -u_1, \ 0, \ 0\ )\\
&+ 2\lambda_2(u_2, \ 0, \ -u_2, \ 0\ )\\
&+ 2\lambda_3(u_3, \ 0, \ 0, \ -u_3)\\
&+ 2\lambda_4(0, \ u_4, \ -u_4, \ 0\ )\\
&+ 2\lambda_5(0, \ u_5, \ 0, \ -u_5)\\
&+ 2\lambda_6(0, \ 0, \ u_6, \ -u_6)\\
&+\textstyle\sum\alpha_j(v_{1j}, \ -v_{2j}, \ v_{3j}, \ -v_{4j})
\end{aligned}$$

with $1=\|u_i\|=\|v_{kj}\|$ for all i, k and j and $1=\|Tx_1\|=2\lambda_1+2\lambda_2+2\lambda_3+\sum\alpha_j$ and so on for the other columns. We easily get $\lambda_1=\lambda_6$, $\lambda_2=\lambda_5$ and $\lambda_3=\lambda_4$. Define S_i and T_j by $T_jx_k=v_{kj}$ for $k=1, \dots, 4$ and

$$\begin{aligned}
S_1x_1 &= u_1 = S_1x_2, & S_1x_3 &= u_6 = S_1x_4\\
S_2x_1 &= u_2 = -S_2x_3, & S_2x_4 &= u_5 = -S_2x_2\\
S_3x_1 &= u_3 = S_3x_4, & S_3x_2 &= -u_4 = S_3x_3,
\end{aligned}$$

and $S_4=S_3$, $S_5=S_2$ and $S_6=S_1$.

Then $\|S_i\|=1=\|T_j\|$ for all i and j and $T=\sum\lambda_iS_i+\sum\alpha_jT_j$. Since T is extreme, we get $T=S_i$ for some i or $T=T_j$ for some j. If $T=S_i$ for some i, then we easily get that $Tx_i\in\partial_e X_1$ for all i. If $T=T_j$ for some j, then

$$(Tx_1, \ -Tx_2, \ Tx_3, \ -Tx_4)\in\partial_e H^4(X)_4.$$

Let P be the projection in X with $P(X)=Y$ and $\ker P=Z$. Then we get

$$\begin{aligned}&(Tx_1, \quad -Tx_2, \quad Tx_3, \quad -Tx_4)\\&=(PTx_1, \quad -PTx_2, \quad PTx_3, \quad -PTx_4)\\&+((I-P)Tx_1, -(I-P)Tx_2, (I-P)Tx_3, -(I-P)Tx_4)\end{aligned}$$

which gives us a convex combination in $H^4(X)_4$. Hence, we may assume $Tx_i=PTx_i$ for all i. Thus T maps l_∞^3 into Y. By the induction hypothesis and Proposition 2.4, we get $Tx_i\in\partial_e Y_1\subseteqq\partial_e X_1$ for all i. The proof is complete.

Remark. It follows from the proof of Proposition 2.5, that if X is finite dimensional with the 3.2.I.P. and if $(u_1, \ldots, u_4)\in\partial_e H^4(X)_4$ with all $u_i\neq 0$, then $u_i\in\partial_e X_1$ for all i.

3. The spaces $L(Y\oplus_\infty R, Z\oplus_1 R)$.

In this section we begin the study of necessary conditions in Theorem 1.3. The results obtained here will be used in the following sections. The main result in this section is the following theorem.

Theorem 3.1. *Assume X and Y are finite-dimensional* CL*-spaces. Suppose there exist projections P in X and Q in Y such that $P(\partial_e X_1)\subseteqq\partial_e X_1$ and $P(X)=l_1^3\oplus_\infty R$ or l_∞^4 and $Q(\partial_e Y_1)\subseteqq\partial_e Y_1$ and $Q(Y)=l_\infty^3\oplus_1 R$ or l_1^4. Then there exist $T\in\partial_e L(X, Y)_1$ and $x\in\partial_e X_1$ such that $Tx\notin\partial_e Y_1$.*

Before we give the proof, we shall prove some special cases. These are contained in the lemmas 3.2, 3.3 and 3.4.

Lemma 3.2. *Let $X=l_\infty^4$ and $Y=l_1^4$. Then there exist $T\in\partial_e L(X, Y)_1$ and $x\in\partial_e X_1$ such that $Tx\notin\partial_e Y_1$.*

Proof. It is easy to see that l_1^k is not a quotient space of any l_∞^n-space when $k\geqq 3$. Thus we get that if $T\in\partial_e L(X, Y)_1$ is such that $Tx\in\partial_e Y_1$ for every $x\in\partial_e X_1$, then $\dim T(X)=1$ or 2. Define $S\in L(X, Y)_1$ by the matrix

$$S=6^{-1}\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & -2 & 1 & 0\\ 1 & 1 & 1 & -1\\ 1 & 0 & -1 & 0\end{pmatrix}.$$

Then $\|Sx\|=1$ for every $x\in\partial_e X_1$. Let $T\in\partial_e\,\mathrm{face}\,(S)$ and assume $Tx\in\partial_e Y_1$ for every $x\in\partial_e X_1$. Then $\dim T(X)=1$ or 2. Let $e_1, \ldots, e_4$ be the natural basis for $Y=l_1^4$. Since $3S(1, -1, 1, -1)=(0, 2, 1, 0)$, we get $T(X)\subsetneqq\mathrm{span}\,(e_1, e_4)$. Similarly, $3S(1, 1, 1, -1)=(1, 0, 2, 0)$ and $3S(1, -1, 1, 1)=(1, 2, 0, 0)$ gives that

$T(X) \nsubseteq \text{span}(e_2, e_4)$ or in span (e_3, e_4). Thus we only have to consider the cases 1, 2 and 3 below.

Case 1. $T(X) \subseteqq \text{span}(e_1, e_2)$.

Since $3S(1, -1, 1, -1) = (0, 2, 1, 0)$, we get $T(1, -1, 1, -1) = e_2$. Similarly, we get $T(1, 1, -1, -1) = -e_2$ and $T(1, 1, 1, -1) = e_1$. We also have $T(1, -1, -1, 1) = e_2$. Thus using that $(1, -1, -1, 1) + (1, -1, 1, -1) = (1, -1, -1, -1) + (1, -1, 1, 1)$ we get $-e_1 = T(1, -1, -1, -1) = e_2$. This is a contradiction.

Case 2. $T(X) \subseteqq \text{span}(e_1, e_3)$

and

Case 3. $T(X) \subseteqq \text{span}(e_2, e_3)$ are treated similarly. Hence, we get that for every $T \in \partial_e \text{face}(S)$, there exists $x \in \partial_e X_1$ such that $Tx \notin \partial_e Y_1$.

Remark. S. Kaijser has shown that the matrix

$$A = 8^{-1} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 2 & -1 & 1 & 0 \\ 0 & 1 & -1 & -2 \\ 1 & 0 & -2 & 1 \end{pmatrix}$$

has the same property as the matrix S.

Lemma 3.3. *Let $X = l_1^3 \oplus_\infty R$ and let $Y = l_1^4$. Then there exist $T \in \partial_e L(X, Y)_1$ and $x \in \partial_e X_1$ such that $Tx \notin \partial_e Y_1$.*

Proof. Let $x_1 = (1, 0, 0, 1)$, $x_2 = (0, 1, 0, 1)$, $x_3 = (0, 0, 1, 1)$, $y_1 = (-1, 0, 0, 1)$, $y_2 = (0, -1, 0, 1)$ and $y_3 = (0, 0, -1, 1)$. Then $x_i + y_i = (0, 0, 0, 2)$ for all i. Define $T \in L(X, Y)_1$ by

$$2Tx_1 = (1, 1, 0, 0), \quad 2Tx_2 = (0, 1, 0, 1) \cdot$$

$$2Tx_3 = (1, 0, 0, 1), \quad 2Ty_1 = (0, 0, 1, 1).$$

Then

$$2Ty_2 = (1, 0, 1, 0), \quad 2Ty_3 = (0, 1, 1, 0).$$

Let $S \in \partial_e \text{face}(T)$. Assume for contradiction that $x \in \partial_e X_1 \Rightarrow Sx \in \partial_e Y_1$. We have to consider four cases.

Case 1. $Sx_1 = e_1$ and $Sy_1 = e_4$.

Since $\text{span}(x_1, y_1, x_3, y_3) = l_\infty^3$ and $Y = l_1^4$, we get $Sx_3, Sy_3 \in \text{span}(e_1, e_4)$. (See the beginning of the proof of Lemma 3.2.) This is impossible.

The cases 2) $Sx_1 = e_1$, $Sy_1 = e_3$, 3) $Sx_1 = e_2$, $Sy_1 = e_3$ and 4) $Sx_1 = e_2$, $Sy_1 = e_4$ are treated similarly. Hence we have shown that for every $S \in \partial_e \text{face}(T)$, there exists $x \in \partial_e X_1$ such that $Sx \notin \partial_e Y_1$.

Considering T^* we get:

Corollary 3.4. *Let $X=l_\infty^4$ and let $Y=l_\infty^3 \oplus_1 R$. Then there exist $T\in\partial_e L(X, Y)_1$ and $x\in\partial_e X_1$ such that $Tx\notin\partial_e Y_1$.*

Lemma 3.5. *Let $X=l_1^3 \oplus_\infty R$ and let $Y=l_\infty^3 \oplus_1 R$. Then there exist $T\in\partial_e L(X, Y)_1$ and $x\in\partial_e X_1$ such that $Tx\notin\partial_e Y_1$.*

Proof. Let $x_i, y_i\in\partial_e X_1$ be as in Lemma 3.2. Let $e_1=(1, -1, -1, 0)$, $e_2=(1, 1, -1, 0)$, $e_3=(1, 1, 1, 0)$, $e_4=(1, -1, 1, 0)$ and $e_5=(0, 0, 0, 1)$. $e_i\in\partial_e Y_1$ and $e_1+e_3=e_2+e_4$. Define $T\in L(X, Y)_1$ by $2Tx_1=e_2+e_5$, $2Tx_2=e_1+e_5$, $Tx_3=e_5$ and $2Ty_1=e_4+e_5$. Then $2Ty_2=e_3+e_5$ and $2Ty_3=e_1+e_3$.

Let $T_i\in\partial_e L(X, Y)_1$ and let $\lambda_i>0$ such that $\sum\lambda_i=1$ and $T=\sum\lambda_i T_i$ where i runs from 1 to some integer p.

Assume for contradiction that $S\in\partial_e L(X, Y)_1$, $x\in\partial_e X_1 \Rightarrow Sx\in\partial_e Y_1$. Then some T_i, say T_1, satisfy $T_1 y_1=e_5$. We have $T_1 x_3=e_5$. Using that $x_1+y_1=x_3+y_3$ and $T_1 y_3\neq e_5$, we get $T_1 x_1=e_2$. But then $T_1 x_2\in\{e_2, e_5\}\cap\{e_1, e_5\}=\{e_5\}$. Similarly $T_1 y_2=e_5$. But then, since $T_1 x_1+T_1 y_1=T_1 y_2+T_1 x_2$, we have obtained a contradiction.

Proof of Theorem 3.1. By lemmas 3.2, 3.3, 3.4 and 3.5, we know that there exists a $T\in\partial_e L(P(X), Q(Y))_1$ such that $Tx\notin\partial_e Q(Y)_1$ for some $x\in\partial_e P(X)_1$. $\|T\cdot P\|=1$. Hence we can find $T_i\in\partial_e L(X, Y)_1$ and $\lambda_i>0$ such that $\sum\lambda_i=1$ and $T\cdot P=\sum\lambda_i T_i$ $(i=1, \dots, p)$. But then we get $T=\sum\lambda_i Q\cdot T_i$. Since T is extreme, this implies that $T=Q\cdot T_1$. Hence $T_1 x\notin\partial_e Y_1$ for some $x\in\partial_e P(X)_1\subseteq\partial_e X_1$. The proof is complete

Corollary 3.6. *Assume X and Y are finite-dimensional* CL*-spaces. If there exist isometries $T\colon l_\infty^4\to X$ and $S\colon l_\infty^4\to Y^*$ such that $x\in\partial_e(l_\infty^4)_1\Rightarrow Tx\in\partial_e X_1$ and $Sx\in\partial_e Y_1^*$, then there exist $U\in\partial_e L(X, Y)_1$ and $x\in\partial_e X_1$ such that $Ux\notin\partial_e Y_1$.*

Proof. $S^*\colon Y\to l_1^4$ is a quotient map such that $S^*(\partial_e Y_1)=\partial_e(l_1^4)_1$. Hence there is a projection Q in Y such that $Q(Y)=l_1^4$ and $Q(\partial_e Y_1)\subseteq\partial_e Y_1$.

Let P be a projection in X such that $\|P\|=1$ and $P(X)=T(l_\infty^4)$. Note that the properties of P that we used in the proof of Theorem 3.1 was $\|P\|=1$ and $\partial_e P(X)_1\subseteq\partial_e X_1$. The proof is complete.

Remark. If we assume in Corollary 3.6 that X and Y have the 3.2.I.P., we get, using Proposition 2.5 and Lemma 6.4, that the corollary is true if we replace l_∞^4 one or both places with $l_1^3\oplus_\infty R$.

Even though we don't need the next result, it is typical for the situation so we include a proof.

Theorem 3.7. *Let Y and Z be finite-dimensional spaces with* 3.2.I.P.
Then

(1) $$T\in\partial_e L(Y\oplus_\infty R, Z\oplus_1 R)_1,\quad x\in\partial_e(Y\oplus_\infty R)_1\Rightarrow Tx\in\partial_e(Z\oplus_1 R)_1$$

if and only if

(2) $$\min(\dim Y, \dim Z)\leqq 2.$$

Proof. (2)⇒(1) easily follows from Theorem 2.1.

Assume next that $\dim Y\geqq 3$ and $\dim Z\geqq 3$. If Y is not an l_1^m-space, then there exists an isometry $T\colon l_\infty^3\to Y$ [5; Theorem 4.3]. Let $S\in\partial_e\operatorname{face}(T)$. Then S is an isometry and by Proposition 2.5, $S(\partial_e(l_\infty^3)_1)\subseteq\partial_e Y_1$. Hence there exists an isometry $U\colon l_\infty^4\to Y\oplus_\infty R$ such that $U(\partial_e(l_\infty^4)_1)\subseteq\partial_e(Y\oplus_\infty R)_1$.

Similarly, either Z is an l_∞^n-space or there exists an isometry $V\colon l_\infty^4\to Z^*\oplus_\infty R$ such that $V(\partial_e(l_\infty^4)_1)\subseteq\partial_e(Z^*\oplus_\infty R)_1$. Now (1)⇒(2) follows from Corollary 3.6 and Theorem 3.1.

4. Necessary conditions when X has the 3.2.I.P.

In this short section we shall prove Theorem 4.1.

Theorem 4.1. *Assume X is finite-dimensional with the* 3.2.I.P. *If X satisfy* (1) *$T\in\partial_e L(X,X)_1$, $x\in\partial_e X_1\Rightarrow Tx\in\partial_e X_1$ then X is isometric to $l_1^m\oplus_1 l_\infty^3\oplus_1\dots\oplus_1 l_\infty^3$ or $l_\infty^m\oplus_\infty l_1^3\oplus_\infty\dots\oplus_\infty l_1^3$ (k copies of l_∞^3 or l_1^3) where $m, k\in\{0, 1, 2, \dots\}$ and* $\dim X=m+3k$.

Proof. By Corollary 3.6, we may assume that there does not exist an isometry $T\colon l_\infty^4\to X$ such that $T(\partial_e(l_\infty^4)_1)\subseteq\partial_e X_1$. We can assume $\dim X\geqq 3$. By Theorem 7.3 in [3] we can write

$$X=l_1^m\oplus_1(Y_1\oplus_\infty Z_1)\oplus_1\dots\oplus_1(Y_p\oplus_\infty Z_p)$$

where $\dim Y_i\geqq\dim Z_i\geqq 1$ and $\dim Y_i+\dim Z_i\geqq 3$ for all i. (We can have $m=0$ or $p=0$.) If $p=0$, there is nothing to prove. So assume $p\geqq 1$.

If one Y_i or Z_i is not an l_1^n-space, then as in the proof of Theorem 3.7, we find an isometry $T\colon l_\infty^4\to Y_i\oplus_\infty Z_i$ such that $T(\partial_e(l_\infty^4)_1)\subseteq\partial_e(Y_i\oplus_\infty Z_i)_1$. This contradicts our assumption above. Hence all Y_i and Z_i are l_1^n-spaces. Using that $l_1^2=l_\infty^2$, we similarly get that $Z_i=R$ for every i. Hence $Y_i\oplus_\infty Z_i=l_1^{k_i}\oplus_\infty R$ for some $k_i\geqq 2$ and all i.

Looking at the lemmas in Section 3 or at Theorem 3.7, it is clear that we cannot have $\dim Y_i>2$ for some i together with $p>1$ or $m\neq 0$. Thus we have either $p>1$ or $m\neq 0$ and then $X=l_1^m\oplus_1 l_\infty^3\oplus_1\dots\oplus_1 l_\infty^3$ or $p=1$ and $m=0$ and $X=l_1^k\oplus_\infty R$ with $k\geqq 3$. In the last case, it follows from Theorem 3.1 that $k\leqq 3$. Thus $X=l_1^3\oplus_\infty R$.

The proof is complete.

5. CL-spaces without the 3.2.I.P.

We shall now characterize CL-spaces without the 3.2.I.P. These results will then be used in the next section where we show that no CL-space without the 3.2.I.P. satisfy (1) in Theorem 1.1.

CL-spaces were characterized in Theorem 2.3. The first result here is well known and its proof can be found in [6] [2].

Theorem 5.1. *The following statements are equivalent:*
(1) X *has the* 3.2.I.P.
(2) X^* *has the* 3.2.I.P.
(3) *If* F *and* G *are disjoint faces of* X_1, *then there exists* $f\in\partial_e X_1^*$ *such that* $f=1$ *on* F *and* $f=-1$ *on* G.

If we compare (3) of Theorem 5.1 with (4) of Theorem 2.3, we see that every CL-space without the 3.2.I.P. contains a pair of disjoint faces F and G of X_1 such that no $f\in X_1^*$ is 1 on F and -1 on G and both F and G consists of more than one point.

Lemma 5.2. *Let* X *be a finite-dimensional* CL-*space without the* 3.2.I.P. *Then there exist a face* N *of* X_1 *and* $x_1, x_2\in\partial_e X_1$ *such that if* $F=\text{face}\left(\frac{x_1+x_2}{2}\right)$, *then* $N\cap F=\emptyset$, *but no* $f\in X_1^*$ *satisfy* $f=1$ *on* N *and* $f=-1$ *on* F.

Proof. By the discussion above and since $\dim X<\infty$, it follows that there exists a minimal face F of X_1 such that there exists a face N of X_1 with the properties: $N\cap F=\emptyset$ and no $f\in X_1^*$ is 1 on N and -1 on F.

Write $N=\text{face}\,(y)$ and let $x_1\in\partial_e F$. By Theorem 2.3, we get that $G=\text{face}\left(\frac{y-x_1}{2}\right)$ is a proper face of X_1. Write $F=\text{face}\,(x)$ and choose $\alpha\in\langle 0, 1]$ and $z\in F$ such that $x=\alpha x_1+(1-\alpha)z$. By choosing α as large as possible, we get $x_1\notin\text{face}\,(z)$. As noted above, F is not a point. Hence $\alpha<1$. Clearly $\text{face}\,(z)$ is a proper subface of F.

If $\text{face}\,(z)\cap G=\emptyset$, then by the minimality of F, there exists $g\in\partial_e X_1^*$ such that $g=1$ on G and $g(z)=-1$. But then $g=1$ on N and $g=-1$ on F. This contradiction shows that $\text{face}\,(z)\cap G\neq\emptyset$. Choose $x_2\in\partial_e G\cap\text{face}\,(z)$, and define $H=\text{face}\left(\frac{x_1+x_2}{2}\right)\subseteqq F$. Then no $f\in X_1^*$ satisfy $f=1$ on N and $f=-1$ on H. Hence by the minimality of F, we get $F=H$. The proof is complete.

Before we proceed to get better characterizations of CL-spaces without the 3.2.I.P. we need a lemma.

Lemma 5.3. *Assume X is a finite-dimensional* CL-*space. Let $p \geqq 2$ and let $y_1, \dots, y_p \in \partial_e X_1$ be such that $F = \text{face}\,(p^{-1}(y_1 + \dots + y_p))$ is a proper face of X_1. If $x_1 \in \partial_e F$, then there exist $x_2, \dots, x_p \in \partial_e F$ such that*

$$y_1 + \dots + y_p = x_1 + \dots + x_p.$$

Proof. There exist $\alpha_1 \in \langle 0, 1]$ and $u_1 \in F$ such that

$$p^{-1}(y_1 + \dots + y_p) = \alpha_1 x_1 + (1 - \alpha_1) u_1.$$

By taking α_1 as large as possible, we get $x_1 \notin \text{face}\,(u_1)$. By Theorem 2.3 there exists $f_1 \in \partial_e X_1^*$ such that $f_1(x_1) = 1$ and $f_1(u_1) = -1$. By Theorem 2.3, we then get

$$\begin{aligned} 2\alpha_1 - 1 &= f_1(\alpha_1 x_1 + (1 - \alpha_1) u_1) \\ &= p^{-1} f_1(y_1 + \dots + y_p) \\ &\in \{1, p^{-1}(p-2), p^{-1}(p-4), \dots, -1\}. \end{aligned}$$

Hence $\alpha_1 \in \{0, p^{-1}, 2p^{-1}, \dots, 1\}$. Thus we can write $\alpha_1 = p^{-1} k_1$ where k_1 is some integer $\geqq 1$. We now have

$$p^{-1}(y_1 + \dots + y_p) = p^{-1} k_1 x_1 + (1 - p^{-1} k_1) u_1.$$

If $u_1 \notin \partial_e F$, then we can choose $x_2 \in \partial_e\, \text{face}\,(u_1)$, $\alpha_2 \in \langle 0, 1 - p^{-1} k_1]$ and $u_2 \in F$ such that

$$p^{-1}(y_1 + \dots + y_p) = p^{-1} k_1 x_1 + \alpha_2 x_2 + (1 - p^{-1} k_1 - \alpha_2) u_2.$$

Choosing α_2 as large as possible, we get $x_2 \notin \text{face}\,(u_2)$. Again using Theorem 2.3 we find $f_2 \in \partial_e X_1^*$ such that $f_2(x_2) = 1$ and $f_2(u_2) = -1$. As in the case with α_1, we find $\alpha_2 = p^{-1} k_2$ where k_2 is some integer $\geqq 1$. Hence

$$y_1 + \dots + y_p = k_1 x_1 + k_2 x_2 + (p - k_1 - k_2) u_2.$$

Proceeding in this manner, we find $x_2, \dots, x_q \in \partial_e F$ and integers $k_1, \dots, k_q \geqq 1$ such that $k_1 + \dots + k_q = p$ and

$$y_1 + \dots + y_p = k_1 x_1 + k_2 x_2 + \dots + k_q x_q.$$

The proof is complete.

The next result is a main theorem in this section. It characterizes CL-spaces without the 3.2.I.P. This theorem together with Theorem 3.1 will be used in the following to show that such spaces cannot satisfy (1) of Theorem 1.1.

Theorem 5.4. *Assume X is a finite-dimensional* CL-*space without the* 3.2.I.P. *Then there exist an integer $p \geqq 2$, a maximal proper face K of X_1 and extreme points $y_1, \dots, y_p, x_1, x_2, z_1, \dots, z_p \in \partial_e K$ such that*

$$y_1 + \dots + y_p + x_1 = x_2 + z_1 + \dots + z_p.$$

Moreover, if $N=\operatorname{face}(p^{-1}(y_1+\ldots+y_p))$, $M=\operatorname{face}(p^{-1}(z_1+\ldots+z_p))$ *and* $F=\operatorname{face}(2^{-1}(x_2-x_1))$, *then* $M\cap N=\emptyset$, $N\cap F=\emptyset$ *and* $-F\cap M=\emptyset$.

Proof. By Lemma 5.2, there exist a minimal integer p such that: There exist $y_1, \ldots, y_p, x_1, x_2 \in \partial_e X_1$ such that if $N=\operatorname{face}(p^{-1}(y_1+\ldots+y_p))$ and

$$F = \operatorname{face}(2^{-1}(x_1+x_2)),$$

then $N\cap F=\emptyset$, but no $f\in X_1^*$ is 1 on F and -1 on F. Then as in the proof of Lemma 5.2, we see that if $G=\operatorname{face}((p+1)^{-1}(y_1+\ldots+y_p-x_1))$, then G is a proper face of X_1 and $x_2\in G$. Let K be a maximal proper face of X_1 such that $G\subseteqq K$.

By Lemma 5.3 there exist $z_1, \ldots, z_p \in \partial_e K$ such that

$$y_1+\ldots+y_p-x_1 = x_2+z_1+\ldots+z_p.$$

Let $M=\operatorname{face}(p^{-1}(z_1+\ldots+z_p))$. If there exists $u\in\partial_e M\cap\partial_e N$, then by Lemma 5.3, we can find $a_i\in\partial_e N$ and $b_i\in\partial_e M$ such that

$$y_1+\ldots+y_p = u+a_1+\ldots+a_{p-1}$$

and

$$z_1+\ldots+z_p = u+b_1+\ldots+b_{p-1}.$$

Hence

$$a_1+\ldots+a_{p-1}-x_1 = x_2+b_1+\ldots+b_{p-1}.$$

Clearly $F\cap\operatorname{face}((p-1)^{-1}(a_1+\ldots+a_{p-1}))=\emptyset$, but no $f\in\partial_e X_1^*$ is 1 on one of these faces and -1 on the other. Thus we have got a contradiction to the minimality of p.

Hence $N\cap M=\emptyset$.

If two z_i are equal, say $z_1=z_2$, then by Theorem 2.3, there exists $f\in\partial_e X_1^*$ such that $f=1$ on N and $f(z_1)=-1$. But then

$$p-1 \leqq f(y_1+\ldots+y_p-x_1) = f(x_2+z_1+\ldots+z_p) \leqq p-3.$$

This contradiction shows that all z_i are different.

If there exists $z\in\partial_e M\cap(-F)$, then by Lemma 5.3 we can write

$$-x_1-x_2 = z+u$$

and

$$z_1+\ldots+z_p = z+b_1+\ldots+b_{p-1}$$

where u, $b_1, \ldots, b_{p-1}\in\partial_e K$. But then

$$y_1+\ldots+y_p = b_1+\ldots+b_{p-1}-u$$

such that $N\cap M\neq\emptyset$. This contradiction shows that $M\cap(-F)=\emptyset$. To conclude the proof, we only have to replace x_1 by $-x_1$.

It is a consequence of Theorem 5.4 that a CL-space without the 3.2.I.P. has dimension $\geqq 5$. In fact, $\dim X \geqq 2p+1$.

Proposition 2.5 says that if X has the 3.2.I.P., then $L(l^3_\infty, X)$ is a CL-space. If X does not have the 3.2.I.P., then we get the following corollary.

Corollary 5.5. *If $L(l^3_\infty, X)$ is a* CL-*space, then the integer p in Theorem* 5.4 *is* 2.

Proof. We use the notation of Theorem 5.4. We have

$$(y_1+\ldots+y_p)+px_1 = (x_2+(p-1)x_1)+(z_1+\ldots+z_p).$$

Define $T\colon l^3_\infty \to X$ by $pT(1,1,1)=y_1+\ldots+y_p$, $T(1,-1,-1)=x_1$, $pT(1,-1,1)=x_2+(p-1)x_1$ and $pT(1,1,-1)=z_1+\ldots+z_p$. Then $\|T\|=1$. Choose $S\in\partial_e\,\mathrm{face}\,(T)$. Then $y=S(1,1,1)\in\partial_e N$, $z=S(1,1,-1)\in\partial_e M$, $x_1=S(1,-1,-1)$ and $x_3=S(1,-1,1)\in\partial_e\,\mathrm{face}\left(\frac{x_1+x_2}{2}\right)$ and $x_1+y=x_3+z$. By Lemma 5.3, there exist extreme points x_4, a_i and b_i such that

$$y_1+\ldots+y_p = y+a_1+\ldots+a_{p-1}$$
$$z_1+\ldots+z_p = z+b_1+\ldots+b_{p-1}$$

and

$$x_1+x_2 = x_3+x_4.$$

Hence

$$a_1+\ldots+a_{p-1}+x_3 = x_2+b_1+\ldots+b_{p-1}.$$

Repeating this procedure, we find $a\in\partial_e N$, $b\in\partial_e M$ and $x_5\in\partial_e\,\mathrm{face}\left(\frac{x_2+x_3}{2}\right)$ such that

$$a+x_3 = x_5+b.$$

But then

$$(y+a)+x_1 = x_5+(z+b)$$

and $\mathrm{face}\left(\frac{y+a}{2}\right)\subseteqq N$ and $\mathrm{face}\left(\frac{z+b}{2}\right)\subseteqq M$. However, these smaller faces satisfy the requirement of Lemma 5.2. Looking at the definition of p, we now see that $p=2$.

There are some CL-spaces such that $L(l^3_\infty, X)$ is not a CL-space. An example of this is the quotient-space $X=l^6_1/U$ where $U=\mathrm{span}\,\{(1,1,1,1,1,1)\}$. In this space, we have that if $x, y\in\partial_e X_1$ with $x+y\neq 0$, then $\mathrm{conv}\,(x,y)$ is a face of X_1. It is easy to see that for every CL-space with this property, we have that $L(l^3_\infty, X)$ is not a CL-space. (In fact, if y, z and x_3 are as in the proof of Corollary 5.5, then $x_1+y=x_3+z$ implies that $y=z\in M\cap N$ or $z=x_1\in M\cap(-F)$. Both are false.)

If X is a CL-space and X is not an l^n_1-space, then it follows from Propositions I.6.11 and II.3.16 in [1] that we can define an integer $p(X)$ as follows: $p=p(X)$ is the smallest integer such that there exist $x_1, \ldots, x_p\in\partial_e X_1$ with the following properties:

(1) $F=\mathrm{face}\,(p^{-1}(x_1+\ldots+x_p))$ is a proper face of X_1

(2) $F\neq\mathrm{conv}\,(x_1, \ldots, x_p)$.

Proposition 5.6. *Let X be a finite-dimensional* CL-*space and assume X is not an l_1^n-space. Then $p=p(X)\geqq 2$ and there exist $x_1, \ldots, x_p, y_1, \ldots, y_p \in \partial_e X_1$ (all different) such that*

$$x_1+\ldots+x_p = y_1+\ldots+y_p$$

and $F=\text{face}\,(p^{-1}(x_1+\ldots+x_p))$ is a proper face of X_1. Moreover, if $z_1, \ldots, z_{p-1} \in \partial_e X_1$ are such that $z_i+z_j \neq 0$ for all i and j, then $\operatorname{conv}(z_1, \ldots, z_{p-1})$ is a proper face of X_1.

The example $X=l_1^6/U$ above satisfy $p(X)=3$. Clearly $L(l_\infty^3, X)$ is a CL-space$\Rightarrow$ $p(X)=2$.

Proof. The statement about $z_1, \ldots, z_{p-1}$ easily follows from the definition of $p(X)$ and Theorem 2.3. The existence of $x_1, \ldots, x_p, y_1, \ldots, y_p$ follows from the definition of $p(X)$ and Lemma 5.3. The proof is complete.

6. Necessary conditions when X is a CL-space

In this section we shall prove that if a CL-space X satisfy (1) of Theorem 1.1, then X has the 3.2.I.P. and we can apply Theorem 4.1. We shall assume that X is not an l_1^n-space such that the number $p(X)$ is defined. (See the text following the proof of Corollary 5.5.) First we show that if X satisfy (1) of Theorem 1.1 then $p(X)=p(X^*)=2$. From this we deduce that $L(l_\infty^3, X)$ and $L(l_\infty^3, X^*)$ are CL-spaces. Using this information we show that if X does not have the 3.2.I.P., then we have a situation like the one decribed in Theorem 3.1.

Theorem 6.1. *Assume X is a finite-dimensional* CL-*space with $p(X)\geqq 3$. Then there exist $T\in\partial_e L(X, X)_1$ and $x\in\partial_e X_1$ such that $Tx\notin\partial_e X_1$.*

Proof. Suppose first that $p(X)$ is an even number, say $p(X)=2k$ where $k\geqq 2$. Let F and $x_1, \ldots, x_{2k}, y_1, \ldots, y_{2k}\in\partial_e F$ be as in Proposition 5.6. By Proposition 5.6, $\operatorname{conv}(x_1, \ldots, x_k, -y_{k+1}, \ldots, -y_{2k-1})$ is a proper face of X_1 not containing y_{2k}. Then there exists, by Theorem 2.3, a $f_2\in\partial_e X_1^*$ such that $f_2(x_i)=1$ and $f_2(y_{k+i})=-1$ for $i=1, \ldots, k$. But then by the equality in Proposition 5.6, $f_2=1$ on $x_1, \ldots, x_k, y_1, \ldots, y_k$ and $f_2=-1$ on $x_{k+1}, \ldots, x_{2k}, y_{k+1}, \ldots, y_{2k}$.

Similarly there exists $f_3\in\partial_e X_1^*$ such that $f_3=1$ on $x_1, \ldots, x_k, y_2, \ldots, y_{k+1}$ and $f_3=-1$ on $x_{k+1}, \ldots, x_{2k}, y_1, y_{k+2}, \ldots, y_{2k}$.

Since F is a proper face of X_1^*, there exists $f_1\in\partial_e X_1^*$ such that $f_1=1$ on F.

Define a map T: $X\to l_\infty^3$ by $T(x)=(f_1(x), f_2(x), f_3(x))$. Define $a_1, a_2, b_1, b_2\in F$ by $ka_1=x_1+\ldots+x_k$, $ka_2=x_{k+1}+\ldots+x_{2k}$, $kb_1=y_1+\ldots+y_k$ and $kb_2=y_{k+1}+\ldots+y_{2k}$. Define a map S: $l_\infty^3\to X$ by $S(1, 1, 1)=a_1$, $S(1, -1, -1)=a_2$, $S(1, 1, -1)=b_1$ and $S(1, -1, 1)=b_2$. Then $\|S\cdot T\|=1$ and $S\cdot T(y_1)=b_1$, $S\cdot T(y_{k+1})=b_2$, $S\cdot T(x_i)=$

$S\cdot T(y_j)=a_1$ for $i=1, ..., k$ and $j=2, ..., k$ and $S\cdot T(x_i)=S\cdot T(y_j)=a_2$ for $i=k+1, ..., 2k$ and $j=k+2, ..., 2k$.

Let $U\in\partial_e \operatorname{face}(S\cdot T)$ and assume for contradiction that $x\in\partial_e X_1 \Rightarrow Ux\in\partial_e X_1$. Then we get $Uy_1\in\{y_1, ..., y_k\}$, $Uy_{k+1}\in\{y_{k+1}, ..., y_{2k}\}$ and $Ux\in\{x_1, ..., x_{2k}\}$ when $x\in\{x_1, ..., x_{2k}, y_2, ..., y_k, y_{k+2}, ..., y_{2k}\}$. Moreover

$$Ux_1+...+Ux_{2k} = Uy_1+...+Uy_{2k}.$$

Either $Uy_2=Ux_i$ for some i or else card $\{Ux_1, ..., Ux_{2k}\}<p(X)$. Thus Uy_1 is in a face generated by less than $p(X)$ extreme points. This contradicts the definition of $p(X)$. Hence $Ux\notin\partial_e X_1$ for at least one $x\in\partial_e F$.

Suppose next that $p(X)$ is an odd number, say $p=p(X)=2k-1$ where $k\geqq 2$. In this case we have to make a small change in the proof above. We start with $x_1, ..., x_p, y_1, ..., y_p$ and F as in Proposition 5.6. Then we write $x_{k+1}=x_{2k}=y_{2k}$. The proof is complete.

Corollary 6.2. *Assume X is a finite-dimensional* CL*-space. If X satisfies* (1) $T\in\partial_e L(X, X)_1, x\in\partial_e X_1\Rightarrow Tx\in\partial_e X_1$, *then both $L(l_\infty^3, X)$ and $L(l_\infty^3, X^*)$ are* CL*-spaces. Moreover, if X is not an l_1^n- or an l_∞^n-space, then $p(X)=p(X^*)=2$.*

Proof. If X is an l_1^n- or an l_∞^n-space, then there is nothing to prove. Hence, we can assume $p(X)$ and $p(X^*)$ are defined. By Theorem 6.1 we get $p(X)=p(X^*)=2$ since X satisfies (1) if and only if X^* satisfies (1).

Assume $p(X)=2$, and let x_1, x_2, y_1, y_2 and F be as in Proposition 5.6. Let $Y=\operatorname{span}(x_1, x_2, y_1, y_2)$. Then $Y=l_\infty^3$. Hence there exists a projection P in X such that $P(X)=Y$ and $\|P\|=1$. Now we proceed as in the proof of Theorem 3.1 to show that if $T\in\partial_e L(l_\infty^3, X)_1$ and $x\in\partial_e(l_\infty^3)_1$, then $Tx\in\partial_e X_1$. Similarly, we show that $L(l_\infty^3, X^*)$ is a CL-space.

The next proposition is a step in order to establish a situation in which we can apply Theorem 3.1.

Proposition 6.3. *Assume that X is a finite-dimensional* CL*-space without the* 3.2.I.P. *and that $L(l_\infty^3, X)$ is a* CL*-space. Then there exists an isometry $T\colon l_1^3\oplus_\infty R\to X$ such that $Tx\in\partial_e X_1$ for all $x\in\partial_e(l_1^3\oplus_\infty R)_1$.*

Proof. By Corollary 5.5, the integer p in Theorem 5.4 is 2. Let $y_1, y_2, x_1, x_2, z_1, z_2, M, N$ and F be as in Theorem 5.4. Then we have

$$(y_1+y_2)+2x_1 = (x_1+x_2)+(z_1+z_2).$$

Using the argument we used in the proof of Corollary 5.5, we find $y\in\partial_e N$, $z\in\partial_e M$ and $x_3\in\partial_e \operatorname{face}\left(\frac{x_1+x_2}{2}\right)$ such that

$$y+x_1 = x_3+z.$$

By Lemma 5.3, we may assume $y=y_1$ and $z=z_1$. Thus

$$y_1+x_1 = x_3+z_1$$

$$y_2+x_3 = x_2+z_2$$

and

$$x_1+x_2 = x_3+x_4$$

for some $x_4\in\partial_e X_1$. Since $N\cap M=\emptyset$, we get $x_3\neq x_1$ and $x_3\neq x_2$. Moreover $M\cap N=\emptyset$ implies that $x_3\notin M\cup N$.

Define $T\colon l_1^3\oplus_\infty R\to X$ by $T(1,0,0,1)=x_2$, $T(0,1,0,1)=x_3$, $T(0,0,1,1)=y_1$ and $T(0,1,0,-1)=x_1$. Then $T(1,0,0,-1)=x_4$ and $T(0,0,1,-1)=z_1$.

That T is an isometry follows from the observations 1)—5) below.

1) There exists $f_1\in\partial_e X_1^*$ such that $f_1=1$ on $x_1, \dots, x_4, y_1, z_1$.
2) Since $z_1\notin N$, there exists $f_2\in\partial_e X_1^*$ such that $f_2=1$ on y_1, y_2, x_2, x_3, z_2 and $f_2=-1$ on z_1, x_1, x_4.
3) Since $x_2\notin N$, there exists $f_3\in\partial_e X_1^*$ such that $f_3=1$ on y_1, y_2, z_1, z_2 and $f_3=-1$ on x_1, x_2, x_3, x_4.
4) Since $z_2\notin N$, there exists $f_4\in\partial_e X_1^*$ such that $f_4=1$ on y_1, y_2, z_1, x_2, x_4 and $f_4=-1$ on z_2, x_1, x_3.
5) Since $N\cap F=\emptyset$, we get $x_2\notin \operatorname{face}\left(\frac{y_1+x_1}{2}\right)$. Hence there exists $f_5\in\partial_e X_1$ such

that $f_5=1$ on y_1, x_1, x_3, z_1, z_2 and $f_5=-1$ on x_2, x_4, y_2.

The proof is complete.

Lemma 6.4. *Assume X is a finite-dimensional* CL-*space and that $L(l_\infty^3, X^*)$ is a* CL-*space. If there exists an isometry $T\colon l_1^3\oplus_\infty R\to X$ such that $Tx\in\partial_e X_1$ for every $x\in\partial_e(l_1^3\oplus_\infty R)_1$, then there exists a projection P in X such that $P(\partial_e X_1)\subseteq\partial_e X_1$ and $P(X)=l_1^3\oplus_\infty R$ or l_∞^4.*

Before we give the proof, let us look upon a consequence of this result.

Theorem 6.5. *Assume X is a finite-dimensional* CL-*space which satisfies* (1) $T\in\partial_e L(X,X)_1$, $x\in\partial_e X_1\Rightarrow Tx\in\partial_e X_1$. *Then X has the* 3.2.I.P.

Proof. This follows from Theorem 3.1, Corollary 6.2, Proposition 6.3 and Lemma 6.4.

It only remains to prove Lemma 6.4. Clearly $T^*\colon X^*\to l_\infty^3\oplus_1 R$ is a quotient map such that $T^*(\partial_e X_1^*)=\partial_e(l_\infty^3\oplus_1 R)_1$.

From now on, we shall assume $L(l_\infty^3, X)$ is a CL-space and that $Q\colon X\to l_\infty^3\oplus_1 R$ is a quotient map such that $Q(\partial_e X_1)=\partial_e(l_\infty^3\oplus_1 R)_1$.

Clearly it suffices to show that either there is a projection P in X such that $P(\partial_e X_1)\subseteq\partial_e X_1$ and $P(X)=l_\infty^3\oplus_1 R$ or there exists an isometry $T\colon l_\infty^4\to X^*$ such that $T(\partial_e(l_\infty^4)_1)\subseteq\partial_e X_1^*$.

Let $x_1=(1,1,1,0)$, $x_2=(1,-1,1,0)$, $x_3=(1,-1,-1,0)$, $x_4=(1,1,-1,0)$ and $x_5=(0,0,0,1)$, and let $N=\operatorname{conv}(x_1,\dots,x_5)$. Let $K=Q^{-1}(N)\cap X_1$. Then N and K are maximal proper faces of $(l_\infty^3\oplus_1 R)_1$ and X_1 respectively. For $i=1,\dots,5$, let $F_i=\{x\in K\colon Q(x)=x_i\}$. Then $\{F_i\}_{i=1}^5$ are disjoint faces of K and $K=\operatorname{conv}(F_1\cup\dots\cup F_5)$.

Case 1. Suppose there exist $a_i\in\partial_e F_i$ such that

$$a_1+a_3=a_2+a_4.$$

Define $T\colon l_\infty^3\oplus_1 R\to X$ by $Tx_i=a_i$, and define P by $P=T\cdot Q$. P is a projection in X and $P(X)=l_\infty^3\oplus_1 R$ and $P(\partial_e X_1)\subseteqq\partial_e X_1$.

Case 2. Here we assume:
(#) If $a_i\in\partial_e F_i$, then $a_1+a_3\neq a_2+a_4$.

Assuming (#), we are going to show that there exists an isometry $T\colon l_\infty^4\to X^*$ such that $T(\partial_e(l_\infty^4)_1)\subseteqq\partial_e X_1$. The proof of this is divided into eight sublemmas.

Choose $y_i\in F_i$ such that $F_i=\operatorname{face}(y_i)$.

Lemma 6.6. $\operatorname{conv}(F_2\cup F_4)$ *is a face of* K.

Proof. Note that $\operatorname{conv}(F_1\cup\dots\cup F_4)$ is a face of F. If $\operatorname{conv}(F_2\cup F_4)$ is not a face of K, then $\operatorname{face}\left(\dfrac{y_2+y_4}{2}\right)\cap(F_1\cup F_3)\neq\emptyset$. Hence there exist $\alpha_i\geqq 0$, $\sum\alpha_i=2$ and $z_i\in F_i$ such that

$$y_2+y_4=\alpha_1 z_1+\dots+\alpha_4 z_4$$

and we can assume $\alpha_1>0$. Let $f\in\partial_e X_1^*$ such that $f=1$ on $F_1\cup F_2$ and $f=-1$ on $F_3\cup F_4$. (Let $f=(0,0,1,1)\cdot Q$.) Then we get

$$0=\alpha_1+\alpha_2-\alpha_3-\alpha_4.$$

Similarly we get

$$0=\alpha_1-\alpha_2-\alpha_3+\alpha_4.$$

Hence $\alpha_1=\alpha_3$, $\alpha_2=\alpha_4$ and $\alpha_1+\alpha_2=1$. Define a map $T\colon l_\infty^3\to X$ by $T(1,1,1)=y_2$, $T(1,-1,-1)=y_4$, $T(1,-1,1)=\alpha_1 z_1+\alpha_2 z_2$ and $T(1,1,-1)=\alpha_3 z_3+\alpha_4 z_4$. Since $L(l_\infty^3,X)$ is a CL-space, we can write $T=\sum\lambda_i T_i$ where $\lambda_i>0$, $\sum\lambda_i=1$ and $T_i(\partial_e(l_\infty^3)_1)\subseteqq\partial_e X_1$. ($i$ is in a finite index set.) Since $\alpha_1>0$ and F_1 is a face of $\operatorname{conv}(F_1\cup F_2)$, there is some T_i, say T_1, such that $T_1(1,-1,1)=a_1\in\partial_e F_1$. Moreover, $T_1(1,1,1)=a_2\in\partial_e F_2$ and $T_1(1,-1,-1)=a_4\in\partial_e F_4$. Hence $T(1,1,-1)=a_3\in\partial_e F_3$ since $a_2+a_4=a_1+a_3$. This contradicts (#).

The proof is complete.

The next three lemmas are proved in exactly the same way as Lemma 6.6 was proved so we omit their proofs.

Lemma 6.7. conv $(F_1 \cup F_3 \cup F_4)$ *is a face of* K.

Lemma 6.8. conv $(F_1 \cup F_3 \cup F_5)$ *is a face of* K.

Lemma 6.9. conv $(F_2 \cup F_3 \cup F_4 \cup F_5)$ *is a face of* K.

We shall say that two subsets A and B of X_1 can be ± 1-separated if there exists $f \in \partial_e X_1^*$ such that $f=1$ on A and $f=-1$ on B.

Lemma 6.10. *The faces* F_1 *and* conv $(F_2 \cup F_3 \cup F_4)$ *can be* ± 1*-separated.*

Proof. Let $G = \operatorname{conv}(F_2 \cup F_3 \cup F_4)$. G is a face of K by Lemma 6.7 and $F_1 \cap G = \emptyset$. If $a \in \partial_e F_1$, then G and $\{a\}$ can be ± 1-separated by Theorem 2.3. If G and F_1 cannot be ± 1-separated, then let face $(a) \subseteq F_1$ be a minimal face of F_1 such that G and face (a) cannot be ± 1-separated. Write $a = \alpha a_1 + (1-\alpha) b$ where $a_1 \in \partial_e F_1$, $b \in F_1$ and $\alpha \in \langle 0, 1 \rangle$. Choosing α as large as possible we can assume $a_1 \notin$ face (b). Clearly we can assume $2\alpha = 1$. By the minimality of face (a), it follows that G and face (b) can be ± 1-separated. From Theorem 2.3 it follows that

$$a_1 \in \operatorname{face}\left(4^{-1}(y_2+y_3+y_4-b)\right).$$

Thus we can write

$$y_2+y_3+y_4-b = \alpha a_1 + (4-\alpha) u$$

where $\|u\| = 1$ and $\alpha \in \langle 0, 3]$. We can write $(4-\alpha)u = (3-\alpha)v - w$ where $v, w \in K$. Hence

$$y_2+y_3+y_4+w = \alpha a_1 + b + (3-\alpha) v.$$

Let $f \in \partial_e X_1^*$ such that $f=1$ on G and $f(b) = -1$. Then $f(a_1) = 1$ such that

$$\begin{aligned} 4 &= f(y_2+y_3+y_4-b) \\ &= \alpha + (3-\alpha) f(v) - f(w) \\ &\leqq \alpha + (3-\alpha) + 1 \\ &= 4. \end{aligned}$$

Hence $f(w) = -1$. Thus $w \in \operatorname{conv}(F_1 \cup F_5)$. Write $(3-\alpha)v = \alpha_1 z_1 + \ldots + \alpha_5 z_5$ where $\alpha_i \geqq 0$ and $z_i \in F_i$. Then $\alpha_1 + \ldots + \alpha_5 = 3 - \alpha$ and

$$y_2+y_3+y_4+w = \alpha a_1 + b + \alpha_1 z_1 + \ldots + a_5 z_5.$$

We use the map Q and get

$$x_2+x_3+x_4+Q(w) = (1+\alpha+\alpha_1)x_1 + \alpha_2 x_2 + \ldots + \alpha_5 x_5.$$

From this it follows that $\alpha_2 = \alpha_4$ and $\alpha_2 + \alpha_3 = 2$. Since $Q(w) \in \operatorname{conv}(x_1, x_5)$, we get $1 \geqq \alpha + \alpha_1 + \alpha_2$. Hence $\alpha_2 < 1$ and $\alpha_3 > 1$. Define $u_i \in K$ by $2u_1 = y_3 + w$, $2u_2 = y_2 + y_4$, $2u_3 = \alpha_2 z_2 + \alpha_3 z_3$ and $2u_4 = \alpha a_1 + b + \alpha_1 z_1 + \alpha_4 z_4 + \alpha_5 z_5$. Then

$$u_1 + u_2 = u_3 + u_4.$$

Now we use that $L(l^3_\infty, X)$ is a CL-space in exactly the same way as we used it in the proof of Lemma 6.6 and find $v_i \in \partial_e \operatorname{face}(u_i)$ such that

$$v_1+v_2 = v_3+v_4.$$

Since $\alpha_3>1$ and $1+\alpha+\alpha_1>1$, we see that we may suppose $v_3\in\partial_e \operatorname{face}(z_3)\subseteqq\partial_e F_3$ and $v_4\in\partial_e F_1$. But by Lemma 6.6, this implies that

$$v_2\in\operatorname{conv}(F_1\cup F_3)\cap\operatorname{conv}(F_2\cup F_4) = \emptyset.$$

This contradiction completes the proof.

We omit the proofs of Lemmas 6.11 and 6.12 since they are similar to the proof of Lemma 6.10.

Lemma 6.11. *F_1 and* $\operatorname{conv}(F_2\cup F_3\cup F_4\cup F_5)$ *can be ± 1-separated.*

Lemma 6.12. $\operatorname{conv}(F_2\cup F_4)$ *and* $\operatorname{conv}(F_1\cup F_3\cup F_5)$ *can be ± 1-separated.*

It remains only one lemma.

Lemma 6.13. *There exists an isometry $T\colon l^4_\infty\to X^*$ such that $T(\partial_e(l^4_\infty)_1)\subseteqq\partial_e X^*_1$.*

Proof. By Lemma 6.12 there exists $g_2\in\partial_e X^*_1$ such that $g_2=1$ on $F_1\cup F_3\cup F_5$ and $g_2=-1$ on $F_2\cup F_4$. Using Q, we see that there exist $g_4, g_5, g_7\in\partial_e X^*_1$ such that $g_4=1$ on K, $g_5=1$ on $F_1\cup F_2\cup F_5$, $g_5=-1$ on $F_3\cup F_4$, $g_7=1$ on $F_1\cup F_4\cup F_5$ and $g_7=-1$ on $F_2\cup F_3$. Define $T\colon l^4_\infty\to X$ by $T(1,1,1,1)=g_4$, $T(1,-1,1,-1)=g_2$, $T(1,1,-1,-1)=g_5$ and $T(1,-1,-1,1)=g_7$. It is straightforward to see that T has the right properties. The proof is complete, and this also completes the proof of Lemma 6.4.

Let us add a last result.

Theorem 6.14. *Assume X and Y are finite-dimensional spaces with the* 3.2.I.P. *The following statements are equivalent:*

(1) $$L(X,Y) \text{ is a CL-space.}$$

(2) $$X = l^m_1\oplus_1 l^3_\infty\oplus_1\dots\oplus_1 l^3_\infty \quad\text{or}\quad Y = l^n_\infty\oplus_\infty l^3_1\oplus_\infty\dots\oplus_\infty l^3_1.$$

Proof. (2)$\Rightarrow$(1) is contained in Theorem 2.1. Assume (2) is not true. Then just as in the proof of Theorem 4.1, we can write $X=U\oplus_1(V\oplus_\infty Z)$ where $\dim(V\oplus_\infty Z)\geqq 4$ and $Y^*=L\oplus_1(M\oplus_\infty N)$ where $\dim(M\oplus_\infty N)\geqq 4$. Then looking at the arguments used in the proofs of Theorems 3.7 and 4.1, we see that we can apply Proposition 2.5, Lemma 6.4 and Theorem 3.1. The proof is complete.

Remark. In [6] we proved that $L(X, Y)$ has the 3.2.I.P. if and only if $X=l^m_1$ or $Y=l^n_\infty$ where X and Y are as in Theorem 6.14.

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