An H^1 multiplier theorem

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Introduction. Let $H^1(\mathbf{R}^d)$ and $H^1(\mathbf{T}^d)$ denote the usual Hardy spaces on Euclidean space and the torus [18], [19, p. 283]. Given a function in $H^1(\mathbf{R}^d)$ its Fourier transform is a continuous function on $\mathbf{R}^d$ which vanishes at the origin. Thus the transform may be integrable with respect to a measure which is singular at the origin.

We have two main results. One is a characterization of all such measures and the other is an application to random Fourier series.

1. Main results. Denote by Λ the integer lattice in $\mathbf{R}^d$ and Q^ε_α the cube $\{x\in\mathbf{R}^d\colon \varepsilon\alpha_j-\varepsilon/2\leqq x_j<\varepsilon\alpha_j+\varepsilon/2\}$ where $\alpha=(\alpha_1, \ldots, \alpha_d)\in\Lambda$ and $\varepsilon>0$.

Theorem 1. *Let μ be a positive Borel measure on $\mathbf{R}^d\setminus\{0\}$. Then*

$$\sup \int |\hat{f}|\, d\mu < \infty \tag{1}$$

where the supremum is taken over all f in $H^1(\mathbf{R}^d)$ of norm 1 *if and only if*

$$\sup_{\varepsilon>0} \left(\sum \mu(Q^\varepsilon_\alpha)^2\right)^{1/2} < \infty. \tag{2}$$

Moreover, the corresponding suprema are equivalent.

Corollary 1. *Let $\{m_\alpha\}_{\alpha\in\Lambda}$ be nonnegative numbers and define a measure on $\mathbf{R}^d\setminus\{0\}$ by $\mu=\sum_{\alpha\neq 0} m_\alpha\delta_\alpha$ where δ_α is a point mass at $x=\alpha$. Then*

$$\sup_{\varepsilon>0} \sum\nolimits_{\alpha\neq 0} |\hat{f}(\alpha)|\, m_\alpha < \infty \tag{3}$$

where the supremum is taken over all f in $H^1(\mathbf{T}^d)$ of unit norm if and only if μ satisfies condition (2).

Remarks. (a) For $d=1$, Corollary 1 is an unpublished result of C. Fefferman, see [1]. It contains in particular the classical inequalities of Hardy [9] and Paley [13]. Theorem 1 is a generalization of this result.

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(b) Theorem 1 can easily be generalized by replacing the 1-norm condition in condition (1) by a p-norm for $p \geqq 1$. Corresponding to condition (2) is

$$(2') \qquad \sup_{\varepsilon>0} \left\{ \sum\nolimits_{\alpha \neq 0} \mu(Q_\alpha^\varepsilon)^{2/2-p} \right\}^{2-p/2p} < \infty, \quad 1 \leqq p < 2$$

$$\sup_{\varepsilon>0} \mu(\varepsilon \leqq |x| \leqq 2\varepsilon)^{1/p} < \infty, \quad p \geqq 2.$$

The case $p=2$ is a result of Stein and Zygmund [20].

The space of functions of bounded mean oscillation (BMO) introduced in [11] are naturally involved in this problem by means of Fefferman's duality theorem [6], [7]. We restrict our attention to functions defined on the circle $(d=1)$. The following corollary is a consequence of this duality and a result in [5, p. 105].

Corollary 2. *Let* $\{m_n\}_{n=-\infty}^{\infty}$ *be a square summable sequence of nonnegative numbers. The function* f *with Fourier series* $\sum m_n e^{in\theta}$ *is in* BMO *if and only if condition* (2) *is satisfied (for the corresponding measure* μ*).*

Theorem 2. *There exists an* l^2*-sequence* $\{m_n\}$ *with the property that* $\sum \lambda_n m_n e^{in\theta}$ *is not in* BMO *for any sequence* $\{\lambda_n\}$ *with* $|\lambda_n|=1$ *for all* n.

Proof. Let $E \subset \mathbf{Z}^+$ be a Sidon set which is not a lacunary set, see [10]. It follows that there exists a sequence of nonnegative numbers $\{m_n\} \in l^2$ supported on the set E which violates condition (2). Let $f \sim \sum m_n e^{in\theta}$ be the corresponding function in L^2.

By Corollary 2 f is not in BMO. If $\{\lambda_n\}$ is sequence with $|\lambda_n|=1$ then there is a measure μ whose Fourier coefficients $\hat{\mu}(n)$ agree with λ_n on the set E [10]. Since BMO is closed under convolution with a measure it follows that $\sum \lambda_n m_n e^{in\theta} \notin$ BMO.

Remarks. (a) The motivation for the last theorem is the well known fact that the random L^2-function is in L^p for $p<\infty$ [21]. Since BMO is contained in all of the L^p-spaces it is natural to extend this result. Theorem 2 provides a strong counterexample to this conjecture. See also [16].

(b) The existence of the sequence $\{m_n\}$ could have been deduced from the converse to Paley's Theorem which was proved by Rudin [14], see also Stein and Zygmund [20].

(c) Another curiosity along the same lines is that a lacunary function (Hadamard gaps) in BMO has some continuity properties, namely, a lacunary function in BMO is in the space of functions with vanishing mean oscillation (VMO) introduced by Sarason [15].

2. Proof of Theorem 1. An atom $a(x)$ corresponding to a cube Q is a measurable function supported on Q which has zero mean and is bounded by $|Q|^{-1}$ $(|\cdot|=$

Lebesgue measure). By a result of Coifman [3] the sufficiency of condition (2) will follow if there is a $c<\infty$ with

$$\int |\hat{a}|\, d\mu \leqq c \tag{4}$$

for all atoms a.

Part of (4) is straightforward. If a is an atom corresponding to a cube of side length δ then by a well-known estimate $|\hat{a}(y)| \leqq c|y|\delta$ for y in Q_0^ε where $\varepsilon=\delta^{-1}$. Here c is a dimensional constant independent of a. Now it is not hard to show that (2) implies

$$\varepsilon^{-1}\int_{Q_0^\varepsilon} |x|\, d\mu(x) \leqq c \quad (\varepsilon>0)$$

and hence (4) will follow from

$$\int_{\mathbf{R}^d \setminus Q_0^\varepsilon} |\hat{a}|\, d\mu \leqq c \tag{4$'$}$$

where ε is related to a as above. This result is now easily seen to be a consequence of condition (2) and the following theorem.

Theorem 3. *There is a constant $c<\infty$ such that if $a(x)$ is an atom corresponding to a cube with side length δ and $\varepsilon=\delta^{-1}$ then*

$$\sum_\alpha \sup_{Q_\alpha^\varepsilon} |\hat{a}|^2 \leqq c.$$

Proof. We only prove the result for $d=1$. The general case involves an iteration technique which is somewhat more complicated to describe. In addition, it suffices to assume that a is smooth and supported in the interval $[-\delta/2, \delta/2]$.

Fix an interval I of length ε and assume that f is continuously differentiable on I. It is elementary that $\sup_I |f-b| \leqq \int_I |f'|$ where b is the average $|I|^{-1}\int_I f$. Hence

$$\sup_I |f|^2 \leqq 2\left[\frac{1}{\varepsilon}\int_I |f|^2 + \varepsilon \int_I |f'|^2\right].$$

Normalizing the Fourier transform so that $\|f\|_2 = \|\hat{f}\|_2$ we obtain

$$\sum_n \sup_{Q_n^\varepsilon} |\hat{a}|^2 \leqq 2\left[\frac{1}{\varepsilon}\int_{-\infty}^{\infty} |\hat{a}|^2 + \varepsilon\int_{-\infty}^{\infty} |\hat{a}'|^2\right]$$

$$= 2\left[\frac{1}{\varepsilon}\int_{-\delta/2}^{\delta/2} |a|^2 + \varepsilon \int_{-\delta/2}^{\delta/2} |2\pi i x a|^2\, dx\right]$$

from which the theorem follows.

Remark. The results in [2], [12] concerning the behavior of $\hat{a}$ follow from Theorem 3.

In order to prove that condition (2) is necessary we require some examples of functions in $H^1(\mathbf{R}^d)$. The following lemma is sufficient for our purposes.

Lemma 4. *Let* $g\in L^2(\mathbf{R}^d)$ *and assume that* $\hat{g}=0$ *on* $|y|\leqq 1$. *If* $f=g\hat{X}_{B(0,1)}$ *then* $f\in H^1$ *and* $\|f\|_{H^1}\leqq c\|g\|_2$. *(Here* $X_{B(0,1)}$ *is the characteristic function for the unit ball centered at the origin.)*

Proof. Assume that $\hat{g}$ is a C^∞-function with compact support in $|y|>1$. Then $\hat{f}$ is the convolution $\hat{g}*X_{B(0,1)}$ and hence is a rapidly decreasing function which vanishes in a neighborhood of the origin. Thus, f is in H^1. If $u\in \mathrm{BMO}(\mathbf{R}^d)$ and b is its average over $B(0,1)$ then by the Schwarz inequality

$$\left|\int fu\right| = \left|\int f(u-b)\right|$$
$$\leqq c\|g\|_2\left\{\int\frac{|u-b|^2}{1+|x|^{d+1}}\,dx\right\}^{1/2}$$
$$\leqq c\|g\|_2\|u\|_{BMO}.$$

The first inequality is a well-known estimate for $\hat{X}_{B(0,1)}$ and the second a slight extension of inequality (1.2) in [7]. By duality the proof is complete.

The proof of Theorem 1 will be complete once we establish the necessity of condition (2). However, if (1) holds with supremum A then from Lemma 4 we deduce that

$$\int [\mu(y+B(0,1))]^2\,dy \leqq cA^2. \tag{5}$$

It follows easily that there is an $M<\infty$, $\delta>0$ for which $\sum_{|\alpha|\geqq M}\mu(Q_\alpha^\delta)^2\leqq cA^2$ where c is a dimensional constant. But then a dilation argument gives this inequality for all $\delta>0$ and (2) now follows in an elementary way. Thus the proof of Theorem 1 is complete.

3. Proof of the Corollary. The space $H_0^1(\mathbf{T}^d)$ is the subspace of $H^1(\mathbf{T}^d)$ consisting of functions with zero mean. Given $f\in H^1(\mathbf{R}^d)$ we define

$$Pf(x) = \sum_{\alpha\in\Lambda} f(x+\alpha).$$

Since $f\in L^1(\mathbf{R}^d)$ we have $Pf\in L^1(\mathbf{T}^d)$ and by the Poisson summation formula it follows that $\hat{f}(\alpha)=(Pf)\hat{\ }(\alpha)$ for $\alpha\in\Lambda$. Here the Fourier coefficients for functions on $\mathbf{T}^d$ are given for $\alpha\in\Lambda$ by

$$\hat{F}(\alpha) = \int_{\mathbf{T}^d} F(x)\,e^{-2\pi i\alpha\cdot x}\,dx$$

where $\mathbf{T}^d$ is identified with the d-fold product of the unit interval.

The proof of the corollary is an immediate consequence of the following theorem.

Theorem 5. $P(H^1(\mathbf{R}^d))=H_0^1(\mathbf{T}^d)$.

Proof. Let φ be a nonnegative rapidly decreasing function for which $\hat{\varphi}$ has support contained in the open unit ball centered at the origin and $\hat{\varphi}(0)=1$. Put $\varphi_\varepsilon(x)=\varepsilon^d\varphi(\varepsilon x)$ for $0<\varepsilon<1$. For a polynomial $F(x)=\sum a_\alpha e^{2\pi i\alpha\cdot x}$ let $f_\varepsilon=F\cdot\varphi_\varepsilon$. Then $f_\varepsilon\in H^1(\mathbf{R}^d)$ and $\hat{f}_\varepsilon(\alpha)=a_\alpha$ for $\alpha\in\Lambda$.

Claim. $\lim_{\varepsilon\to 0}\|f_\varepsilon\|_{H^1(\mathbf{R}^d)}\leqq\|F\|_{H^1(\mathbf{T}^d)}$.

We start with the easily derived fact that

$$\lim_{\varepsilon\to 0}\int_{\mathbf{R}^d} g|\varphi_\varepsilon| = \int_{\mathbf{T}^d} g\,dx \tag{6}$$

for all continuous functions g on T^d. Observe that $\|\varphi_\varepsilon\|_1=1$.

Let S_j, R_j denote the jth Riesz transforms on $\mathbf{T}^d$, $\mathbf{R}^d$. Then by (6) we obtain

$$\limsup_{\varepsilon\to 0}\left[\|f_\varepsilon\|_1+\textstyle\sum_1^d\|R_jf_\varepsilon\|_1\right]\leqq\|F\|_{H^1(\mathbf{T}^d)}$$

$$+\limsup_{\varepsilon\to 0}\textstyle\sum_1^d\|R_jf_\varepsilon-(S_jF)\varphi_\varepsilon\|_1$$

so that we must show that the second term on the right is zero. Since F is a polynomial it suffices to fix $\alpha\in\Lambda$ with $\alpha\neq 0$, put

$$h_\varepsilon(y-\alpha) = (y_j/|y|-\alpha_j|\alpha|)\,\hat{\varphi}_\varepsilon(y-\alpha)$$

for some $1\leqq j\leqq d$ and show that $\lim_{\varepsilon\to 0}\|\hat{h}_\varepsilon\|_1=0$.

Now $\hat{\varphi}_\varepsilon$ is supported in the ball of radius ε centered at the origin. Thus, we may assume that $h_\varepsilon(y)=m(y)\hat{\varphi}_\varepsilon(y)$ where m is smooth, all derivatives up to order $d+1$ are bounded by a dimensional constant, and $|m(y)|\leqq c|y|$. The conditions on m imply that $\|Dh_\varepsilon\|_1\leqq c$ where $D=\partial^{d+1}/\partial y_1^{d+1}+\ldots+\partial^{d+1}/\partial y_d^{d+1}$. Hence $|\hat{h}_\varepsilon(x)|\leqq c|x|^{-(d+1)}$. Clearly, $\lim_{\varepsilon\to 0}\|h_\varepsilon\|_1=0$ so that $\lim_{\varepsilon\to 0}\|\hat{h}_\varepsilon\|_\infty=0$ and thus the above estimate implies that $\lim_{\varepsilon\to 0}\|\hat{h}_\varepsilon\|_1=0$. This proves the claim.

To complete the proof we fix F in $H_0^1(\mathbf{T}^d)$ and note that there are polynomials $F_n\in H_0^1(\mathbf{T}^d)$ with $\sum\|F_n\|_{H^1}<\infty$ and $F=\sum F_n$. Using the above we find $f_n\in H^1(\mathbf{R}^d)$ with $\sum\|f_n\|_{H^1(\mathbf{R}^d)}<\infty$ and $Pf_n=F_n$. Thus, $Pf=F$ where $f=\sum f_n$ is a function in $H^1(\mathbf{R}^d)$.

Remarks. (a) Theorem 5 is an extension of a result of deLeeuw [4], see Goldberg [8] for a similar result.

(b) A sharpening of the lemma is that given $F\in H_0^1(\mathbf{T}^d)$ and $\varepsilon>0$, there exist $f\in H^1(\mathbf{R}^d)$ with $Pf=F$ and

$$\|F\|_{H^1(\mathbf{T}^d)}\leqq\|f\|_{H^1(\mathbf{R}^d)}\geqq(1+\varepsilon)\|F\|_{H^1(\mathbf{T}^d)}.$$

This is best possible since $\|Pf\|_{H^1(\mathbf{T}^d)}<\|f\|_{H^1(\mathbf{R}^d)}$ in general.

(c) A similar argument to the above shows that $P(L^1(\mathbf{R}^d))=L_0^1(\mathbf{T}^d)$.

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