Schauder's existence theorem for α-Dini continuous data

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1.

In 1934 J. Schauder (see [14]) proved his well known theorem on the existence of classical solutions to linear elliptic partial differential equations of second order. In this article we shall establish the following improvement of Schauder's theorem: A classical solution exists, if the given data (coefficients, boundary values, right hand side) are uniformly continuous and their modulus of continuity is bounded by some function θ which owns the following two properties:

i) $$\int_0^1 \theta(\tau)/\tau \, d\tau < \infty.$$

ii) There is an $0<\alpha<1$, such that $\theta(\tau)/\tau^\alpha$ is monotonically decreasing on some interval $(0, T]$.

For notations we refer to **16.** below. We shall always use the summation convention.

2.

In order to give a precise statement, let us introduce the following notions: Let $\zeta\colon [0, \infty)\to[0, \infty)$ be a monotonically increasing function, $\lim_{t\to 0+} \zeta(t)=0$, $\zeta(0)=0$, $\zeta(t)>0$, if $t>0$; B a real Banach space equiped with the norm $\|\cdot\|_B$; $A\subset\mathbf{R}^n$ a nonvoid, open set. $C^{0,\zeta}(A)=C^{0,\zeta}(A, B)$ is the set of all bounded continuous functions $f\colon A\to B$, for which

(2.1) $$[f]_\zeta := \sup_{x,y\in A, x\neq y} \|f(x)-f(y)\|_B/\zeta(\|x-y\|)$$

is finite. It is easy to prove, that $C^{0,\zeta}(A)$ becomes a Banach space under the norm:

(2.2) $$\|f\|_{0,\zeta} := \|f\|_0+[f]_\zeta.$$

If $k\in\mathbf{N}$, let $C^{k,\zeta}(A)$ be the set of k-times uniformly continuously differentiable functions $f\colon A\to B$, such that f and the derivatives up to order $k-1$ are uniformly Lipschitz continuous and any derivative of order k is in $C^{0,\zeta}(A)$. On $C^{k,\zeta}(A)$ we shall use the norm:

$$\|f\|_{k,\zeta}:=\|f\|_0+\sum\nolimits_{1\leq|\beta|\leq k}\|D^\beta f\|_0+\sum\nolimits_{|\beta|=k}[D^\beta]_\zeta. \tag{2.3}$$

Indeed $C^{k,\zeta}$ is a Banach space. If $\zeta=t^\alpha$, $0<\alpha\leqq 1$, we shall simply write $\|\cdot\|_{k,\alpha}$ and $C^{k,\alpha}$.

3.

We formulate our assumptions.

i) Let $\theta\colon [0,\infty)\to[0,\infty)$, $\theta(0)=0$, $\theta(t)>0$, if $t>0$, be a monotonically increasing Dini-function; that is: $\int_0^1 \theta(t)/t\,dt<\infty$. We assume that there is a $T>0$ and $\alpha\in(0,1)$, such that $\theta(t)/t^\alpha$ is monotonically decreasing on $(0,T)$.

ii) Let $\Omega\subset\mathbf{R}^n$ be a nonvoid, open, bounded and connected set. We make the following assumptions about its boundary $\partial\Omega$: To any $x\in\partial\Omega$ there exists an open neighbourhood U_x and a C^2-diffeomorphism $\varphi_x=\colon U_x\to B(1)$ with the properties:

a) $\varphi_x(x)=0,\quad \varphi_x(U\cap\Omega)=B^+(1)=\{y\in B(1)\colon y^n>0\}$
$\varphi_x(U_x\cap\partial\Omega)=\{y\in B(1)\colon y^n=0\}$.

b) φ_x is in $C^{2,\theta}(U_x)$, φ_x^{-1} is in $C^{2,\theta}(B(1))$. Further we shall assume, that there is a constant $d_1>0$, such that the Lipschitz constants of φ_x, of φ_x^{-1} and of all their derivatives as well as $\|\varphi_x\|_{2,\theta}$ and $\|\varphi_x^{-1}\|_{2,\theta}$ are bounded by d_1.

iii) There are n^2+n+1 functions $m_{ij}, m_i, m\in C^{0,\theta}(\Omega,\mathbf{R})$, $i,j=1,\dots,n$, $m(x)\leqq 0$, $m_{ij}(x)=m_{ji}(x)$, and constants $0<e_0<e_1$, such that for any $\xi\in\mathbf{R}^n$, $x\in\Omega$:

$$e_0\cdot\|\xi\|^2\leqq m_{ij}(x)\cdot\xi^i\xi^j\leqq e_1\cdot\|\xi\|^2. \tag{3.1}$$

For any $w\in H^2(\Omega)$ write

$$Mw(x):=m_{ij}(x)\cdot w_{x^ix^j}(x)+m_i(x)\cdot w_{x^i}(x)+m(x)\cdot w(x). \tag{3.2}$$

4.

The theorem to be proven is

Assume 3.i)—3.iii), *let* $h\in C^{0,\theta}(\Omega)$, $g\in C^{2,\theta}(\Omega)$, *then there exists a unique* $w\in C^2(\Omega)$, *such that* $Mw=h$ *on* Ω *and* $w|\partial\Omega=g|\partial\Omega$. *For any* $x_1,x_2\in\Omega$, $1\leqq i,j\leqq n$ *the estimate*

$$|u_{x^ix^j}(x_1)-u_{x^ix^j}(x_2)|\leqq c_1\cdot\int_0^{\|x_1-x_2\|}\theta(t)/t\,dt \tag{4.1}$$

is valid.

Remarks.

a) If $\theta(t)=t^\alpha$, this is Schauder's theorem.

b) Recently Burch [1] proved Weyl's lemma for equations in divergence form, the coefficients of which satisfy some Dini conditions.

c) Obviously it suffices to prove the theorem for $g\equiv 0$.

d) Choose $0<\alpha<1<\beta$, then the function

$$\theta(t):=\begin{cases}(-\ln t)^{-\beta}, & \text{if} \quad 0<t\leqq \exp(-\beta/\alpha)\\ (\beta/\alpha)^{-\beta}, & \text{if} \quad t\geqq \exp(-\beta/\alpha)\end{cases}$$

satisfies 3i), since for $0<t<\exp(-\beta/\alpha)$

$$D_t\theta(t)=(\beta/t)(-\ln t)^{-\beta-1}\geqq 0,$$

$$\theta(t)/t=\frac{1}{\beta-1}D_t(-\ln t)^{1-\beta},$$

$$D_t\big(\theta(t)\cdot t^{-\alpha}\big)=t^{-\alpha-1}\theta(t)\big(\beta(-\ln t)^{-1}-\alpha\big)\leqq 0.$$

5.

i) Let ζ be as in **2**. We shall call ζ an α-function, $0<\alpha<1$, on $(0, T]$ if $\zeta(t)/t^\alpha$ is monotonically decreasing on $(0, T]$, $T>0$. This implies:

(5.1) $$\zeta(t)\leqq t^\alpha\cdot\tau^{-\alpha}\cdot\zeta(\tau), \quad 0<\tau\leqq t\leqq T.$$

Inserting $t=c\cdot\tau$, $c>1$, we obtain:

(5.2) $$\zeta(c\cdot\tau)\leqq c^\alpha\cdot\zeta(\tau), \quad 0<\tau\leqq T/c.$$

ii) Now let θ be as in **3**.i), that is: θ is àn α-Dini function. We introduce the function

(5.3) $$\delta(t):=t^{-1}\cdot\int_0^t\theta(\tau)\,d\tau.$$

Since $\delta(t)\leqq\theta(t)$ and

(5.4) $$D\delta(t)=t^{-1}\big(\theta(t)-\delta(t)\big)\geqq 0,$$

δ is a monotonically increasing, absolutely continuous Dini function. Integrating the inequality

(5.5) $$\theta(\tau)\geqq t^{-\alpha}\theta(t)\cdot\tau^\alpha, \quad 0<\tau\leqq t\leqq T$$

with respect to τ over $(0, t)$, we obtain:

5.6) $$\delta(t)\geqq\theta(t)/(1+\alpha), \quad 0<t\leqq T.$$

We infer, that

(5.7) $$\frac{t\cdot D\delta(t)}{\delta(t)} = -1+\frac{\theta(t)}{\delta(t)} \leqq \alpha, \quad 0<t\leqq T.$$

Multiplying (5.7) by $t^{-\alpha-1}$, we arrive at $D(t^{-\alpha}\delta(t))\leqq 0$, $0<t\leqq T$, which implies, that δ is indeed an α-Dini function.

iii) We need the auxiliary functions

(5.8) $$\omega(t) = \int_0^t \theta(\tau)/\tau\, d\tau, \quad \sigma(t) = \int_0^t \delta(\tau)/\tau\, d\tau.$$

From (5.7) we infer by integration:

(5.9) $$\delta(t) \leqq \alpha\cdot\sigma(t), \quad 0<t\leqq T$$

while (5.3) and (5.6) imply, that

(5.10) $$\sigma(t) \leqq \omega(t) \leqq (1+\alpha)\sigma(t), \quad 0<t\leqq T.$$

iv) Observe, that by (5.2):

(5.11) $$\omega(c\cdot t) = \int_0^{ct} \theta(\tau)/\tau\, d\tau$$
$$= \int_0^t \theta(c\cdot s)/s\, ds \leqq c^\alpha\cdot\omega(t), \quad c>1,\ 0<t<T/c$$

and the same inequality holds for σ.

6.

i) An open set $\emptyset\neq A\subset\mathbf{R}^n$ is said to own the property C, if there exist two positive constants K, ϱ_0, such that for any $x\in A$ and $0<\varrho\leqq\varrho_0$:

(6.1) $$\mathscr{L}\langle B(x,\varrho)\cap A\rangle \geqq K\cdot\varrho^n.$$

It is easy to prove, that any open set with compact Lipschitz boundary owns the property C. Thus **3**.ii) implies property C for Ω.

ii) If $f\in C^{0,\theta}(A)$, then for any $x\in A$, $\varrho>0$:

(6.2) $$v(f,x,\varrho,A) \leqq 2\omega_n[f]_\theta^2\cdot\varrho^n\cdot\theta^2(\varrho)$$

(see (16.3)). The proof is obvious.

iii) *If A owns property C, $f\in L_\infty(A,\mathbf{R})$, θ a Dini function* (see **3**.i)) *and* (6.2) *holds for any $x\in A$, $0<\varrho\leqq\varrho_0$, with $2\omega_n[f]_\theta^2$ replaced by some constant L^2, then f*

coincides $\mathcal{L}$-almost everywhere on A with an uniformly continuous function $\tilde f$, for which the inequality

(6.3) $$|\tilde f(x)-\tilde f(y)| \leqq c_2\cdot\left(\int_0^{2\|x-y\|}\theta(t)/t\,dt+\theta(2\|x-y\|)\right),$$

$c_2=c_3(\varrho_0, K, n)\cdot(L+\|f\|_0)$, *holds whenever* $x, y\in A$.

Proof. Let $0<\varrho\leqq\varrho_0$, $x\in A$,

$$F(\varrho):=\int_0^\varrho\int_{B(x,\zeta)\cap A} f\,d\mathcal{L}\,dr=\int_{B(x,\varrho)\cap A}(\varrho-\|x-z\|)f(z)\,d\mathcal{L}z$$

(6.4) $$l(\varrho):=\mathcal{L}\langle B(x,\varrho)\cap A\rangle$$

$$g(\varrho):=\int_0^\varrho l(r)\,dr=\int_{B(x,\varrho)\cap A}(\varrho-\|x-z\|)\,d\mathcal{L}z.$$

The property C implies:

(6.5) $$g(\varrho)\geqq K\cdot\varrho^{n+1},\quad 0<\varrho\leqq\varrho_0.$$

We calculate:

(6.6) $$g(\varrho)\cdot\int_{B(x,\varrho)\cap A} f\,d\mathcal{L}-l(\varrho)\cdot F(\varrho)$$
$$=\int_{B(x,\varrho)\cap A} f(z)\cdot\big(g(\varrho)-l(\varrho)(\varrho-\|x-z\|)\big)\,d\mathcal{L}z$$
$$=\int_{B(x,\varrho)\cap A} f(z)\cdot\Phi(z)\,d\mathcal{L}z,$$

where

(6.7) $$\int_{B(x,\varrho)\cap A}\Phi(z)\,d\mathcal{L}z=0,$$
$$\int_{B(x,\varrho)\cap A}\Phi^2\,d\mathcal{L}=l^2(\varrho)\cdot\int_{B(x,\varrho)\cap A}(\varrho-\|x-z\|)^2\,d\mathcal{L}z-l(\varrho)\,g(\varrho)^2$$
$$\leqq\omega_n^3\cdot\varrho^{3n+2}.$$

We derive:

(6.8) $$\left|\int_{B(x,\varrho)\cap A} f(z)\cdot\Phi(z)\,d\mathcal{L}z\right|$$
$$=\left|\int_{B(x,\varrho)\cap A}\big(f(z)-m(f,x,\varrho,A)\big)\cdot\Phi(z)\,d\mathcal{L}z\right|$$
$$\leqq v(f,x,\varrho,A)^{1/2}\cdot\int_{B(x,\varrho)\cap A}\Phi^2\,d\mathcal{L}^{1/2}$$
$$\leqq\omega_n^{3/2}\cdot\varrho^{1+3n/2}\cdot v(f,x,\varrho,A)^{1/2}$$

and

(6.9) $$|D_\varrho(F(\varrho)/g(\varrho))|$$
$$=g^{-2}(\varrho)\left|g(\varrho)\cdot\int_{B(x,\varrho)\cap A} f\,d\mathcal{L}-l(\varrho)\cdot F(\varrho)\right|$$
$$\leqq(\omega_n^{3/2}/c_3^2\cdot\varrho^{1+n/2})\cdot v(f,x,\varrho,A)^{1/2}$$
$$\leqq c_4\cdot L\cdot\theta(\varrho)/\varrho,\quad 0<\varrho\leqq\varrho_0.$$

Integration implies for arbitrary $0<\varrho_1\leqq\varrho_2\leqq\varrho_0$:

$$\left|\frac{F(\varrho_2)}{g(\varrho_2)}-\frac{F(\varrho_1)}{g(\varrho_1)}\right|\leqq c_4\cdot L\cdot\int_0^{\varrho_2}\theta(\varrho)/\varrho\,d\varrho. \tag{6.10}$$

Thus the limit $\tilde{f}(x):=\lim_{\varrho\to 0}F(\varrho)/g(\varrho)$ exists and equals $f(x)$ $\mathscr{L}$-almost everywhere by Lebesgue's theorem. From (6.10) we derive

$$|\tilde{f}(x)-F(\varrho)/g(\varrho)|\leqq c_4\cdot L\cdot\int_0^{\varrho}\theta(t)/t\,dt,\quad 0<\varrho\leqq\varrho_0 \tag{6.11}$$

and from (6.6), (6.8):

$$\begin{aligned}&|m(f,x,\varrho,A)-F(\varrho)/g(\varrho)|\\ &\leqq g(\varrho)^{-1}\cdot l(\varrho)^{-1}\cdot\omega_n^{3/2}\cdot\varrho^{1+3n/2}\cdot L\cdot\varrho^{n/2}\cdot\theta(\varrho)\\ &\leqq c_5\cdot L\cdot\theta(\varrho),\quad 0<\varrho\leqq\varrho_0.\end{aligned} \tag{6.12}$$

For $x,y\in A$, $\varrho:=2\|x-y\|\leqq\varrho_0$, we have

$$\begin{aligned}&K^{1/2}\cdot(\varrho/2)^{n/2}|m(f,x,\varrho,A)-m(f,y,\varrho,A)|\\ &\leqq\int_{B(x,\varrho/2)\cap A}|m(x,\varrho)-m(y,\varrho)|^2\,d\mathscr{L}^{1/2}\\ &\leqq\int_{B(x,\varrho)\cap A}(f-m(x,\varrho))^2\,d\mathscr{L}^{1/2}+\int_{B(y,\varrho)\cap A}(f-m(y,\varrho))^2\,d\mathscr{L}^{1/2}\\ &\leqq 2L\cdot\varrho^{n/2}\cdot\theta(\varrho),\quad 0<\varrho\leqq\varrho_0.\end{aligned} \tag{6.13}$$

From (6.11), (6.12), (6.13) we derive (6.3) for some specific constant $\tilde{c}_2$, as long as $\|x-y\|\leqq\varrho_0/2$. If $\|x-y\|>\varrho_0/2$,

$$|\tilde{f}(x)-\tilde{f}(y)|\leqq 2\|f\|_0\theta^{-1}(\varrho_0)\cdot\theta(2\|x-y\|). \tag{6.14}$$

This implies the statement.

iv) If θ is an α-Dini function, we derive by some simple argument (see (6.14)) from (5.6), (5.9), (5.10), (5.11) the estimate

$$\begin{aligned}&|\tilde{f}(x)-\tilde{f}(y)|\leqq c_6\cdot(L+\|f\|_0)\,\omega(\|x-y\|)\\ &c_6=c_6(K,\varrho_0,n,\alpha,T).\end{aligned} \tag{6.15}$$

v) *Remark.* Spanne's result (see [15]) might be useful to obtain a slightly better estimate in specific domains.

7.

i) If $f, g \in C^{0,\zeta}(A, \mathbf{R})$, then $f \cdot g \in C^{0,\zeta}(A)$ and obviously:

$$\|f \cdot g\|_{0,\zeta} \leqq 2\|f\|_{0,\zeta} \cdot \|g\|_{0,\zeta}. \tag{7.1}$$

ii) If $f\colon A \to \mathbf{R}$ is a bounded uniformly continuous function and there is a $\varrho_1 > 0$, such that $x, y \in A$ with $\|x-y\| \leqq \varrho_1$ implies $|f(x)-f(y)| \leqq M \cdot \zeta(\|x-y\|)$, then $f \in C^{0,\zeta}(A)$ and obviously

$$[f]_\zeta \leqq \max\{M, 2\|f\|_0 \cdot \zeta(\varrho_1)^{-1}\}. \tag{7.2}$$

iii) If ζ is an α-function on $[0, T]$ (see **5**.i)) and $f \in C^{0,\beta}(A)$, $\alpha \leqq \beta \leqq 1$, then $f \in C^{0,\zeta}(A)$ and

$$[f]_\zeta \leqq \max\{\|f\|_{0,\alpha} \cdot T, 2\|f\|_0\}/\theta(T). \tag{7.3}$$

Indeed, if $x, y \in A$, $0 \neq \|x-y\| \leqq T$, then by (5.1)

$$\begin{aligned} |f(x)-f(y)| &\leqq \|f\|_{0,\beta} \cdot \|x-y\|^\beta \\ &\leqq \|f\|_{0,\beta} \cdot \|x-y\|^{\beta-\alpha} \frac{\|x-y\|^\alpha}{\zeta(\|x-y\|)} \cdot \zeta(\|x-y\|) \\ &\leqq \|f\|_{0,\beta} \cdot T^{\beta-\alpha}(T^\alpha/\zeta(T)) \cdot \zeta(\|x-y\|). \end{aligned}$$

Now the statement comes from **7**.ii).

iv) If ζ is an α-function, $f \in C^{0,\zeta}(A)$, $\psi\colon B \to A$ Lipschitz continuous with Lipschitz constant L, then $f \circ \psi \in C^{0,\zeta}(B)$ with

$$[f \circ \psi]_\zeta \leqq \max\{[f]_\zeta \cdot L^{*\alpha}, 2\|f\|_0 \cdot \zeta(T/L^*)^{-1}\}, \quad L^* = \max\{L, 1\}, \tag{7.4}$$

Indeed, if $x, y \in B$, $\|x-y\| \leqq T/L^*$, then by (5.2):

$$|f(\psi(x)) - f(\psi(y))| \leqq [f]_\zeta \cdot \zeta(L^* \cdot \|x-y\|) \leqq [f]_\zeta \cdot L^{*\alpha} \cdot \zeta(\|x-y\|).$$

Now apply **7**.ii).

8.

If $(f_\nu)_{\nu \in \mathbf{N}}$ *is a bounded sequence in* $C^{k,\zeta}(A)$, $\emptyset \neq A \subset \mathbf{R}^n$ *bounded, open and connected, then there exists a subsequence* $(f_{\nu'})$ *converging in* $C^k(A)$ (see (**16**)) *to some function* $f_0 \in C^{k,\zeta}(A)$ *with*

$$[D^\beta f_0]_\zeta \leqq \sup_{\nu \in \mathbf{N}} [D^\beta f_\nu]_\zeta, \quad |\beta| = k. \tag{8.1}$$

Proof by induction.

In the case $k=0$ this is essentially the Arzela—Ascoli theorem. Assume the theorem to be proven for $j=k-1\geqq 0$ and $f_\nu \in C^{k,\zeta}(A)$, $\nu\in\mathbf{N}$. Then for any $i=1, \ldots, n$ $f_{\nu,x^i}\in C^{j,\zeta}(A)$ and thus a subsequence f_{ν,x^i} converges in $C^j(A)$ to a function $f_i\in C^j(A)$, $i=1, \ldots, n$. Certainly for some $x_0\in A$ there is a subsequence $f_{\nu''}$ of $f_{\nu'}$, such that $f_{\nu''}(x_0)$ converges. Now the statement follows from a familiar theorem (see [3, I.3.6]). (8.1) is obvious in any case.

9.

i) *Assume, that* θ *is an* α*-function and* Ω *satisfies* 3.ii). *If* $f\in C^{0,\theta}(\Omega)$, *then there exists a sequence of uniformly Lipschitz continuous functions on* Ω *converging uniformly to* f *and*

(9.1) $$\|f_\nu\|_{0,\theta}\leqq d_2\|f\|_{0,\theta},\quad d_2=d_2(\Omega).$$

If f *is nonpositive, so is any* f_ν, $\nu\in\mathbf{N}$.

ii) *Proof.* Assume first, that $g\in C^{0,\theta}(\mathbf{R}^n)$ and has compact support, then the functions $g_\nu(x):=m(g, x, 1/\nu)$ (see (16.1)) are Lipschitz continuous and converge uniformly to g on $\mathbf{R}^n$, further

(9.2) $$\|g_\nu\|_{0,\theta}\leqq\|g\|_{0,\theta},\quad \nu\in\mathbf{N}.$$

iii) Now cover $\bar\Omega$ by a finite number of open sets Ω_i, $i=0, \ldots, k$, such that $\bar\Omega_0\subset\Omega$ and each Ω_i, $i=1, \ldots, k$ is in the domain of a boundary chart φ_i (see 3.ii)). Let be $\eta_i\in C_0^\infty(\Omega_j)\cap C^\infty(\mathbf{R}^n)$, $\eta_i\geqq 0$, such that $\sum_{i=0}^k \eta_i(x)^2=1$ for any $x\in\Omega$ (see [2, I.4.1], [4, p. 35]). Now set

(9.3) $$\tilde h_i(z):=f(\varphi_i^{-1}(z)),\quad i=1, \ldots, k,$$

$z\in B^+(1)$. We shall use

$$s(z):=\begin{cases} z, & \text{if}\quad z\in B^+(1)\\ (z^1, \ldots, z^{n-1}, -z^n) & \text{if}\quad z\in B(1)\setminus B^+(1)\end{cases}$$

and define

(9.4) $$h_i(z):=\tilde h_i(s(z)).$$

Since φ_i^{-1} and s are Lipschitz continuous, we infer from 7.iv)

(9.5) $$\|h\|_{0,\theta}\leqq c_7\cdot\|f\|_{0,\theta},\quad c_7=c_7(d_1, T, \alpha).$$

Further define:

(9.6)
$$g_i(z):=\begin{cases}\eta_i(\varphi^{-1}(z))\cdot h_i(z), & \text{if}\quad z\in B(1)\\ 0, & \text{otherwise},\ i=1, \ldots, k\end{cases}$$

$$g_0(z):=\begin{cases}\eta_0(z)\cdot f(z), & z\in\Omega_0\\ 0, & \text{otherwise}.\end{cases}$$

By (7.1) $g_i \in C^{0,\theta}(\mathbf{R}^n)$ and

$$\|g_i\|_{0,\theta} \leqq c_8 \cdot \|f\|_{0,\theta}, \quad c_8 = c_8(d_1, T, \alpha, \eta_i). \tag{9.7}$$

iv) Now approximate g_i by Lipschitz continuous functions $g_{i,\nu}$, $\nu \in \mathbf{N}$, as described above and set

$$\tag{9.8} \begin{aligned} \tilde{f}_{i,\nu}(x) &:= \begin{cases} \eta_i(x) \cdot g_{i,\nu}(\varphi_i(x)), & x \in \Omega_i \\ 0, & \text{otherwise, } i = 1, \dots, k \end{cases} \\ \tilde{f}_{0,\nu}(x) &:= \begin{cases} \eta_0(x) \cdot g_{0,\nu}(x), & x \in \Omega_0 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Certainly $\tilde{f}_{i,\nu}$ is Lipschitz continuous and converges uniformly on Ω to $f \cdot \eta_i^2$. Further by (9.2), 7.i), 7.iv):

$$\|\tilde{f}_{i,\nu}\|_{0,\theta} \leqq c_9 \cdot \|f\|_{0,\theta}, \quad c_9 = c_9(d_1, T, \alpha, \eta_i), \ \nu \in \mathbf{N}, \ i = 0, \dots, k. \tag{9.9}$$

Now define

$$f_\nu(x) := \sum_{i=0}^k f_{i,\nu}(x), \quad x \in \Omega \tag{9.10}$$

and the statement follows.

v) Later we shall need the following *remark.* If $(m_{ij}(x))_{i,j=1}^{\nu}$ is the matrix function of 3.iii) satisfying (3.1), and any coefficient m_{ij}, $1 \leqq i, j \leqq n$, is approximated by Lipschitz continuous functions $m_{ij}^{(\nu)}$ as described before, then (3.1) holds for any $(m_{ij}^{(\nu)}(x))_{i,j=1}^n$, $\nu \in \mathbf{N}$, $x \in \Omega$. Indeed the settings in (9.3) and (9.4) do not change the ellipticity constants e_0, e_1, (9.6) changes both by a factor $\eta_i(\varphi^{-1}(z))$ or $\eta_0(z)$, which might be zero, the approximation in **9**.iv) does not change ellipticity constants, since (3.1) stays unchanged by taking means. (9.8) again alters the constants by a factor $\eta_i(x)$, which is observed to be the same than the one before. The sum in (9.10), however, implies, that all the factors accumulate to 1, since η_i^2 is a partition of unity.

10.

If the bounded open set $\emptyset \neq A \subset \mathbf{R}^n$ *has Lipschitz boundary, then for every* $\varepsilon > 0$, *there exists a* $c(\varepsilon, \Omega) > 0$, *such that for every* $f \in C^{0,\zeta}(A)$:

$$\sum_{i=1}^n \|f_{x^i}\|_0 \leqq \varepsilon \cdot \sum_{i=1}^n [f_{x^i}]_\zeta + c(\varepsilon, \Omega) \cdot \|f\|_0. \tag{10.1}$$

Proof. We first recall a proposition in [6, p. 136, 6.7.ii]: There are two positive constants L, ϱ_1 such that for any $y \in A$ with $\operatorname{dist}(y, \partial A) \leqq \varrho_1$ and any $0 < \varrho \leqq \varrho_1$ there exists a $x \in B(y, \varrho)$ such that $\bar{B}(x, \varrho/L) \subset \Omega$. Now choose ϱ at least so small, that $\zeta(\varrho(1+1/L)) \leqq \varepsilon$. If $\operatorname{dist}(y, \partial A) > \varrho_1$, there are for every $i \in \{1, \dots, n\}$ two points $y_1, y_2 \in \partial B(y, \varrho)$ and a $\bar{y} \in B(y, \varrho)$, such that

(10.2) $$|f_{x^i}(\bar{y})| = \frac{1}{2\varrho}|f(y_1)-f(y_2)| \leqq \|f\|_0/\varrho,$$

thus

(10.3) $$|f_{x^i}(y)| \leqq |f_{x^i}(\bar{y})| + |f_{x^i}(y)-f_{x^i}(\bar{y})| \leqq \|f\|_0/\varrho + [f_{x^i}]_\zeta \cdot \zeta(\varrho).$$

If $\operatorname{dist}(y, \partial A) \leqq \varrho_1$, there are $y_1, y_2 \in \partial B(x, \varrho/L)$, $\bar{y} \in B(x, \varrho/L)$:

(10.4) $$|f_{x^i}(\bar{y})| = \frac{L}{2\varrho}|f(y_1)-f(y_2)| \leqq L\|f\|_0/\varrho.$$

Since $\|y-\bar{y}\| \leqq \|y-x\| + \|x-\bar{y}\| \leqq \varrho(1+1/L)$ we conclude

(10.5) $$|f_{x^i}(y)| \leqq |f_{x^i}(y)-f_{x^i}(\bar{y})| + |f_{x^i}(\bar{y})| \leqq \zeta(\varrho(1+1/L)) \cdot [f_{x^i}]_\zeta + \frac{L}{\varrho}\|f\|_0,$$

which finally implies the statement.

11.

The proof of the theorem proceeds as follows:

i) We approximate the coefficients m_{ij}, m_i, m and the right hand side h by Lipschitz continuous functions $m_{ij}^{(\nu)}$, $m_i^{(\nu)}$, $m^{(\nu)}$, $h^{(\nu)}$, $\nu \in \mathbf{N}$ as described in **9**.

ii) Any equation $M^{(\nu)}u^{(\nu)} := m_{ij}^{(\nu)}u_{x^ix^j}^{(\nu)} + m_i^{(\nu)}u_{x^i}^{(\nu)} + m^{(\nu)}u^{(\nu)} = h^{(\nu)}$, $u^{(\nu)}|\partial\Omega = 0$ owns an unique solution $u^{(\nu)} \in C^{2,\alpha}(\Omega)$ by Schauder's theorem.

iii) We shall prove below, that

(11.1) $$\sum_{i,j=1}^n [u_{x^ix^j}^{(\nu)}]_\sigma \leqq d_3 \cdot \left(\|h^{(\nu)}\|_{0,\delta} + \|u^{(\nu)}\|_{1,\alpha} + \sum_{i,j=1}^n \|u_{x^ix^j}^{(\nu)}\|_0\right),$$

$d_3 = d_3(\Omega, \delta, \underline{\mu^{(\nu)}})$ where δ, σ are the same in **5**.i) and **5**.iii) and

$$\mu^{(\nu)} := \max_{1 \leqq i,j,k \leqq n} \{\|m_{ij}^{(\nu)}\|_{0,\delta}, \|m_k^{(\nu)}\|_{0,\delta}, \|m^{(\nu)}\|_{0,\delta}\}.$$

Here as later we underlined the quantity $\mu^{(\nu)}$ in order to emphasize, that d_3 depends monotonically increasing on $\mu^{(\nu)}$.

iv) Now we apply (10.1) with $f = u_{x^i}^{(\nu)}$, $i = 1, \ldots, n$ and ε small and infer:

(11.2) $$\sum_{1 \leqq i,j \leqq n} \|u_{x^ix^j}^{(\nu)}\|_0 \leqq \varepsilon \cdot \sum_{1 \leqq i,j \leqq n} [u_{x^ix^j}^{(\nu)}]_\delta + c(\varepsilon, \Omega) \cdot \sum_{1 \leqq i \leqq n} \|u_{x^i}^{(\nu)}\|_0$$

thus the terms $\|u_{x^ix^j}^{(\nu)}\|_0$ in the brackets on the right hand side of (11.1) may be dropped on changing from d_3 to a constant $d_4 = d_4(\Omega, \delta, \underline{\mu^{(\nu)}})$.

v) Now choose $p:=n/(1-\alpha)$ and recall

(11.3) $$\|u^{(\nu)}\|_{H^{2,p}} \leqq d_4 \cdot (\|h^{(\nu)}\|_{L_p} + \|u^{(\nu)}\|_{L_2})$$
$$d_4 = d_4(\Omega, \underline{\mu}^{(\nu)})$$

from [9; § 3.11, line (11.8)], [7], [8] and [10; chap. 36]. From [13; § 3, Th. I] we infer

(11.4) $$\|u^{(\nu)}\|_{L_2} \leqq c_{10}(\Omega) \cdot \|\underline{\mu}^{(\nu)}\|_0 \leqq d_5 \cdot \|M^{(\nu)} u^{(\nu)}\|_{L_p}$$
$$d_5 = d_5(\Omega, \underline{\mu}^{(\nu)}).$$

We infer from (11.2), (11.3) and the Morrey—Kondrashev theorem [11; Th. 3.6.6.]:

(11.5) $$\|u^{(\nu)}\|_{1,\alpha} \leqq d_{5*} \|h^{(\nu)}\|_0,$$
$$d_{5*} = d_{5*}(\Omega, \underline{\mu}^{(\nu)}).$$

Thus the term $\|u^{(\nu)}\|_{1,\alpha}$ in the brackets on the right side of (11.1) may be dropped on changing the constant.

vi) From (11.1), (11.2) and (11.5) we conclude

(11.6) $$\|u^{(\nu)}\|_{2,\sigma} \leqq d_6 \cdot \|h^{(\nu)}\|_{0,\delta}$$
$$d_6 = d_6(\Omega, \underline{\mu}^{(\nu)}).$$

(9.1) however implies, that the right hand side is now bounded by a constant depending on $\|h\|_{0,\delta}$ and $\underline{\mu}:=\max_{1\leqq i,j,k\leqq n}\{\|m_{ij}\|_{0,\delta}, \|m_k\|_{0,\delta}, \|m\|_{0,\delta}\}$. Since $C^{0,\theta}(\Omega)=C^{0,\delta}(\Omega)$ by (5.3), (5.6) this means, that $(u^{(\nu)})_{\nu\in\mathbf{N}}$ is bounded in $C^{2,\sigma}(\Omega)$. By **8.** thus a subsequence $(u^{(\nu')})$ exists, which converges to some $u_0\in C^{2,\sigma}(\Omega)$ in $C^2(\Omega)$ (see (**16**)). Since the coefficients $m_{ij}^{(\nu')}$, $m_k^{(\nu')}$, $m^{(\nu')}$ and the functions $h^{(\nu')}$ converge uniformly on Ω, u_0 is the desired solution of

$$Mu = h, \quad u|\partial\Omega = 0.$$

(5.10) implies the final part of the statement.

vii) The rest of this paper is concerned with proving the estimate (11.1). For simplicities sake we make the following change in notation: $\underline{\mu}:=\underline{\mu}^{(\nu)}$, $u:=u^{(\nu)}\subset C^{2,\alpha}(\Omega)\subset C^{2,\delta}$ (by **7**.iii)), $a_{ij}:=m_{ij}^{(\nu)}$, $a_k:=m_k^{(\nu)}$, $a:=m^{(\nu)}$, $f:=h^{(\nu)}$. Observe, that by **9**.v), (3.1) is valid for $(a_{ij})_{i,j=1}^n$.

12.

i) First write

(12.1) $$a_{ij}(x)\cdot u_{x^i x^j}(x) = f(x) - a_k(x)\cdot u_{x^k}(x) - a(x)\cdot u(x) =: f_1(x).$$

By (11.4), **7**.iii) and **7**.i) we have:

(12.2) $$\|f_1\|_{0,\delta} \leqq d_7(\underline{\mu})\cdot\|f\|_{0,\delta}.$$

ii) Now choose some fixed point $x_0 \in \Omega$ and let S be the symmetric, positive definite matrix, such that

$$S^{-2} = \big(a_{ij}(x_0)\big)_{i,j=1}^n.$$

We infer from (3.1), that $\|S\| \leqq e_0^{-1/2}$, $\|S^{-1}\| \leqq e_1^{1/2}$. Consider the transformation

$$\begin{aligned} y &= y_0 + S[x-x_0], \quad y_0 := x_0, \\ x &= x_0 + S^{-1}[y-y_0] \end{aligned} \tag{12.3}$$

and the functions:

$$\begin{aligned} v(y) &:= u(x_0 + S^{-1}[y-x_0]), \\ \underline{\alpha}_{ij}(y) &:= S_{i\nu} a_{\nu\mu}(x) \cdot S_{j\mu}, \\ f_2(y) &= f_1(x) \end{aligned} \tag{12.4}$$

and conclude:

$$\underline{\alpha}_{ij}(y) \cdot v_{y^i y^j}(y) = f_2(y), \quad \underline{\alpha}_{ij}(y_0) = \underline{\delta}_{ij}. \tag{12.5}$$

From **7**.iv) we infer, that

$$\begin{aligned} \|(\alpha_{ij})\|_{0,\delta} &\leqq c_{10}(T, e_0) \cdot \|(a_{ij})\|_{0,\delta} \\ \|f_2\|_{0,\delta} &\leqq c_{11}(T, e_0) \cdot \|f_1\|_{0,\delta}. \end{aligned} \tag{12.6}$$

Defining $\alpha_{ij}(y) := \underline{\delta}_{ij} - \underline{\alpha}_{ij}(y)$, we obtain

$$\Delta v(y) = f_2(y) + \alpha_{ij}(y) \cdot v_{y^i y^j}(y) \tag{12.7}$$

for $y \in B(y_0, \varrho_1)$, $\varrho_1 := \varrho_1(x_0) = \operatorname{dist}(x_0, \partial\Omega)/\sqrt{e_1}$.

13.

i) We shall prove for any $\bar{x} \in \partial\Omega$:

$$B(\bar{x}, 1/2d_1) \subset U_{\bar{x}} \text{ (see 3.ii)).} \tag{13.1}$$

There is a maximal $\varrho > 0$, such that $B(\bar{x}, \varrho) \subset U_{\bar{x}}$. Assume, that $\varrho < 1/2d_1$. By **3**.ii): $\varphi_{\bar{x}}(B(\bar{x}, \varrho)) \subset B(1/2)$. The set $V := \varphi_{\bar{x}}^{-1}(\bar{B}(1/2))$ is compact and $\operatorname{dist}(x, \partial U_{\bar{x}})$ attains a positive minimum $\varepsilon > 0$ on V. Since $B(\bar{x}, \varrho) \subset V$, this implies $B(\bar{x}, \varrho + \varepsilon) \subset U_{\bar{x}}$, which is a contradiction to the choice of ϱ.

ii) Choose $x_0 \in \Omega$, such that $\operatorname{dist}(x_0, \partial\Omega) \leqq \varrho_2 := \min\{1/2d_1, \sqrt{e_0}/2d_1^3 \sqrt{e_1}\}$. If $\bar{x} \in \partial\Omega$, such that $\|\bar{x} - x_0\| = \operatorname{dist}(x_0, \partial\Omega)$, then $x_0 \in U_{\bar{x}}$. From now on, we shall write U and φ instead of $U_{\bar{x}}$ and $\varphi_{\bar{x}}$, $\psi := \varphi^{-1}$. Define $z_0 := \varphi(x_0)$, $J_{ki}(z) := \psi^k_{z^i}(z)$, $\tilde{u}(z) := u(\psi(z))$, $z \in B^+(1)$, $b_{ij}(z) := a_{\mu\nu} \cdot (\psi(z)) \cdot J_{j\nu}^{-1}(z) \cdot J_{i\mu}^{-1}(z)$. We have $\tilde{u} \in C^{2,\delta}(B^+(1))$ by **7**.iv), **3**.ii) and for $z \in B^+(1)$:

$$\begin{aligned} b_{ij}(z) \cdot \tilde{u}_{z^i z^j}(z) &= f_1(\psi(z)) + u_{x^k}(\psi(z)) \cdot \psi^k_{z^i z^j}(z) \cdot b_{ij}(z) \\ &=: f_3(z). \end{aligned} \tag{13.2}$$

3.ii), **7**.iv) and **7**.i) imply:

(13.3) $$\|(b_{ij})\|_{0,\delta} \leqq c_{12}(d_1, \delta) \cdot \|(a_{ij})\|_{0,\delta}$$

and thus by (11.4), **3**.ii), **7**.i), (12.2):

$$\|f_3\|_{0,\delta} \leqq d_8(\underline{\mu}, d_1, \delta) \cdot \|f\|_{0,\delta}.$$

Now let S be the symmetric, positive definite matrix, such that $S^{-2} = (b_{ij}(z_0))^n_{i,j=1}$. Since ψ and φ are Lipschitz continuous with Lipschitz constant d_1, we have $\|(J_{ki})\| \leqq d_1$, $\|(J^{-1}_{ki})\| \leqq d_1$ and thus by the definition of (b_{ij}): $\|S^{-1}\| \leqq d_1 \cdot \sqrt{e_1}$, $\|S\| \leqq d_1/\sqrt{e_0}$. Further let O be an orthogonal matrix, such that

$$OS[\{y \in \mathbf{R}^n:\ y^n \geqq 0\}] = \{y \in \mathbf{R}^n:\ y^n \geqq 0\}.$$

Then $B(1/d_1\sqrt{e_1}) \subset OS[B(1)]$ and on defining $y_0 := OSz_0$ we obtain:

$$\begin{aligned} \|y_0\| &\leqq \|z_0\| \cdot d_1/\sqrt{e_0} \\ &\leqq \|\varphi(\bar{x}) - \varphi(x_0)\| \cdot d_1/\sqrt{e_0} \\ &\leqq \|\bar{x} - x_0\| \cdot d_1^2/\sqrt{e_0} \leqq 1/2d_1 \cdot \sqrt{e_1} =: \varrho_3. \end{aligned}$$

We shall use:

$$\tilde{v}(y) := \tilde{u}(S^{-1}O^{-1}y) \in C^{2,\delta}(B^+(2\varrho_3)),$$

$$\underline{b}_{ij}(y) = O \cdot S \cdot \big(b_{ij}(S^{-1} \cdot O^{-1}y)\big) \cdot S \cdot O^T$$

and $f_4(y) := f_3(S^{-1} \cdot O^{-1}y)$ to calculate

(13.4) $$\underline{b}_{ij}(y)\tilde{v}_{y^iy^j}(y) = f_4(y), \quad \underline{b}_{ij}(y_0) = \underline{\delta}_{ij}$$

(13.5) $$\|(\underline{b}_{ij})\|_{0,\delta} \leqq c_{13}(d_1, e_1, \delta) \cdot \|(a_{ij})\|_{0,\delta}$$
$$\|f_4\|_{0,\delta} \leqq d_9(\underline{\mu}, d_1, \delta, e_1) \cdot \|f\|_{0,\delta}.$$

iii) Let y_1 be the orthogonal projection of y_0 onto $\{y \in \mathbf{R}^n:\ y^n = 0\}$ and define $\underline{v}(y) := \tilde{v}(y) - \frac{1}{2} f_4(y_1) \cdot (y^n)^2 \in C^{2,\delta}(B^+(2\varrho_3))$. Since $u(x) = 0$ if $x \in \partial\Omega$ we still have $\underline{v}(y) = 0$ if $y^n = 0$. Further we have

(13.6) $$\underline{b}_{ij}(y) \cdot v_{y^iy^j}(y) = f_4(y) - f_4(y_1) =: f_5(y)$$
$$\|f_5\|_{0,\delta} \leqq 2\|f_4\|_{0,\delta}.$$

Now define $\underline{\beta}_{ij}(y) := \underline{\delta}_{ij} - \underline{b}_{ij}(y)$ and observe, that

(13.7) $$\Delta\underline{v}(y) = f_5(y) - \underline{\beta}_{ij}(y) \cdot \underline{v}_{y^iy^j}(y) := f_6(y).$$

We have $\underline{\beta}_{ij}(y_0) = 0$, $[(\underline{\beta}_{ij})]_\delta = [(\underline{b}_{ij})]_\delta$

$$\big\|\big(\underline{\beta}_{ij}(y)\big)\big\| \leqq 1 + \big\|\big(\underline{b}_{ij}(y)\big)\big\|.$$

We recall the function S in **9**.iii) and define

$$\Sigma\hat{h}(y) = \operatorname{sgn}(y^n) \cdot \hat{h}(s(y)), \quad y \in B(2\varrho_3),$$

for any function $\hat{h}$ on $B^+(2\varrho_3)$, and

$$v := \Sigma \underline{v},\ f_8 := \Sigma f_6,\ f_7 := \Sigma f_5.$$

Since $\underline{v}(y)$ and $\underline{v}_{y^i}(y)$, $i \neq n$ equal zero if $y^n = 0$, we first observe, that $v \in C^1(B(2\varrho_3))$ and that the first derivatives are still Lipschitz continuous, thus $v \in H^{2,\infty}(B(2\varrho_3))$ and

$$v_{y^iy^j}(y) = \begin{cases} (\Sigma \underline{v}_{y^iy^j})(y), & \text{if} \quad i \neq n \neq j \text{ or } i = n = j \\ \underline{v}_{y^iy^j}(s(y)), & \text{otherwise} \end{cases}$$

$\mathscr{L}$-almost everywhere on $B(2\varrho_3)$. Thus v is a strong solution of

$$\Delta v = f_8 \quad \text{on } B(2\varrho_3) \quad \text{(see [5, 4.5.6])}. \tag{13.8}$$

Define

$$\beta_{ij}(y) = \begin{cases} \underline{\beta}_{ij}(s(y)), & \text{if} \quad i \neq n \neq j \text{ or } i = n = j \\ (\Sigma \underline{\beta}_{ij})(y) & \text{otherwise}, \ y \in B(2\varrho_3) \end{cases}$$

and conclude, that $\sum (\underline{\beta}_{ij} \cdot \underline{v}_{y^iy^j})(y) = \beta_{ij}(y) \cdot v_{y^iy^j}(y)$ $\mathscr{L}$-almost everywhere. Thus $f_8 = \beta_{ij} \cdot v_{y^iy^j} + f_7$. For any $y \in B(y_0, \varrho_3) \cap B^-(1)$ we have $\|s(y) - y_1\| = \|y - y_1\| \leqq \|y - y_0\|$ and $\|y_1 - y_0\| \leqq \|y - y_0\|$. Since $f_5(y_1) = 0$, we infer

$$\begin{aligned} |f_7(y) - f_7(y_0)| &\leqq |f_5(s(y))| + |f_5(y_0)| \\ &= |f_5(s(y)) - f_5(y_1)| + |f_5(y_0) - f_5(y_1)| \\ &\leqq [f_5]_\delta \big(\delta(\|s(y) - y_1\|) + \delta(\|y_0 - y_1\|)\big) \\ &\leqq 2[f_5]_\delta \cdot \delta(\|y - y_0\|). \end{aligned} \tag{13.9}$$

This inequality now holds for all $y \in B(y_0, \varrho_3)$. Since further $\|s(y) - y_0\| \leqq \|y - y_0\|$ we obtain similarly for any $y \in B(y_0, \varrho_3)$

$$\begin{aligned} |\beta_{ij}(y)| &\leqq [\underline{\beta}_{ij}]_\delta \cdot \delta(\|y - y_0\|) \\ &\leqq c_{14}(d_1, e_1, \delta) \cdot \|(a_{ij})\|_\delta \cdot \delta(\|y - y_0\|). \end{aligned} \tag{13.10}$$

14.

i) We notice, that (12.7) and (13.8) are of the same type and proceed to prove the following proposition.

Let $v \in H^{2,\infty}(B(y_0, \varrho_4), \mathbf{R})$, $0 < \varrho_4$, *be a strong solution of*

$$\Delta v(y) = f_{10}(y) := f_9(y) + \gamma_{ij}(y) \cdot v_{y^iy^j}(y) \tag{14.1}$$

where the functions f_9 and γ_{ij} own the properties:

$$(14.2)\qquad \begin{aligned} &|f_9(y_0)-f_9(y_1)| \leqq K_1\cdot\delta(\|y_0-y_1\|)\\ &|\gamma_{ij}(y_0)-\gamma_{ij}(y_1)| \leqq K_2\cdot\delta(\|y_0-y_1\|)\\ &y_1\in B(y_0,\varrho_4),\quad K_i=K_i(d_1,e_0,e_1,\delta,\underline{\mu}),\quad i=1,2.\end{aligned}$$

Then there is a $0<\varrho_5<\varrho_4$, $\varrho_5=\varrho_5(\delta,n,\alpha,\mu)$ *depending monotonically decreasing on* μ *and a constant* $d_{11}=d_{11}(d_1,e_0,e_1,\delta,\alpha,\underline{\mu})$, *such that for any* $0<r\leqq\varrho_5$

$$(14.3)\qquad r^{-n}\cdot\delta(r)^{-2}\cdot\int_{B(y_0,r)}\sum_{i,j=1}^{n}(v_{y^iy^j}-m(v_{y^iy^j},y_0,r))^2\,d\mathscr{L} \leqq d_{11}(\|f\|_{0,\delta}+\|v_{y^iy^j}\|_0).$$

ii) *Proof.* For $0<\varrho\leqq\varrho_4$ define

$$(14.4)\qquad \begin{aligned}\tau_{ki}(\varrho) &:= m(v_{y^iy^j},y_0,\varrho)/(1+\underline{\delta}_{ki})\\ &=(\omega_n\cdot\varrho^{n+1})^{-1}\cdot\int_{\partial B(y_0,\varrho)}v_{y^k}\cdot(y^i-y_0^i)\,d\mathscr{H}/(1+\underline{\delta}_{ki})\end{aligned}$$

$$\sigma_k(\varrho):=\partial m(v_{y^k},y,\varrho).$$

The function

$$(14.5)\qquad v_\varrho(y):=v(y)-\sigma_k(\varrho)\cdot(y^k-y_0^k)-\tau_{ki}(\varrho)\cdot(y^k-y_0^k)\cdot(y^i-y_0^i)$$

strongly solves

$$(14.6)\qquad \Delta v_\varrho(y)=f_{10}(y)-m(f_{10},y_0,\varrho)=:f_{11}(y).$$

Let $\varphi\in C_0^\infty(B(y_0,\varrho_4))$. Multiplying (14.6) with φ_{y^k} and integrating by parts twice yields:

$$(14.7)\qquad \int_{B(y_0,\varrho_4)}v_{\varrho,y^ky^i}\cdot\varphi_{y^i}\,d\mathscr{L}=\int_{B(y_0,\varrho_4)}f_{10}\cdot\varphi_{y^k}\,d\mathscr{L}$$

and by some continuity argument, this equation holds for any $\varphi\in\mathring{H}^{1,1}(B(y_0,\varrho_4))$. For $0<\varepsilon<r\leqq\varrho_4$ define

$$\xi_{r,\varepsilon}(t):=\begin{cases}1, & \text{if}\quad 0<t<r-\varepsilon\\ (r-t)/\varepsilon, & \text{if}\quad r-\varepsilon\quad t<r\\ 0, & \text{if}\quad t>r\end{cases}$$

and choose

$$(14.8)\qquad \varphi(y):=v_{\varrho,y^k}(y)\cdot\xi_{r,\varepsilon}(\|y-y_0\|),$$

(14.7) implies

(14.9)
$$\int_{B(y_0,r)} v^2_{\varrho,y^k y^i}(y)\cdot\xi_{r,\varepsilon}(\|y-y_0\|)\,d\mathscr{L}$$
$$=\frac{1}{\varepsilon}\int_{B(y_0,r)\setminus B(y_0,r-\varepsilon)} v_{\varrho,y^k y^i}(y)\cdot v_{\varrho,y^k}(y)\cdot(y^i-y_0^i)/\|y-y_0\|\,d\mathscr{L}$$
$$+\int_{B(y_0,r)} f_{11}(y)\cdot v_{\varrho,y^k y^k}(y)\cdot\xi_{r,\varepsilon}(\|y-y_0\|)\,d\mathscr{L}$$
$$-\frac{1}{\varepsilon}\int_{B(y_0,r)\setminus B(y_0,r-\varepsilon)} f_{11}(y)\cdot v_{\varrho,y^k}(y)\cdot(y^k-y_0^k)/\|y-y_0\|\,d\mathscr{L}.$$

Inserting (14.5) we observe, that any integral is a sum of products, each of which consists of factors, which are either $\sigma_k(\varrho)$ or $\sigma_k^2(\varrho)$ or $\sigma_{ki}(\varrho)$ or $\sigma_{ki}^2(\varrho)$, or an integral, that does not depend on ϱ. Thus there exists one set $N\subset(0,\varrho_4)$, $\mathscr{L}_1\langle N\rangle=0$, not depending on ϱ, such that for any $r\notin N$ and any $\varrho\in(0,\varrho_4)$ the limit $\varepsilon\to 0$ exists. Then it is legitimate to set $\varrho=r\notin N$. Using the notation $w_\varrho:=(v_{\varrho,y^1},\ldots,v_{\varrho,y^n})$ and summing over k we obtain:

(14.10)
$$\int_{B(y_0,\varrho)}\|\nabla w_\varrho\|^2\,d\mathscr{L}=\frac{1}{2\varrho}\int_{\partial B(y_0,\varrho)}(w_\varrho^2)_{y^i}\cdot(y^i-y_0^i)\,d\mathscr{H}$$
$$+\int_{B(y_0,\varrho)} f_{11}^2\,d\mathscr{L}-\frac{1}{\varrho}\int_{\partial B(y_0,\varrho)} f_{11}\cdot w_\varrho^k\cdot(y^k-y_0^k)\,d\mathscr{H}.$$

Now notice, that

(14.11)
$$w_\varrho^k(y)=v_{y^k}-\sigma_k(\varrho)-\tau_{ki}(\varrho)\cdot(1+\delta_{ik})(y^i-y_0^i)$$

and

(14.12)
$$\int_{\partial B(y_0,\varrho)}(w_\varrho^k)^2\,d\mathscr{H}=\int_{\partial B(y_0,\varrho)} v_{y^n}^2\,d\mathscr{H}-(n\cdot\omega_n\cdot\varrho^{n-1})\cdot\sigma_k(\varrho)^2$$
$$-(\omega_n\cdot\varrho^{n+1})\cdot\sum\nolimits_{i=1}^n[\tau_{ki}(\varrho)\cdot(1+\delta_{ik})]^2.$$

If $h\in H^{1,1}$, then

(14.13)
$$\varrho\cdot\int_{\partial B(y_0,\varrho)} h\,d\mathscr{H}=\int_{B(y_0,\varrho)} h_{y^i}(y)\cdot(y^i-y_0^i)\,d\mathscr{L}y+n\cdot\int_{B(y_0,\varrho)} h\,d\mathscr{L},$$

which implies, that $\int_{\partial B(y_0,\varrho)} h\,d\mathscr{H}$ is an absolutely continuous function of ϱ and

(14.14)
$$D_\varrho\int_{\partial B(y_0,\varrho)} h\,d\mathscr{H}=\frac{n-1}{\varrho}\int_{\partial B(y_0,\varrho)} h\,d\mathscr{H}$$
$$+\frac{1}{\varrho}\int_{\partial B(y_0,\varrho)} h_{y^i}(y)\cdot(y^i-y_0^i)\,d\mathscr{H}y.$$

Using (14.4) it is now easy to calculate from (14.12)

$$D_\varrho \int_{\partial B(y_0,\varrho)} w_\varrho^2\, d\mathcal{H} = \frac{n-1}{\varrho} \int_{\partial B(y_0,\varrho)} w_\varrho^2\, d\mathcal{H} + \tag{14.15}$$

$$+ \frac{1}{\varrho} \int_{\partial B(y_0,\varrho)} (w_\varrho)^2_{y^i} \cdot (y^i - y_0^i)\, d\mathcal{H} y.$$

Inserting this in (14.10) we arrive at

$$2\int_{B(y_0,\varrho)} \|\nabla w_\varrho\|^2 d\mathcal{L} = -\frac{n-1}{\varrho} \int_{\partial B(y_0,\varrho)} w_\varrho^2\, d\mathcal{H} - D_\varrho \int_{\partial B(y_0,\varrho)} w_\varrho^2\, d\mathcal{H} \tag{14.16}$$

$$+ 2\int_{B(y_0,\varrho)} f_{11}^2\, d\mathcal{L} - \frac{2}{\varrho} \int_{\partial B(y_0,\varrho)} f_{11} \cdot w_\varrho^k \cdot (y^k - y_0^k)\, d\mathcal{H}.$$

iii) We recall, that $f_{11}(y) = f_{10}(y) - m(f_{10}, y_0, \varrho)$, $f_{10} := f_g - \gamma_{ij} \cdot v_{y^i y^j}$ and introduce the notation $g_e := \gamma_{ij} \cdot v_{y^i y^j} - m(\gamma_{ij} \cdot v_{y^i y^j}, y_0, \varrho)$. We have

$$f_{11}^2 \leqq 2\{(f_9 - m(f_9, y_0, \varrho))^2 + g_e^2\}, \tag{14.17}$$

$$\int_{B(y_0,\varrho)} g_\varrho^2\, d\mathcal{L}^{1/2} \leqq \int_{B(y_0,\varrho)} (\gamma_{ij} \cdot v_{y^i y^j} - m(\gamma_{ij}) \cdot m(v_{y^i y^j}))^2\, d\mathcal{L}^{1/2} \tag{14.18}$$

$$= \int_{B(y_0,\varrho)} \{\gamma_{ij}(v_{y^i y^j} - m(v_{y^i y^j})) + m(v_{y^i y^j}) \cdot (\gamma_{ij} - m(\gamma_{ij}))\}\, d\mathcal{L}^{1/2}$$

$$\leqq \int_{B(y_0,\varrho)} \{\gamma_{ij}(v_{y^i y^j} - m(v_{y^i y^j}))\}^2\, d\mathcal{L}^{1/2}$$

$$+ \int_{B(y_0,\varrho)} \{m(v_{y^i y^j})(\gamma_{ij} - m(\gamma_{ij}))\}^2\, d\mathcal{L}^{1/2}$$

$$\leqq c_{15} \cdot \delta(\varrho) \cdot \int_{B(y_0,\varrho)} \|\nabla w_\varrho\|^2\, d\mathcal{L}^{1/2} + \|v_{y^i y^j}\|_0 \cdot v(\gamma_{ij}, y_0, \varrho)^{1/2}.$$

From (14.17), (14.18), (14.2) and some simple calculations we conclude

$$\int_{B(y_0,\varrho)} f_{11}^2\, d\mathcal{L} \leqq 4 \cdot d_{10}^2 \cdot \|f\|_{0,\delta} \cdot \delta(\varrho)^2 \varrho^n \tag{14.19}$$

$$+ 2c_{15}^2 \cdot \delta(\varrho)^2 \cdot \|(a_{ij})\|_{0,\delta} \cdot \int_{B(y_0,\varrho)} \|\nabla w_\varrho\|^2\, d\mathcal{L}$$

$$+ 2c_{16} \cdot \delta(\varrho)^2 \cdot \varrho^n \cdot \|(a_{ij})\|_{0,\delta} \cdot \sum_{i,j=1}^n \|v_{y^i y^j}\|_0^2,$$

$$c_{16} = c_{16}(d_1, e_1, e_0, \delta, n).$$

iv) The last term in (4.16) is bounded above by

$$\frac{1-\alpha}{\varrho}\int_{\partial B(y_0,\varrho)} w_\varrho^2\, d\mathscr{H}+\frac{\varrho}{1-\alpha}\int_{\partial B(y_0,\varrho)} f_{11}^2\, d\mathscr{H}. \tag{14.20}$$

Now choose $0<\varrho_5\leqq\varrho_4$ so small, that

$$\delta^2(\varrho_5)\leqq\eta/(n+2\alpha)\cdot 4c_{15}^2\cdot\|(a_{ij})\|_\delta, \tag{14.21}$$

where

$$\eta=(1-\alpha)/12.$$

Now we derive from (4.16), (4.19), (4.20)

$$(2-\eta)\int_{B(y_0,\varrho)}\|\nabla w_\varrho\|^2\, d\mathscr{L} \tag{14.22}$$

$$= -\frac{n-2+\alpha}{\varrho}\int_{\partial B(y_0,\varrho)} w_\varrho^2\, d\mathscr{H}-D_\varrho\int_{\partial B(y_0,\varrho)} w_\varrho^2\, d\mathscr{H}$$

$$+4c_{16}\|(a_{ij})\|_\delta\cdot\delta(\varrho)^2\varrho^n\sum_{i,j}\|v_{y\,y^j}\|_0^2$$

$$+8\cdot d_{10}^2\cdot\|f\|_{0,\delta}\cdot\delta(\varrho)^2\varrho^n+\frac{\varrho}{1-\alpha}\int_{\partial B(y_0,\varrho)} f_{11}^2\, d\mathscr{H}.$$

v) It is essential to observe, that for any $0<\varrho<\varrho_4$ the inequality

$$2\int_{\partial B(y_0,\varrho)} w_\varrho^2\, d\mathscr{H}\leqq\varrho\cdot\int_{B(y_0,\varrho)}\|\nabla w_\varrho\|^2\, d\mathscr{L} \tag{14.23}$$

holds. Indeed the eigenvalue problem

$$\text{(E)}\qquad \Delta u(y)=0 \quad\text{on}\quad B(y_0,\varrho),\quad \lambda\cdot u(y)=\big(\nabla u(y)\cdot y\big)\quad\text{on}\quad \partial B(y_0,\varrho)$$

owns in any dimension the eigenvalues $\lambda_i=i$, $i=0,1,2,\dots$ and the restrictions of the corresponding eigenfunctions $u_j^{(i)}$ to $\partial B(y_0,\varrho)$ form a complete orthogonal basis of $L_2(\partial B(y_0,\varrho))$. Actually u is an eigenfunction of (E) if and only if $\tilde u:=u|\partial B(y_0,\varrho)$ is an eigenfunction of the Laplace—Beltrami operator on $\partial B(y_0,\varrho)$ corresponding to the eigenvalue $\gamma:=\lambda(\lambda+n-2)$ and it is $u(y)=\|y-y_0\|^\lambda\cdot\tilde u([y-y_0]/\|y-y_0\|)$. The eigenvalues λ_i are characterised by the Rayleigh quotient:

$$\text{(R)}\qquad \lambda_i=\min\Big\{\int_{B(y_0,\varrho)}\|\nabla z\|^2\, d\mathscr{L}: z\in H_2^1(B(y_0,\varrho)),\ \int_{\partial B(y_0,\varrho)} z\cdot u_j^{(k)}\, d\mathscr{H}=0$$

$$\text{for all eigenfunctions } u_j^{(k)} \text{ of some eigenvalue } \lambda_k,\ k<i\Big\}.$$

The eigenfunctions corresponding to $\lambda_0=0$ are the constant functions, the space of eigenfunctions corresponding to $\lambda_1=1$ is spaned by $u_j^{(1)}(y):=y^j-y_0^i$, $j=1,\dots,n$ (see [16], [12]). Thus (R) implies (14.23) by the definition of the τ_{ki} and σ_k (see (14.4)).

Multiply (14.23) by $-\dfrac{3+\alpha}{2\varrho}$ and add the resulting line to (14.22):

$$\left(\frac{1-\alpha}{2}-\eta\right)\int_{B(y_0,\varrho)}\|\nabla w_\varrho\|^2\,d\mathscr{L} \tag{14.24}$$

$$\leqq -\frac{n+1+2\alpha}{\varrho}\int_{\partial B(y_0,\varrho)} w_\varrho^2\,d\mathscr{H}+D_\varrho\int_{\partial B(y_0,\varrho)} w_\varrho^2\,d\mathscr{H}$$

$$+\left(4c_{16}\|(a_{ij})\|_{0,\delta}\cdot\sum\nolimits_{i,j=1}^{n}\|v_{y^iy^j}\|_0^2+8d_{10}^2\|f\|_{0,\delta}\right)\cdot\delta(\varrho)^2\cdot\varrho^n$$

$$+\frac{\varrho}{1-\alpha}\int_{\partial B(y_0,\varrho)} f_{11}^2\,d\mathscr{H}.$$

Consider now the two continuous functions

$$\xi(\varrho):=\eta\cdot\int_{B(y_0,\varrho)}\|\nabla w_\varrho\|^2\,d\mathscr{L} \tag{14.25}$$

and

$$\zeta(\varrho):=2(n+2+2\alpha)\left(c_{16}\|(a_{ij})\|_{0,\delta}\sum\nolimits_{i,j=1}^{n}\|v_{y^iy^j}\|_0+2d_{10}^2\cdot\|f\|_{0,\delta}\right)\cdot\delta(\varrho)^2\cdot\varrho^n.$$

The set $E:=\{\varrho\in(0,\varrho_5)\colon \zeta(\varrho)<\xi(\varrho)\}$ is an open set and thus the union of an at most denumerable sequence of open disjoint, nonempty intervalls (r'_ν, r_ν), $\nu\in\mathbf{N}$, $0<r'_\nu<r_\nu\leqq\varrho_5$. For any $r'_\nu<\varrho<r_\nu$ we obtain by adding the inequality $-\xi(\varrho)<-\zeta(\varrho)$ to (14.24):

$$\left(\frac{1-\alpha}{2}-2\eta\right)\int_{B(y_0,\varrho)}\|\nabla w_\varrho\|^2\,d\mathscr{L} \tag{14.26}$$

$$\leqq -\frac{n+1+2\alpha}{\varrho}\int_{\partial B(y_0,\varrho)} w_\varrho^2\,d\mathscr{H}+D_\alpha\int_{\partial B(y_0,\varrho)} w_\varrho^2\,d\mathscr{H}$$

$$+\frac{\varrho}{1-\alpha}\int_{\partial B(y_0,\varrho)} f_{11}^2\,d\mathscr{H}$$

$$-2(n+2\alpha)\left(c_{16}\|(a_{ij})\|_{0,\delta}\cdot\sum\nolimits_{i,j=1}^{n}\|v_{y^iy^j}\|_0+2d_{10}^2\|f\|_{0,\delta}\right)\cdot\delta(\varrho)^2\cdot\varrho^n.$$

Observe now, that by (5.7) for $q>0$

$$D_\varrho\left(\varrho^{-q}\cdot\delta(\varrho)^{-2}\right)=\varrho^{-(q+1)}\cdot\delta(\varrho)^{-2}\left(-q-2\cdot\frac{\varrho\cdot D_\varrho\delta(\varrho)}{\delta(\varrho)}\right) \tag{14.27}$$

$$\geqq -(q+2\alpha)\cdot\varrho^{-(q+1)}\cdot\delta(\varrho)^{-2}.$$

We first use $q=n+1$ to obtain

$$\varrho^{-(n+1)}\cdot\delta(\varrho)^{-2}\cdot\left(-\frac{n+2+2\alpha}{\varrho}\int_{\partial B(y_0,\varrho)} w_\varrho^2\,d\mathscr{H}+D_\varrho\int_{\partial B(y_0,\varrho)} w_\varrho^2\,d\mathscr{H}\right) \tag{14.28}$$

$$\leqq D_\varrho\left(\varrho^{-(n+1)}\delta(\varrho)^{-2}\cdot\int_{\partial B(y_0,\varrho)} w_\varrho^2\,d\mathscr{H}\right).$$

Thus multiplying (14.26) by $\varrho^{-(n+1)}\cdot\delta(\varrho)^2$ and integrating over (r, r_ν), $r_{\nu'}<r<r'_\nu$, we arrive at:

$$\left(\frac{1-\alpha}{2}-2\eta\right)\int_r^{r_\nu}\delta(\varrho)^{-2}\cdot\varrho^{-(n+1)}\int_{B(y_0,\varrho)}\|\nabla w_\varrho\|^2\,d\mathscr{L}\,d\varrho \tag{14.29}$$
$$\leqq\left[\varrho^{-(n+1)}\delta(\varrho)^{-2}\int_{\partial B(y_0,\varrho)}w_\varrho^2\,d\mathscr{H}\right]_{\varrho=r}^{\varrho=r_\nu}$$
$$-2(n+2\alpha)\left(c_{16}\|(a_{ij})\|_{0,\delta}\cdot\textstyle\sum_{i,j=1}^n\|v_{y^iy^j}\|_0+2d_{10}^2\|f\|_{0,\delta}\right)\cdot\delta(\varrho)^2\cdot\varrho^n$$
$$+\frac{1}{1-\alpha}\int_r^{r_\nu}\varrho^{-n}\cdot\delta(\varrho)^{-2}\int_{\partial B(y_0,\varrho)}f_{11}^2\,d\mathscr{H}\,d\varrho.$$

Indeed f_{11} depends on ϱ (see (14.6)), but it is easy to see, that

$$D_\varrho\int_{B(y_0,\varrho)}f_{11}^2\,d\mathscr{H}=\int_{\partial B(y_0,\varrho)}f_{11}^2\,d\mathscr{H},$$

and thus by integrating by parts the last integral in (14.29) can be transformed to

$$\left[\varrho^{-n}\delta(\varrho)^{-2}\int_{B(y_0,\varrho)}f_{11}^2\,d\mathscr{L}\right]_{\varrho=r}^{\varrho=r_\nu} \tag{14.30}$$
$$-\int_r^{r_\nu}D_\varrho\left(\varrho^{-n}\delta(\varrho)^{-2}\right)\cdot\int_{B(y_0,\varrho)}f_{11}^2\,d\mathscr{L}\,d\varrho.$$

Dropping one negative term and using (14.27) with $q=n$ and (14.19) we find the upper bound

$$r_\nu^{-n}\cdot\delta(r_\nu)^{-2}\int_{B(y_0,r_\nu)}f_{11}^2\,d\mathscr{L} \tag{14.31}$$
$$+(n+2\alpha)\int_r^{r_\nu}\varrho^{-(n+1)}\delta(\varrho)^{-2}\int_{B(y_0,\varrho)}f_{12}^2\,d\mathscr{L}\,d\varrho$$
$$\leqq 4d_{10}^2\|f\|_{0,\delta}+2c_{16}\|(a_{ij})\|_{0,\delta}\cdot\textstyle\sum_{i,j=1}^n\|v_{y^iy^j}\|_0^2$$
$$+2c_{15}^2\|(a_{ij})\|_{0,\delta}\,r_\nu^{-n}\int_{B(y_0,r_\nu)}\|\nabla w_\varrho\|^2\,d\mathscr{L}$$
$$+(n+2\alpha)\cdot 2c_{15}^2\|(a_{ij})\|_{0,\delta}\cdot\delta(r_\nu)^2\int_r^{r_\nu}\varrho^{-(n+1)}\delta(\varrho)^{-2}\cdot\int_{B(y_0,\varrho)}\|\nabla w_\varrho\|^2\,d\mathscr{L}\,d\varrho$$
$$+(n+2\alpha)\left(4d_{10}^2\|f\|_{0,\delta}+2c_{16}\|(a_{ij})\|_{0,\delta}\cdot\textstyle\sum_{i,j=1}^n\|v_{y^iy^j}\|_0^2\right)\cdot\int_r^{r_\nu}\varrho^{-1}\,d\varrho.$$

Using this estimate in (14.29) we obtain (see (14.21))

$$\left(\frac{1-\alpha}{2}-3\eta\right)\cdot\int_r^{r_\nu}\delta(\varrho)^{-2}\cdot\varrho^{-(n+1)}\cdot\int_{B(y_0,y)}\|\nabla w_\varrho\|^2\,d\mathscr{L}\,d\varrho \tag{14.32}$$
$$\leqq\left[\varrho^{-(n+1)}\delta(\varrho)^{-2}\int_{\partial B(y_0,\varrho)}w_\varrho^2\,d\mathscr{H}\right]_{\varrho=r}^{\varrho=r_\nu}$$
$$+c_{17}\left(\|f\|_{0,\delta}+\textstyle\sum_{i,j=1}^n\|v_{y^iy^j}\|_0\right)+c_{18}\cdot r_\nu^{-n}\int_{B(y_0,r_\nu)}\|\nabla w_\varrho\|^2\,d\mathscr{L},$$
$$c_{17}=c_{17}(d_1,e_1,e_0,\delta,\underline{\mu}),\quad c_{18}=c_{18}(d_1,e_1,e_0,\delta,\underline{\mu}).$$

Now insert η from (14.21) and add

$$-[(1-\alpha)/4(n+2\alpha)]\int_r^{r_\nu} \varrho^{-n}\delta(\varrho)^2\int_{\partial B(y_0,\varrho)}\|\nabla w_\varrho\|^2\, d\mathscr{H}d\varrho$$

to the left side of (14.32). Using (14.27) with $q=n$, we obtain the lower bound

$$-\frac{1-\alpha}{4(n+2\alpha)}\Big[\int_r^{r_\nu} D_\varrho\Big(\varrho^{-n}\delta(\varrho)^{-2}\int_{\partial B(y_0,\varrho)}\|\nabla w_\varrho\|^2\, d\mathscr{L} + \varrho^{-n}\delta^{-2}\int_{\partial B(y_0,\varrho)}\|\nabla w_\varrho\|^2\, d\mathscr{H}\Big].\tag{14.33}$$

Now again it is easy to observe, that

$$D_\varrho\int_{B(y_0,\varrho)}\|\nabla w_\varrho\|^2\, d\mathscr{L} = \int_{\partial B(y_0,\varrho)}\|\nabla w_\varrho\|^2\, d\mathscr{H}\tag{14.34}$$

and we hence obtain

$$-\frac{1-\alpha}{4(n+2\alpha)}\Big[\varrho^{-n}\cdot\delta(\varrho)^{-2}\int_{B(y_0,\varrho)}\|\nabla w_\varrho\|^2\, d\mathscr{L}\Big]_{\varrho=r}^{\varrho=r_\nu}$$
$$\leqq \Big[\varrho^{-(n+1)}\delta(\varrho)^{-2}\cdot\int_{\partial B(y_0,\varrho)} w_\varrho^2\, d\mathscr{H}\Big]_{\varrho=r}^{\varrho=r_\nu} + c_{17}\big(\|f\|_{0,\delta}+\textstyle\sum_{i,j=1}^n\|v_{y^iy^j}\|_0\big)+c_{18}\cdot r_\nu^{-n}\displaystyle\int_{B(y_0,r_\nu)}\|\nabla w_\varrho\|^2\, d\mathscr{L}.$$

Dropping one negative term on the right and using (14.23) once more, we arrive at

$$r^{-n}\delta(r)^{-2}\int_{B(y_0,r)}\|\nabla w_r\|^2\, d\mathscr{L} \leqq c_{19}\Big(\|f\|_{0,\delta}+\textstyle\sum_{i,j=1}^n\|v_{y^iy^j}\|_0 + r_\nu^{-n}\cdot\displaystyle\int_{B(y_0,r_\nu)}\|\nabla w_{r_\nu}\|^2\, d\mathscr{L}\Big),\tag{14.35}$$
$$c_{19}=c_{19}(d_1, e_1, e_0, \delta, \underline{\mu}).$$

However, by the definition of E, either $r_\nu=\varrho_5$ or

$$r_\nu^{-n}\delta(r_\nu)^{-2}\int_{B(y_0,r_\nu)}\|\nabla w_{r_\nu}\|^2\, d\mathscr{L} \leqq c_{20}\big(\|f\|_{0,\delta}+\textstyle\sum_{i,j=1}^n\|v_{y^iy^j}\|_0\big),\tag{14.36}$$
$$c_{20}=c_{20}(d_1, e_1, e_0, \delta, \alpha, \underline{\mu}).$$

So we achieved, for all $0<r\leqq\varrho_5$, the inequality

$$r^{-n}\delta(r)^{-2}\cdot\int_{B(y_0,r)}\|\nabla w_r\|^2\, d\mathscr{L} \leqq d_{11}(\|f\|_{0,\delta}+\|v_{y^iy^j}\|_0),\tag{14.37}$$
$$d_{11}=\max\{c_{20}, c_{19}(1+c_{20})\}.$$

15.

i) From **12.** and **13.** we infer, that

$$u_{x^i x^j}(x) = v_{y^k y^l}(\lambda(x)) \cdot \lambda^k_{x^i}(x) \cdot \lambda^l_{x^j}(x) + v_{y^k}(\lambda(x)) \cdot \lambda^k_{x^i x^j}(x), \tag{15.1}$$

in some vicinity of x_0, where either $\lambda(x) = y_0 + S[x - x_0]$ or $\lambda(x) = O \cdot S[\varphi(x)]$. We first remark, that by **3.** and the definitions of v

$$\|v\|_{1,\alpha} \leqq c_{21}(d_1, e_0, e_1) \cdot \|u\|_{1,\alpha},$$

thus by (11.5), **3.** and the device of (14.18) we derive from (15.1)

$$v(u_{x^i x^j}(x), x_0, \varrho) \leqq c_{22}(d_1, d_5, e_1, e_0) \cdot \left(\sum_{k,l=1}^n (v(v_{y^k y^l}(\lambda(x)), x_0, \varrho) + \|v_{y^k y^l}\|_0) + \|f\|_0\right), \tag{15.2}$$

which implies by some simple estimates

$$v(u_{x^i x^j}, x_0, \varrho, \Omega) \leqq c_{23}(d_1, e_1, e_0, \Omega, \underline{\mu}) \cdot \left(\sum_{k,l=1}^n (v(v_{y^k y^l}, x_0, c_{24}\varrho) + \|v_{y^k y^l}\|_0) + \|f\|_0\right), \quad 1 < c_{24} = c_{24}(d_1, e_1, e_0). \tag{15.3}$$

This, however, implies by (14.37)

$$v(u_{x^i x^j}, x_0, \varrho, \Omega) \leqq d_{12}\left(\sum_{k,l} \|v_{y^k y^l}\|_0 + \|f\|_{0,\delta}\right) \cdot \varrho^n \cdot \delta(\varrho)^2, \quad d_{12} = d_{12}(\Omega, \delta, e_1, e_0, \underline{\mu}), \quad 0 < \varrho \leqq \varrho_6 = \varrho_5 / c_{24}. \tag{15.4}$$

ii) On the other hand, we have

$$v_{y^k y^l}(y) = u_{x^i x^j}(\lambda^{-1}(y)) \cdot \lambda^{-1}(y)^i_{y^k} \cdot \lambda^{-1}(y)^j_{y^l} + u_{x^i}(\lambda^{-1}(y)) \cdot \lambda^{-1}(y)^i_{y^k y^l}, \tag{15.5}$$

which by (11.5) and **3.** implies:

$$\|v_{y^k y^l}\|_0 \leqq c_{25}(d_1, d_5) \cdot \left(\sum_{i,j=1}^n \|u_{x^i x^j}\|_0 + \|f\|_{0,\delta}\right).$$

Inserting this in (15.4):

$$v(u_{x^i x^j}, x_0, \varrho, \Omega) \leqq d_{13} \cdot \left(\sum_{k,l} \|u_{x^k x^l}\|_0 + \|f\|_{0,\delta}\right) \cdot \varrho^n \cdot \delta(\varrho)^2, \quad d_{13} = d_{13}(\Omega, \delta, e_1, e_0, \underline{\mu}), \quad 0 < \varrho \leqq \varrho_6. \tag{15.6}$$

Now the desired inequality (11.1) follows from **6.**iii).

16. Notations

$$B(x, \varrho) := \{y \in \mathbf{R}^n : \|x-y\| < \varrho\}, \quad B(\varrho) := B(0, \varrho),$$

$$B^{\pm}(\varrho) := B(\varrho) \cap \{y \in \mathbf{R}^n : y^n \gtrless 0\},$$

$$\delta_{ij} := \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

A bar denotes the closure of a set, a ∂ the boundary. $\mathscr{L}_q$ is the q-dimensional Lebesque measure, $\mathscr{L} := \mathscr{L}_n$, $\omega_n := \mathscr{L}\langle B(1)\rangle$; $\mathscr{H}$ is the $(n-1)$-dimensional Hausdorff measure. If $h \in L_2(A)$, we write

$$m(h, y, \varrho, A) := \mathscr{L}\langle B(y, \varrho) \cap A\rangle^{-1} * \int_{B(y,\varrho)\cap A} h \, d\mathscr{L}, \tag{16.1}$$

$$\partial m(h, y, \varrho, A) := \mathscr{H}\langle \partial B(y, \varrho)\rangle^{-1} \cdot \int_{\partial B(y,\varrho)} h \, d\mathscr{H}, \tag{16.2}$$

$$v(h, y, \varrho, A) := \int_{B(y,\varrho)\cap A} (h - m(y, h, \varrho, A))^2 \, d\mathscr{L} \tag{16.3}$$

We drop some of the quantities in the brackets, if their choice is obvious. Frequently we shall use, that

$$v(h, y, \varrho, A) = \min_{k \in \mathbf{R}} \int_{B(y,\varrho)\cap A} (h-k)^2 \, d\mathscr{L}.$$

For the definition of the Sobolev spaces $H^{2,p}$ see [11; Ch. 3], the spaces C^k and $C^{k,\alpha}$ are introduced in [6; § 4.1].

If h is differentiable, Jh denotes its Jacobi matrix.

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