Extensions of a fixed point theorem of Meir and Keeler

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1. Introduction

Meir and Keeler [9] established a fixed point theorem which is a remarkable generalization of the Banach contraction principle.

A selfmap g of a metric space (X, d) is called a weakly uniformly strict contraction or simply an (ε, δ)-contraction if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in X$,

(1) $$\varepsilon \leqq d(x, y) < \varepsilon + \delta \quad \text{implies} \quad d(gx, gy) < \varepsilon.$$

Meir and Keeler proved that an (ε, δ)-contraction g of a complete metric space X has a unique fixed point η in X and $\{g^n x\}_{n=1}^{\infty}$ converges to η for all $x \in X$ [9]. The class of (ε, δ)-contractions clearly contains the classes of (Banach) contractions and nonlinear contractions investigated by Browder [3] and by Boyd and Wong [2].

A fixed point of a selfmap g of X can be considered as a common fixed point of g and 1_X, the identity map of X. In certain cases, we can replace 1_X by a continuous selfmap f of X and consider common fixed point of f and g. Jungck [8] adopted this idea and obtained a useful generalization of the Banach contraction principle to commuting selfmaps. More recently, Park [10] extended these facts and obtained a number of results on commuting selfmaps.

Let f be a continuous selfmap of a metric space (X, d) and C_f denote the class of selfmaps g of X such that $fg = gf$ and $gX \subset fX$. A selfmap g of X is called an (ε, δ)-*f-contraction* if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for all $x, y \in X$,

(2) $$\varepsilon \leqq d(fx, fy) < \varepsilon + \delta \quad \text{implies} \quad d(gx, gy) < \varepsilon,$$

and (2′) $gx = gy$ whenever $fx = fy$.

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In this paper we show that an (ε, δ)-f-contraction g in C_f has a unique common fixed point with f whenever X is complete, and that this extends fixed point theorems of Meir and Keeler [9], Edelstein [5], Browder [3], Boyd and Wong [2], Jungck [8], Park [10], Jeong [7], and Chung [4]. Some related results are also obtained.

2. Fixed point theorems

Let f and g be selfmaps of a metric space (X, d). Given a point x_0 in X, we consider a sequence $\{fx_n\}_{n=1}^{\infty}$ recursively given by the rule $fx_n = gx_{n-1}$, $n=1, 2, \ldots$. Such a sequence is called an *f-iteration* of x_0 under g.

Note that for an (ε, δ)-f-contraction g, we have

$$d(gx, gy) < d(fx, fy) \quad \text{for} \quad x, y \in X, \quad fx \neq fy. \tag{3}$$

Lemma 2.1. *Let f be a selfmap of a metric space X and g be an (ε, δ)-f-contraction. If there exists an $x_0 \in X$ and an f-iteration $\{fx_n\}_{n=1}^{\infty}$ of x_0 under g, then $\{d(fx_n, fx_{n+1}) | n=1, 2, \ldots\}$ is monotone decreasing to* 0.

Proof. Suppose $\inf \{d(fx_n, fx_{n+1})\} = r$ for some $r > 0$. Then by (3), we have

$$d(fx_n, fx_{n+1}) = d(gx_{n-1}, gx_n) < d(fx_{n-1}, fx_n),$$

so $\{d(fx_n, fx_{n+1})\}_{n=1}^{\infty}$ is a decreasing sequence and, hence, $\lim_n d(fx_n, fx_{n+1}) = r$. By (2), there exists a $\delta > 0$ such that

$$r \leqq d(fx, fy) < r + \delta \quad \text{implies} \quad d(gx, gy) < r.$$

Since $\lim_n d(fx_n, fx_{n+1}) = r$, there exists a positive integer N such that for every $m \geqq N$, we have

$$r \leqq d(fx_m, fx_{m+1}) < r + \delta. \tag{4}$$

Then for every $m \geqq N$, we have

$$d(fx_{m+1}, fx_{m+2}) = d(gx_m, gx_{m+1}) < r,$$

which contradicts (4). Therefore, we have $\lim_n d(fx_n, fx_{n+1}) = 0$.

Lemma 2.2. *Let g be an (ε, δ)-f-contraction commuting with f. If there exists a $\xi \in X$ such that $f\xi = g\xi$, then $f\xi$ is the unique common fixed point of f and g.*

Proof. Let $f\xi = g\xi = \eta$, and suppose $f\eta \neq \eta$. Then by (3)

$$d(\eta, f\eta) = d(g\xi, fg\xi) = d(g\xi, gf\xi) < d(f\xi, ff\xi) = d(\eta, f\eta),$$

which is a contradiction. Hence we have $f\eta = \eta$ and $g\eta = gf\xi = fg\xi = f\eta = \eta$. Therefore, $f\xi$ is a common fixed point of f and g. Let η' be a common fixed point of f

and g such that $\eta \neq \eta'$. Then by (3)

$$d(\eta, \eta') = d(g\eta, g\eta') < d(f\eta, f\eta') = d(\eta, \eta'),$$

which is a contradiction. Therefore η is unique.

Now we have our main result.

Theorem 2.3. *Let f be a selfmap of a metric space X and g be an (ε, δ)-f-contraction commuting with f. If a point $x_0 \in X$ has an f-iteration $\{fx_n\}_{n=1}^{\infty}$ under g with a cluster point $\xi \in X$ at which f is continuous, then $\{fx_n\}$ converges to ξ, and $f\xi$ is the unique common fixed point of f and g.*

Proof. By Lemma 2.2, it is sufficient to show that we can find a point ξ in X such that $f\xi = g\xi$. If $d(fx_n, fx_{n+1}) = 0$ for some n, then $fx_{n+1} = gx_n = fx_n$, and we are done. Suppose $d(fx_n, fx_{n+1}) \neq 0$ for every n. We now claim that $\{fx_n\}$ is a Cauchy sequence. Suppose not. Then there exists an $\varepsilon > 0$ and a subsequence $\{fx_{n_i}\}$ of $\{fx_n\}$ such that

$$d(fx_{n_i}, fx_{n_{i+1}}) > 2\varepsilon. \tag{5}$$

By (2), there exists $0 < \delta < \varepsilon$ such that

$$\varepsilon \leqq d(fx, fy) < \varepsilon + \delta \quad \text{implies} \quad d(gx, gy) < \varepsilon.$$

Since $\lim_n d(fx_n, fx_{n+1}) = 0$ by Lemma 2.1, there exists a positive integer N such that for every $m \geqq N$, we have

$$d(fx_m, fx_{m+1}) < \delta/6. \tag{6}$$

Then by (5) and (6), for every $n_i > N$, we can find m such that $n_i < m < n_{i+1}$ and

$$\varepsilon + \frac{\delta}{3} \leqq d(fx_{n_i}, fx_m) < \varepsilon + \delta. \tag{7}$$

Then

$$\begin{aligned} d(fx_{n_i}, fx_m) &\leqq d(fx_{n_i}, fx_{n_i+1}) + d(fx_{n_i+1}, fx_{m+1}) + d(fx_{m+1}, fx_m) \\ &< \frac{\delta}{6} + d(gx_{n_i}, gx_m) + \frac{\delta}{6} \\ &< \varepsilon + \frac{\delta}{3}, \end{aligned}$$

which contradicts (7), and hence $\{fx_n\}$ is a Cauchy sequence. Since $\{fx_n\}$ clusters at $\xi \in X$, it converges to ξ. Since f is continuous at ξ, $\{ffx_n\} = \{fgx_{n-1}\} = \{gfx_{n-1}\}$ converges to $f\xi$.

Suppose $ffx_m = ffx_{m+1} = ffx_{m+2} = \ldots$ for some m. Then $\{ffx_n\}$ converges to ffx_m and $ffx_m = ffx_{m+1} = fgx_m = gfx_m$. Hence fx_m is a coincidence point of f and g, and $ffx_m = f\xi$. Thus we are done. Suppose that we can not find an m satisfying $ffx_m = ffx_{m+1} = \dots$. Then for any $\varepsilon > 0$, there exists an N such that for every $m \geqq N$, $d(ffx_m, f\xi) < \varepsilon/2$, and we can find an $n \geqq N$ such that $ffx_n \neq f\xi$. Then we have

$$\begin{aligned} d(f\xi, g\xi) &\leqq d(f\xi, fgx_n) + d(fgx_n, g\xi) \\ &= d(f\xi, ffx_{n+1}) + d(gfx_n, g\xi) \\ &< \varepsilon/2 + d(ffx_n, f\xi) < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Therefore $f\xi = g\xi$, and this completes our proof.

Note that if $gX \subset fX$, then every $x_0 \in X$ has an f-iteration under g. Therefore, from Theorem 2.3, we have

Theorem 2.4. *Let f be a continuous selfmap of a complete metric space X and g be an (ε, δ)-f-contraction in C_f. Then f and g have a unique common fixed point η in X, and, for any x_0 in X, every f-iteration of x_0 under g converges to some $\xi \in X$ satisfying $f\xi = \eta$.*

Proof. An f-iteration $\{fx_n\}$ of x_0 under g is Cauchy as in the proof of Theorem 2.3. Since X is complete, $\{fx_n\}$ converges to some $\xi \in X$. Now Theorem 2.4 follows from Theorem 2.3.

Remark. In case $f = 1_X$, Theorem 2.4 is reduced to the result of Meir and Keeler [9]. In case $g = f^2$, Theorem 2.4 is reduced to the main result of Chung [4].

Corollary 2.5. *Let f be a continuous selfmap of a complete metric space X and g be in C_f. If g^N is an (ε, δ)-f-contraction for some positive integer N, then f and g have a unique common fixed point.*

Proof. Clearly we have $g^N f = f g^N$ and $g^N X \subset fX$, and hence $g^N \in C_f$. Applying Theorem 2.3, we have a unique common fixed point η of f and g^N. Then we have $fg\eta = gf\eta = g\eta$ and $g^N g\eta = gg^N\eta = g\eta$. Hence $g\eta$ is also a common fixed point of f and g^N. This implies $g\eta = \eta$ because of the uniqueness. Suppose η and η' are common fixed points of f and g. Then $g^N\eta = \eta = f\eta$ and $g^N\eta' = \eta' = f\eta'$. Since f and g^N have a unique common fixed point, we have $\eta = \eta'$.

Corollary 2.6. *If f is a bijective continuous selfmap of a complete metric space X, and for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$,*

$$\varepsilon \leqq d(fx, fy) < \varepsilon + \delta \quad \textit{implies} \quad d(x, y) < \varepsilon,$$

then f has a unique fixed point.

Proof. In Theorem 2.4, we set $g = 1_X$.

Corollary 2.7. *Let f be a continuous selfmap of a complete metric space X and $\{g_\lambda\}_{\lambda\in\Lambda}$ a commuting family of selfmaps in C_f. If each g_λ is an (ε, δ)-f-contraction, then there exists a unique point $\eta\in X$ such that $f\eta=g_\lambda\eta=\eta$ for every $\lambda\in\Lambda$.*

Proof. For each λ, g_λ and f have a unique common fixed point, say η. For any $\mu\in\Lambda$, $g_\lambda(g_\mu\eta)=g_\mu(g_\lambda\eta)=g_\mu\eta=g_\mu(f\eta)=f(g_\mu\eta)$ implies $g_\mu\eta=\eta$ by the uniqueness.

In certain case the continuity of f in Theorem 2.4 can be relaxed to that of some iterate of f.

Corollary 2.8. *Let f be a selfmap of a complete metric space X such that f^k is continuous for some positive k. Let g: $f^{k-1}X\to X$ be a map such that $gf^{k-1}X\subset f^kX$ and $gf=fg$ whenever both sides are defined. If gf^{k-1} is an (ε, δ)-f^k-contraction, then f and g have a unique common fixed point.*

Proof. By Theorem 2.4, gf^{k-1} and f^k have a unique common fixed point η. From $gf^{k-1}(f\eta)=g(f^k\eta)=g\eta$ and $gf^{k-1}(f\eta)=f(gf^{k-1}\eta)=f\eta$, we have $f\eta=g\eta$. From $f^k(f\eta)=f(f^k\eta)=f\eta$, we know that $f\eta$ is also a common fixed point of gf^{k-1} and f^k. Therefore, we have $\eta=f\eta=g\eta$. The uniqueness is clear.

Remark. The class of (ε, δ)-f-contractions contains the classes of selfmaps satisfying (2′) and one of the following conditions:

(8) There exists a map ϕ: $[0, \infty)\to[0, \infty)$ which is upper-semicontinuous from the right such that $\phi(t)<t$ for all $t>0$ and

$$d(gx, gy) < \phi(d(fx,fy)), \quad fx \neq fy.$$

(9) There exists a nondecreasing map ϕ: $[0, \infty)\to[0, \infty)$ which is continuous from the right such that $\phi(t)<t$ for all $t>0$ and

$$d(gx, gy) < \phi(d(fx,fy)), \quad fx \neq fy.$$

(10) There exists an $\alpha\in[0, 1)$ such that

$$d(gx, gy) \leqq \alpha d(fx,fy).$$

Note that (10)⇒(9)⇒(8)⇒(2)⇒(3) and that (8) and (10) are investigated by Jeong [7] and Jungck [8], and particular types of (9) by Park [10]. Therefore, certain results in [7], [8], [10] are consequences of ours. Note also that for $f=1_X$, (8) and (9) reduce to non-linear contractive type conditions of Boyd—Wong [2] and Browder [3], respectively. Avramescu [1] obtained some results for (10) with $g=1_X$ and f surjective. We can easily obtain an extended form of a result in [6] with respect to (ε, δ)-f-contractions.

Finally, we consider compact metric spaces.

Theorem 2.9. *Let f and g be continuous selfmaps of a compact metric space X such that $g \in C_f$ and (3) and (2′) hold. Then f and g have a unique common fixed point η in X, and, for each x_0 in X, any f-iteration of x_0 under g converges to some $\xi \in X$ satisfying $f\xi = \eta$.*

Proof. Given $\varepsilon > 0$, consider

$$\inf \{d(fx, fy) - d(gx, gy) | \varepsilon \leqq d(fx, fy)\} = \delta(\varepsilon).$$

Since X is compact, this infimum is achieved for some $(a, b) \in X \times X$ with $d(fa, fb) \geqq \varepsilon$. Since (3) holds, we have $\delta(\varepsilon) > 0$. This shows that g is an (ε, δ)-f-contraction. Therefore, Theorem 2.9 follows from Theorem 2.4.

Remark. Theorem 2.9 was proved in [10]. For $f = 1_X$, Theorem 2.9 is reduced to a result of Edelstein [5].

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