Two problems on removable sets for analytic functions

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Introduction. Two elementary theorems on continuation of analytic functions across a compactum $C \subseteq R^2$ are stated, and to each an example is given to show that the sufficient condition is best possible. In I. we use methods from the theory of singular integrals to estimate a certain sum of analytic functions. In II. we use Fourier analysis in R^2 to estimate certain sums of functions that seem immune to direct methods (especially their higher derivatives). We make explicit use of the "symbols" of the operators $\partial/\partial y$ and $\partial/\partial \bar{z}$; we are impelled to a complicated construction of the exceptional set by the necessity of evaluating certain integrals arising as Fourier transforms.

I. A compact set C in the plane is called *non-removable* if there is a uniformly continuous, nonconstant analytic function defined on $R^2 - C$. By a theorem of Besicovitch [1], C is removable if C is of σ-finite Hausdorff 1-measure, hence *a fortiori* if C is a rectifiable curve. (The theorem must be changed if uniform continuity is replaced by uniform boundedness.) When C is the graph of a function $y = f(x)$, $-1 \leqq x \leqq 1$, then C is removable if f has bounded variation, or is differentiable everywhere (an elementary theorem). Our purpose is to show that this last statement cannot be improved.

Theorem. *Let $h(u)$ be positive and increasing for $0 < u \leqq 2$, and $\lim u^{-1}h(u) = +\infty$ at $u = 0+$. Then there is a curve $y = f(x)$, $-1 \leqq x \leqq 1$, non-removable in the definition stated, such that $|f(x_1) - f(x_2)| \leqq h(x_1 - x_2)$ whenever $-1 \leqq x_2 < x_1 \leqq 1$.*

It is clear that C has finite Hausdorff measure for the function $H(u) \equiv h^{-1}(u)$, and that $h^{-1}(u)$ can be any function $H(u)$ subject to the relation $H(u) = o(u)$. This corollary of our Theorem is due to Carleson [2]; however, the sets constructed by Carleson are disconnected and in fact do not appear to lie on any Jordan curves with interesting properties.

2. Let $\psi(t)=1-|t|$ for $|t|<1$, $\psi(=0$ oth$t)$erwise, and then

$$F_0(z) = \int_{-\infty}^{\infty} \psi(t)(z-t)^{-1}\,dt, \quad z \notin [-1, 1].$$

Then $|F_0| \leqq C$ off $[-1, 1]$ and $F_0(z)=z^{-1}+\frac{1}{6}z^{-3}+\dots$ near infinity. We note the following approximation for F_0: when $|z|>4$ and $|\xi|\leqq 2$, then $F_0(z+\xi)= z^{-1}+O(z^{-2})$.

Let now $0<r<1$ and $a_k=r\int_k^{k+1}\psi(rt)\,dt$, so that $\sum a_k=1$ and $|a_{k+1}-a_k|< r^2$; moreover $a_k=0$ when $|k|>r^{-1}+1$. Let w be any complex number and $g(z)= \sum a_k F_0(z-kw)$.

Lemma. *$|g(z)|\leqq C'(r|w|^{-1}+r)$ on the domain of holomorphy of g.*

Proof. Clearly $g(\infty)=0$, $|g|\leqq C$, and g is analytic off the line segments $[kw-1, kw+1]$ with $|k|\leqq 2r^{-1}$. It is sufficient therefore to prove the inequality for numbers z such that $|z-vw|<2$ with $|v|\leqq 2r^{-1}$. We can suppose that $|w|\geqq 4r$ (since $|g|\leqq C$), and then omit from the sum the indices satisfying the inequality $|k-v|\leqq 4|w|^{-1}$, because $|a_k|<r$. For the remaining indices k we observe that $z-kw=(v-k)w+z-vw$ so that

$$F_0(z-kw) = (v-k)^{-1}w^{-1}+O(|w|^{-2}|v-k|^{-2}).$$

Summation of the error $|w|^{-2}|v-k|^{-2}$, over the range $|k-v|>4|w|^{-1}$, yields $O(|w|^{-1})$, which is then multiplied by r. Again $|a_k-a_v|<r^2|k-v|$, and the range of summation is taken as $4|w|^{-1}<|k-v|\leqq 4r^{-1}$. The principal term $a_v w^{-1}(v-k)^{-1}$ sums up to 0, and the error to $O(r|w|^{-1})$.

We apply this inequality to the function $\sum a_k\cdot\lambda F_0(\lambda z-\lambda rk)$, wherein $0<r<1$ and $\lambda=wr^{-1}$. We find the inequality $|G(z)|\leqq|\lambda|\cdot|C'|(r|w|^{-1}+r)=|C'|+|C'||w|$, on the open set where G is analytic. Suppose further that $r\to 0+$, $|\lambda|\to+\infty$, $|w|= |\lambda|r<1$. Then $G(z)$ tends uniformly to F_0 on each compact set K disjoint from $[-1, 1]$; to see this we observe that if r is small and $a_k\neq 0$, then $|z-rk|\geqq\delta>0$ for all z in K, with a constant δ independent of k, r. Then $\lambda F_0(\lambda z-\lambda rk)=(z-rk)^{-1}+ O(\lambda^{-1})$, so $G(z)$ may be compared with a Riemann sum for $F_0(z)$. At the same time $G(z)$ remains uniformly bounded, by the requirements on λ, r and λr. The open set G on which G is analytic tends to $R^2-[-1, 1]$.

(The lemma and its applications remain valid if ψ is replaced by a Dini-continuous function; we have only to estimate a_k-a_v more carefully.)

3. The curve $y=f(x)$ will be a limit of curves $y=f_n(x)$, $-1\leqq x\leqq 1$, with $f_0\equiv 0$, $f_n(-1)=f_n(1)=0$ and $|f_n(x_1)-f_n(x_2)|\leqq(1-2^{-n})h(|x_1-x_2|)$. We suppose that φ_n is analytic off the curve $y=f_n(x)$, $|x|\leqq 1-2^{-n-1}$, and that $\varphi_n=z^{-1}+O(z^{-2})$ near infinity. Also, each φ_n is a sum $\sum_j b_j F_0(\alpha_n z+c_j)$, with some α_n depending

on n and each $|b_j|\leqq 3^{1-n}$. The singular lines of the different functions $F_0(\alpha_n z+c_j)$ are disjoint line segments contained in the curve $y=f_n(x)$, so $\operatorname{Re}\alpha_n\neq 0$. For φ_0 we choose $2F_0(2z)$.

The singular line of $F_0(\alpha_n z+c_j)$ has end points $\alpha_n^{-1}(c_j-1)$, $\alpha_n^{-1}(c_j+1)$; if $\alpha_n=Re^{i\varphi}$ then this line has slope $-\tan\varphi$. We replace F_0 by a sum

$$\sum a_k\cdot\lambda F_0(\lambda z-k\lambda r),$$

obtaining (for fixed j) a sum $\sum a_k\cdot\lambda F_0(\lambda\alpha_n z+\lambda c_j-\lambda kr)$. We shall take r very small, $|\lambda|=(6r)^{-1}$, and choose the argument of λ in the following way. The singular line of $F_0(\lambda\alpha_n z+\lambda c_j-k\lambda r)$ is represented parametrically by the equation $z=(\lambda\alpha_n)^{-1}t+k\alpha_n^{-1}r-\alpha_n^{-1}c_j$, $-1\leqq t\leqq 1$. Taking a number z^* on the singular line with a different index k^*, we see that $|\operatorname{Re}(z-z^*)|\geqq|\operatorname{Re}(\alpha_n^{-1}r)|-2|\operatorname{Re}(\lambda\alpha_n^{-1})|$. Now α_n is not purely imaginary, as observed before, so that $\operatorname{Re}(\alpha_n^{-1}r)|\geqq\delta_n r$, for a certain $\delta_n>0$. Setting $\lambda=(6r)^{-1}e^{i\psi}$, we calculate $\operatorname{Re}(\lambda\alpha_n)^{-1}=6rR^{-1}\cos(\varphi+\psi)$. We therefore choose ψ so that $\cos(\varphi+\psi)\neq 0$, $|\cos(\varphi+\psi)|<R\delta_n/24$.

Thus as k varies, but j is held fixed, the singular lines have projections on the x-axis separated by intervals of length at least $r\delta_n/2$. Hence the lines can be completed to a graph with slope at most M_n, independent of r.

When $a_k\neq 0$, then $|kr|\leqq 1+r$, and so each point on the associated line is within $7r|\alpha_n|^{-1}$ of the singular line for $F_0(\alpha_n z+c_j)$. For small r, therefore, the singular lines in the full sum $\varphi_{n+1}=\sum\sum b_j a_k\cdot\lambda F_0(\lambda\alpha_n z+\lambda c_j-k\lambda r)$ are disjoint and are contained in a graph $y=f_{n+1}(x)$ with $|f_{n+1}(x_1)-f_{n+1}(x_2)|\leqq(1-2^{-n-1})h(|x_1-x_2|)$. Here we use for the first and only time the hypothesis that $u=o(h(u))$ as $u\to 0+$. Taking r small enough, we can ensure that the φ_{n+1} is analytic outside the strip $|x|\leqq 1-2^{-n-2}$. Moreover, $|b_j||a_k||\lambda|\leqq 3^{1-n}(6r)^{-1}2r=3^{-n}$.

We assert that for small enough $r>0$, $|\varphi_n-\varphi_{n+1}|\leqq C''3^{-n}$, with C'' independent of n. By the Lemma, we certainly have

$$|b_j F_0(\alpha_n z+c_j)-b_j\sum a_k\cdot F_0(\lambda\alpha_n z+\lambda c_j-k\lambda r)|\leqq C'3^{-n}$$

for each fixed j. The singular lines of the functions $F_0(\alpha_n z+c_j)$ can be separated by Jordan curves; near the j-th line, all the other differences tend uniformly to 0 with r, so we can attain a bound $C''3^{-n}=2C'3^{-n}$.

Now $f=\lim f_n(x)$ exists uniformly, as does $\varphi=\lim\varphi_n$ outside the union of all the curves. To prove that φ is continuous in the entire plane, we fix a number z_0 and observe that for each $n\geqq 1$, φ_n is a sum $\varphi_n^1+\varphi_n^2$, where $|\varphi_n^1|\leqq C3^{-n}$ and φ_n^2 is continuous on a neighborhood of z_0. Plainly $\varphi=2^{-1}+O(z^{-2})$ near infinity and φ is analytic except on the curve $y=f(x)$.

II. In this part we examine whether there is a function φ, analytic in R^2-C but not a polynomial there, such that a certain derivative $\varphi^{(n)}$ is bounded. The examples of Carleson [2] suggest that Hausdorff measures alone cannot give interesting results here, for $n \geqq 1$. We state an elementary theorem on removable singularities, and then prove that the condition named is best possible.

Let $h(t)$ be positive, continuous, and increasing for $0 < t < +\infty$; we say that C has a *perimeter of h-measure* 0 if to each $\varepsilon > 0$ we can find closed, rectifiable Jordan curves Γ_i in $R^2 - C$, of lengths L_i, so that $\sum h(L_i) < \varepsilon$, and $2\pi i = \sum \int_{\Gamma_i} (\zeta - z)^{-1} d\zeta$ for each $z \in C$. Because h is positive and increasing, we see that C is a Cantor set, and cannot separate $R^2 - C$.

Theorem. *(a) Let φ be analytic in $R^2 - C$, and $\varphi^{(n)}$ bounded, for a certain $n = 0, 1, 2, \ldots$. If C has a perimeter of h-measure 0, $h(t) \equiv t^{n+1}$, then φ is a polynomial of degree at most n.*

(b) Suppose that $\liminf h(t) t^{-n-1} = 0$ *as* $t \to 0+$. *Then there is a compactum C, with a perimeter of h-measure 0, and a function φ, analytic in $R^2 - C$, such that $\varphi, \ldots, \varphi^{(n)}$ are bounded and uniformly continuous in the plane, but φ is not constant.*

Proof of (a). Subtracting a polynomial from φ, we can assume that $\varphi(z) = \sum_0^\infty a_k z^{-k-1}$ in some region $|z| > R$. Then

$$2\pi i a_k = \sum \int_{\Gamma_i} f(\zeta) \zeta^k \, d\zeta.$$

Each integral is 0 unless Γ_i surrounds a point in C, and we can assume that $h(L_i) \leqq 1$ for each Γ_i, so that the Γ_i fall within some circle $|z| < R_1$. On each curve we integrate ζ^k n times, choosing a primitive q_i on Γ_i so that $|q_i(\zeta)| = O(\text{diam}\, \Gamma_i)^n = O(L_i)^n$. The integral is then $(-1)^n \int \varphi^{(n)}(\zeta) q_i(\zeta) d\zeta = O(L_i)^{n+1}$. As we can make the sum $\sum L_i^{n+1}$ arbitrarily small, each $a_k = 0$, so that $\varphi = 0$ for $|z| > R$; and then $\varphi \equiv 0$ because $R^2 - C$ is connected.

For the proof of (b), we treat separately the cases $n = 0$ and $n \geqq 1$, as they differ in important details. We use the operator $\bar\partial = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$ so that $f' = -i\partial f/\partial y$ on the interior of the set $\bar\partial f = 0$, that is, the domain of analyticity of f. In place of singular integrals we have resort to the Fourier transform, using the symbol $e(t) \equiv e^{2\pi i t}$. Let $F(u, v) \in L^1(R^2) \cap L^\infty(R^2)$ and

$$\hat F(x, y) \equiv \iint e(ux + vy) F(u, v)\, du\, dv.$$

A solution of the equation $\bar\partial g = \hat F$ is given by

$$g(x, y) = (i\pi)^{-1} \iint e(ux + vy)(u + iv)^{-1} F(u, v)\, du\, dv.$$

This is an absolutely convergent integral because $F \in L^1 \cap L^\infty$. We have also

$$\partial g(x, y)/\partial y = 2 \iint e(ux + vy) v (u + iv)^{-1}\, du\, dv,$$

and this equals ig' when $\bar\partial g = 0$.

Proof for $n=0$. To each pair of integers M, N $(M>N>2)$, we construct a function $g_{M,N}$ of class $C^\infty(R^2)$, period 1 in each variable, and mean-value 1. Let $f_N(x, y)$ have mean-value 1 and

(i) $f(x,y)=0$ unless $|x|<N^{-1}$, $|y|<N^{-1}$ (Modulo 1),

(ii) $|f_{xx}|+|f_{yy}| < C_1 N^2 \cdot N^2 = C_1 N^4$.

The Fouier coefficients a_{pq} of f satisfy

(iii) $a_{00}=1$, $|a_{pq}|\leqq 1$ always

$$|a_{pq}| \leqq C_1 N^2(p^2+q^2)^{-1}.$$

Then we find from (iii)

(iv) $\sum\sum |a_{pq}|(1+|p|+|q|)^{-1} \leqq C_2 N$.

To find this estimate, we use the first inequality when $p^2+q^2\leqq N^2$, and the second when $p^2+q^2>N^2$. We set $g_{M,N}(x,y)=f(Mx, My)$.

Lemma. *Let* $F(u, v)$ *be a rapidly decreasing function on* R^2. *Then*

$$\hat{F}(x, y) g_{M,N}(x, y)$$

is the Fourier transform of a rapidly decreasing function $G_{M,N}$.

If $N\to+\infty$, $M\to+\infty$ *while* $N=o(M)$, *then*

$$\iint |F(u,v)-G_{M,N}(u,v)|\,|u+iv|^{-1}\,du\,dv \to 0.$$

Proof. The first statement follows from the smoothness of G as we shall see presently. Recalling that a_{pq} are the Fourier coefficients of f_N, we have $g_{M,N} = \sum\sum a_{pq} e(Mpx+Mqy)$, and we find that $G_{M,N}$ is a sum

$$\sum\sum a_{p,q} F(u-Mp, v-Mq).$$

It will be sufficient, therefore, to study the sum of integrals

$$\sum\sum' |a_{p,q}| \iint |F(u-Mp, v-Mq)|\,|u+iv|^{-1}\,du\,dv.$$

(As usual, a sum $\sum\sum'$ means that $(0,0)$ is excluded.) Now the integral is $O(|Mp|+|Mq|)^{-1}$ as we see by dividing it into the domain $(|u+iv|>|Mp|/2+|Mq|/2)$ and its complement, and using the rapid decrease of F. We arrive at the estimate $M^{-1}\sum\sum'|a_{pq}|(|p|+|q|)^{-1}=O(M^{-1}N)$ by (iv), and this is $o(1)$.

We now explain how this estimate leads to the function φ and the set C. We take F so that $\hat{F}$ has compact support and then define φ_0 as a Fourier transform so that $\bar{\partial}\varphi_0=\hat{F}$. Then we take $M>N>2$ and define φ_1 so that $\bar{\partial}\varphi_1=F\cdot g_{M,N}$. The support of $\hat{F}\cdot g_{M,N}$ is covered by squares of side $2M^{-1}N^{-1}$, and the number of these that meet the support of $\hat{F}$ is $O(M^2)$, as the centers of the squares are at the points $(M^{-1}k, M^{-1}l)$. For the h-measure of this system of boundaries we find $O(M^2)h(8M^{-1}N^{-1})$. To make this sum small, together with NM^{-1}, we find $t>0$ so that $h(t^2)<\varepsilon^4 t^2$, and define $M\cong 10(\varepsilon t)^{-1}$, $N\cong \varepsilon t^{-1}$, so $M^2h(8M^{-1}N^{-1})<100\varepsilon^2$ and $NM^{-1}<\varepsilon^2$.

We now choose sequences $(M_1, N_1), \ldots, (M_j, N_j)$ so that $M_j^2 h(8M_j^{-1}N_j^{-1})=o(1)$. We define φ_p by the equation $\bar\partial\varphi_j=F\cdot g_{M_1N_1}\cdots g_{M_jN_j}$; then $\varphi_j=\hat G_j$ and the construction can be carried out so that $G_j\to G_\infty\neq 0$ in $L^1(R^2)$. Then $\varphi_j\to\varphi\not\equiv c$ uniformly and φ is analytic outside a set C with a perimeter of h-measure 0.

Proof for $n\geqq 1$.

For $n\geqq 1$ it doesn't seem to be possible to use squares, or even rectangles, for the covering of C. We shall therefore need a function more complicated than $g_{M,N}$ used for $n=0$. Let $H(s,t)$ be a smooth function of period 1 and mean-value 1, and let $H(s,t)=0$ if $|s|\leqq 1/3$ or $|t|\leqq 1/3$. As before, F is a function in $L^1\cap L^\infty$, such that $\hat F(x,y)$ has compact support.

Lemma. *Let* $s=0, 1, 2, 3, \ldots$ *and* $F(u,v)=O(1+|u|+|v|)^{-s-3}$. *For all real numbers* α, β *we have*

$$I_s(\alpha,\beta)\equiv\iint |F(u-\alpha, v-\beta)|\,|v|^s|u+iv|^{-1}\,du\,dv$$
$$\leqq c(F)(1+|\alpha|)^{-1}(1+|\beta|)^s.$$

Proof. We rewrite the integral and use the inequality $|v+\beta|^s<2^s|v|^s+2^s|\beta|$, obtaining a majorant

$$2^s\iint |F(u,v)|\,(|v|^s+|\beta|^s)\cdot|(u+\alpha)+i(v+\beta)|^{-1}\,du\,dv.$$

We divide the (u,v) plane into two subsets.

(i) $(u+\alpha)^2+(v+\beta)^2>(1+|\alpha|)^2/4$. The integral over this set is $O(1+|\alpha|)^{-1}(1+|\beta|^s)$ because $|F(u,v)|(1+|v|^s)$ is integrable.

(ii) $(u+\alpha)^2+(v+\beta)^2\leqq(1+|\alpha|)^2/4$. The integral of $|(u+\alpha)+i(v+\beta)|^{-1}$ over this region is $\pi(1+|\alpha|)$. On this subset $|u|\geqq\frac{1}{2}|\alpha|-\frac{1}{2}$; considering separately the cases $|\alpha|\geqq 2$ and $|\alpha|\leqq 2$, we conclude that $|F(u,v)|=O(1+|\alpha|)^{-2}(1+|\beta|)^{-s}$, so the cofactor of $|(u+\alpha)+i(v+\beta)|^{-1}$ on this subset is $O(1+|\alpha|)^{-2}(1+|\beta|^s)$.

We use this observation about $p+q\sqrt2$, when p, q are integers and $p^2+q^2\geqq 1$: $|(p+q\sqrt2)(p-q\sqrt2)|=|p^2-2q^2|\geqq 1$, whence $(|p|+|q|)|p+q\sqrt2|>1/2$.

To select pairs of integers M_j, N_j, we begin with a sequence of integers $N_j\to+\infty$ so that $h(8N_j^{-1})<j^{-2}N_j^{-n-1}$ and choose $M_j\cong jN_j^n$, so that $M_jN_jh(8N_j^{-1})=o(1)$, $N_j^n=o(M_j)$. In place of the functions $g_{M,N}$ used for $n=0$, we use $H(Mx, M\sqrt2x+Ny)$. This is zero along each line $Mx=k$ or $M\sqrt2x+Ny=l$, and these divide the plane into parallelograms of area $M^{-1}N^{-1}$ and perimeter $<8N^{-1}$ (since $M>N$). The number of these meeting a fixed bounded set is $O(MN)$, and this leads to a sum

$MNh(8N^{-1})$ of the type just considered. Let H have Fourier coefficients b_{pq}, with $b_{00}=1$; if $F\in L^1$ then $H(Mx, M\sqrt{2}x+Ny)\hat{F}(x,y)$ is the Fourier transform of

$$\sum\sum b_{pq}F(u-Mpx-M\sqrt{2q}, v-Nq).$$

To estimate the function ψ defined by $\bar{\partial}\psi=H(Mx, M\sqrt{2}x+Ny)\hat{F}(x,y)$, along with $\partial^s\psi/\partial y^s$, $1\leqq s\leqq n$, we have to consider sums

$$\sum\sum{}'|b_{pq}|\iint |F(u-Mp-M\sqrt{2q}, v-Nq)|\,|v|^s|u+iv|^{-1}du\,dv,$$

for $0\leqq s\leqq n$, or

$$\sum\sum{}'|b_{pq}|\,I_s(Mp+M\sqrt{2}q, Nq).$$

But $(Mp+M\sqrt{2}q)\geqq M(|p|+|q|)^{-1}/2$, so the cofactor of $|b_{pq}|$ has order of magnitude $(|p|+|q|)M^{-1}(1+|Nq|)^s<(1+|p|+|q|)^{s+1}M^{-1}N^s$ (for $q\neq 0$) and $|p|M^{-1}$ for $q=0$. Since the coefficients b_{pq} decrease rapidly, $\sum\sum b_{pq}(|p|+|q|)^{s+1}<+\infty$, and the construction can be completed for $n\geqq 1$.

The complicated function $H(Mx, M\sqrt{2}x+Ny)$ is introduced, to handle the terms with $p\neq 0$, $q\neq 0$ in the Fourier expansion of $H(s,t)$. For almost all real numbers ξ, $|p\xi+q|\geqq c(|p|+|q|)^{-2}$, for example, and this would be a sufficient lower bound.

III. In this section we consider briefly the class of sets (FS): C is FS if some element $\varphi\neq 0$ of FL^1 is analytic outside C.

Theorem. *Let E be a compact set of R, and suppose that $E\times[0,1]$ is of class FS. Then E has positive logarithmic capacity.*

Proof. There exist test functions $\theta(x)$ and $\psi(y)$ such that

$$\iint \bar{\partial}(\theta(x)\psi(y))\varphi(x,y)\,dx\,dy\neq 0;$$

in the opposite case, φ would be entire, while $\varphi(\infty)=0$. We define a distribution $T\neq 0$ carried by E: $T(\theta)\equiv\iint\bar{\partial}(\theta(x)\psi(y))\varphi(x,y)dxdy$. Inasmuch as φ belongs to FL^1, we have the bound

$$\begin{aligned}|T(\theta)| &\leqq C\sup|u+iv|\,|\hat{\theta}(u)\cdot\hat{\psi}(v)|\\ &\leqq C_1\sup(1+|u|)|\hat{\theta}(u)|.\end{aligned}$$

(The converse is true and easily proved: if E carries a distribution $T\neq 0$ with this bound, then $E\times[0,1]$ is of class FS. This applies, for example, to sets E of positive Hausdorff dimension.)

We define a function $\Phi(u+iv)$ harmonic outside E: $\Phi(u+iv)=T_x(\log|w-x|)$; T_x means that T operates on the variable x. If Φ were continuable to be harmonic in the plane, then $\Phi_u-i\Phi_v$ would be entire. For large w

$$\Phi_u-i\Phi_v=T_x(w-x)^{-1}=\sum_0^\infty w^{-n-1}T(x^n).$$

Assuming that $\Phi_u - i\Phi_v$ is entire, we obtain $T(x^n)=0$ for $n=0, 1, 2, \dots$, so that $T=0$ by Weierstrass' theorem, a contradiction.

To prove the boundedness of $\Phi(w)$ near E, let $\chi(x)$ be a test function, equal to 1 on a neighborhood of E.

Then $\int \chi(x) \log |w-x| e(tx) dt = O(1) \log (e+|w|)$; we use this when $|t|<1$. For $|t|>1$ we use the observe that when $v\neq 0$

$$\chi'(x) \log |w-x| + \chi(x) \operatorname{Re} (w-x)^{-1}$$

has Fourier transform bounded by the same quantity; hence $\chi(x) \log |w-x|$ has a Fourier transform bounded by $O(1) \cdot \log (e+|w|) \cdot (1+|t|)^{-1}$. This completes the proof that E has positive logarithmic capacity; in particular $E\times[0, 1]$ need not be of class FS when E is perfect, but $E\times[0, 1]$ must be non-removable [2]. The case $n=0$ of (b) shows that Besicovitch's theorem [1] is best possible for sets of class FS. We state two problems on this class:

Is every set of positive Lebesgue measure necessarily of class FS? Every set $E\times[0, 1]$, where E has positive logarithmic capacity?

References

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