The Franklin system is an unconditional basis in H_1

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It was an open question if H_1 has an unconditional basis (cf. [6], [9] p. 75). The positive answer was recently provided by B. Maurey [10], but his proof does not indicate how to construct such a basis. Later L. Carleson [2] constructed an explicite sequence in BMO whose biorthogonal functionals form an unconditional basis in H_1. In the present note we apply Carleson's proof to the Franklin system. In this way we provide a new unconditional basis for H_1 and we obtain new information about the Franklin system.

Our proof is a minor modification of Carleson's but orthonormality of Franklin functions permits us to replace the most delicate part of Carleson's considerations by much simpler argument.

The Franklin system is an orthonormal set of piecewise linear functions on interval $[0, 1]$. We will index those functions with dyadic intervals. If $\underline{v}=(j2^{-k}, (j+1)2^{-k})$ $k=0, 1, 2, \ldots, j=0, 1, \ldots, 2^k-1$, then $f_{\underline{v}}$ is a piecewise linear function having nodes at points

$$i2^{-(k+1)}, \quad i = 0, 1, \ldots, 2j+2$$

and

$$i2^{-k}, \quad i = i+2, \ldots, 2^k.$$

The first two Franklin functions, i.e. the constant function and the function $2\sqrt{3}\,(x-1/2)$ are not covered by this notation, but it does not matter. The letter $\underline{v}$, $\underline{w}$, $\underline{z}$ will always stand for dyadic intervals.

The Franklin system was investigated in detail by Z. Ciesielski [3] and [4]. In particular he proved the estimates (1) and (2) (cf. [4] Th. 1) which are basic for our work. Let us introduce the following notation:

If t is a point or I is an interval in $[0, 1]$ and $\underline{v}$ is a dyadic interval then

$$r(t, \underline{v}) = d(t, \underline{v})|\underline{v}|^{-1}$$
$$r(I, \underline{v}) = d(I, \underline{v})|\underline{v}|^{-1}$$

where d is the usual distance and $|\underline{v}|$ denotes the length of $\underline{v}$.

The estimates of Ciesielski are: there exist a constant q, $0<q<1$ and a constant C such that

$$|f_{\underline{v}}(t)| \leqq \frac{C}{|\underline{v}|} q^{r(t, \underline{v})} \tag{1}$$

$$|f_{\underline{v}}(t_1)-f_{\underline{v}}(t_2)| \leqq C|\underline{v}|^{-3/2}|t_1-t_2|q^{([t_1, t_2], \underline{v})}. \tag{2}$$

We will work with the space BMO of functions of bounded mean oscillation on interval $[0, 1]$. Let us recall that $f\in BMO$ if

$$\sup_{I\subset[0,1]} \frac{1}{|I|}\int_I \left|f-\frac{1}{|I|}\int_I f\right| < \infty.$$

It is known that

$$\sup\left(\frac{1}{|I|}\int_I \left|f-\frac{1}{|I|}\int_I f\right|^2\right)^{1/2}$$

is an equivalent norm on BMO.

Our H_1 is an atomic H_1. Let us recall that an atom is a function f with $\operatorname{supp} f\subset I$ such that $|f(t)|\leqq|I|^{-1}$ and $\int_0^1 f=0$. We say that $f\in H_1$ if and only if $f=\sum a_i f_i$ where f_i are atoms and $\sum |a_i|<\infty$. The norm is defined as infimum over all such representations of $\sum |a_i|$. The Fefferman—Stein theorem asserts that such H_1 coincides with the space of functions integrable together with its Hilbert transform and that $H_1^*=BMO$. Moreover if VMO denotes the BMO closure of continuous functions we have $VMO^*=H_1$. The excellent reference for all this is [5].

Our main result is the following

Theorem. *The BMO norm of $f=\sum_{\underline{v}} (\int f f_{\underline{v}}) f_{\underline{v}}$ is equivalent to*

$$\sup_{\underline{w}}\left(\frac{1}{|\underline{w}|}\sum_{\underline{v}\subset\underline{w}}\left|\int f f_{\underline{v}}\right|^2\right)^{1/2}.$$

In particular the Franklin system is an unconditional basis in VMO and H_1.

The proof of the above Theorem is contained in the following three lemmas.

Lemma 1. *If a sequence $\{a(\underline{v})\}$ satisfies*

$$\sum_{\underline{v}\subset\underline{w}} |a(\underline{v})|^2 \leqq A|\underline{w}| \quad \text{for all } \underline{w} \tag{3}$$

then $f=\sum_{\underline{v}} a(\underline{v})f_{\underline{v}}\in BMO$ and $\|f\|_{BMO}\leqq C=C(A)$.

Proof. Let us fix an arbitrary interval $I\subset[0, 1]$ and take two adjacent dyadic intervals $\underline{w}_1$ and $\underline{w}_2$ such that $|w_1|=|w_2|$, $\underline{w}_1, \underline{w}_2\subset 4I$ and $\underline{W}=\underline{w}_1\cup\underline{w}_2\supset 2I$. Let us decompose

$$f=\sum_{\underline{v}\subset\underline{W}}+\sum_{\substack{\underline{v}\cap\underline{W}=\emptyset\\ |\underline{v}|\le|I|}}+\sum_{\underline{v}:|\underline{v}|>|I|}a(\underline{v})f_{\underline{v}}=\sum\nolimits_1+\sum\nolimits_2+\sum\nolimits_3.$$

If we write $\sum_1=a(\underline{W})f_{\underline{W}}+\sum_{\underline{v}\subset\underline{w}_1}a(\underline{v})f_{\underline{v}}+\sum_{\underline{v}\subset\underline{w}_2}a(\underline{v})f_{\underline{v}}$ (the first summand does not appear if $\underline{W}$ is not a dyadic interval) we infer from (3)

$$\int_I\left|\sum\nolimits_1\right|^2\le\int_0^1\left|\sum\nolimits_1\right|^2\le A(2+\sqrt{2})\,|I|. \tag{4}$$

Observe that (3) implies $|a(\underline{v})|^2\le A|\underline{v}|$. For $x\in I$ we have by (1)

$$\left|\sum\nolimits_2(x)\right|\le\sum|a(\underline{v})|\,|f_{\underline{v}}(x)|\le A\sum_{|\underline{v}|\le|I|}|\underline{v}|^{1/2}\sum_{\substack{\underline{v}\cap\underline{W}=\emptyset\\ |\underline{v}|=\text{const}}}|f_{\underline{v}}(x)|\le CA. \tag{5}$$

If we fix $x_0\in I$ we obtain using (2)

$$\left|\sum\nolimits_3(x)-\sum\nolimits_3(x_0)\right|\le\sum_{\underline{v}:|\underline{v}|\ge|I|}|a(\underline{v})|\,|f_{\underline{v}}(x)-f_{\underline{v}}(x_0)|$$

$$\le A\sum_{|\underline{v}|>|I|}|\underline{v}|^{1/2}\sum_{|\underline{v}|=\text{const}}|\underline{v}|^{-3/2}|I|q^{r(I,\underline{v})}\le CA|I|\sum_{|\underline{v}|>|I|}|\underline{v}|^{-1}\le CA. \tag{6}$$

Clearly (4), (5) and (6) implies

$$\frac{1}{|I|}\int_I\left|f-\sum\nolimits_3(x_0)\right|^2\le CA.$$

This completes the proof of the Lemma.

Lemma 2. $\|f_{\underline{v}}\|_{H_1}\le C|\underline{v}|^{1/2}$.

Proof. Let us consider the space of all mean zero step functions constant on intervals $(k2^{-N},(k+1)2^{-N})$ $k=0, 1, \dots, 2^N-1$, where $|\underline{v}|=2^{-N}$.

There exists a system of functions in this space $\{\chi_j^k\}$ $j=1, 2, \dots, 2^{N-k}$, $k=1, 2, \dots, N-1$ satisfying the following conditions:

(i) $|\operatorname{supp}\chi_j^k|=2^{-N+k}$ and for every k, $\bigcup_j\operatorname{supp}\chi_j^k=[0, 1]$, in particular supports of χ_j^k and χ_s^k are disjoint for $s\ne j$

(ii) χ_j^k takes only the values 0, $+1$ and -1

(iii) χ_j^{k+1} is constant on supports of all χ_s^k.

By $\chi_{\underline{v}}^k$ we denote such χ_j^k that $\operatorname{supp}\chi_j^k\supset\underline{v}$.

(iv) $$d\left(\underline{v},\ \{t:\chi_{\underline{v}}^k(t)\ne\chi_{\underline{v}}^k|\underline{v}\}\right)\ge\left(\frac{k}{2}-2\right)|\underline{v}|$$

(v) for each k the supports of χ_j^k are intervals, except for at most one function $\chi_e^k\ne\chi_{\underline{v}}^k$. For this function $\chi_e^k=1$ is an interval adjacent from the left to $\operatorname{supp}\chi_{\underline{v}}^k$ and $\chi_{\underline{e}}^k=-1$ is an interval adjacent from the right to $\operatorname{supp}\chi_{\underline{v}}^k$.

Remark. Condition (iv) is the crucial one. The rest is to ensure that we get the orthogonal, Haar-like system.

We present only the construction of our system. It is obvious that conditions (i)—(v) will be satisfied.

To start the inductive proof we define

$$\chi_j^1(t) = \begin{cases} 1 & \text{for } \ 2j2^{-N} < t < (2j+1)2^{-N} \\ -1 & \text{for } \ (2j+1)2^{-N} < t < (2j+2)2^{-N} \\ 0 & \text{otherwise.} \end{cases}$$

Having constructed χ_j^k's we define χ_j^{k+1} as follows:

a) If $\operatorname{supp} \chi_{\underline{v}}^k$ is an interval adjacent to an endpoint of $[0, 1]$ we put

$$(*) \qquad \chi_j^{k+1}(t) = \begin{cases} 1 & \text{for } \ t \in \operatorname{supp} \chi_{2j-1}^k \\ -1 & \text{for } \ t \in \operatorname{supp} \chi_{2j}^k \\ 0 & \text{otherwise.} \end{cases}$$

b) If $\operatorname{supp} \chi_{\underline{v}}^k$ is not adjacent to an endpoint of $[0, 1]$ we consider two cases:

b.1. There exists χ_e^k, the function with disjoint support. We define

$$\chi_{\underline{v}}^{k+1}(t) = \begin{cases} 1 & \text{for } \ t \in \operatorname{supp} \chi_{\underline{v}}^k \\ -1 & \text{for } \ t \in \operatorname{supp} \chi_e^k \\ 0 & \text{otherwise.} \end{cases}$$

The rest of χ_j^{k+1}'s we define as in $(*)$ whenever it is possible (i.e. we have to keep the supports disjoint from the support of $\chi_{\underline{v}}^{k+1}$). The union of supports of χ_j^{k+1}'s so defined is either $[0, 1]$, in this case the construction is finished, or we have left $\operatorname{supp} \chi_j^k$ and $\operatorname{supp} \chi_{j+3}^k$, adjacent from the left and from the right to the $\operatorname{supp} \chi_{\underline{v}}^{k+1}$. In this case we define

$$\chi_e^{k+1}(t) = \begin{cases} 1 & \text{for } t \in \operatorname{supp} \chi_j^k \\ -1 & \text{for } \ t \in \operatorname{supp} \chi_{j+3}^k \\ 0 & \text{otherwise.} \end{cases}$$

b.2. There is no χ_e^k. We pick $\chi_j^k \neq \chi_{\underline{v}}^k$ with $\operatorname{supp} \chi_j^k$ closest to $\underline{v}$. We define

$$\chi_{\underline{v}}^{k+1}(t) = \begin{cases} 1 & \text{for } \ t \in \operatorname{supp} \chi_{\underline{v}}^k \\ -1 & \text{for } \ t \in \operatorname{supp} \chi_j^k \\ 0 & \text{otherwise.} \end{cases}$$

The rest of the construction is done as in b.1.

Now we are ready to estimate $\|f_{\underline{v}}\|_{H_1}$. By $h_{\underline{w}}$ we will denote the Haar function such that $\operatorname{supp} h_{\underline{w}} = \underline{w}$, normalised in L_∞. Since $\{h_{\underline{w}}\}_{|\underline{w}| < |\underline{v}|} \cup \{\chi_j^k\}_{j,k} \cup \{1\}$ is a complete orthogonal system and each χ_j^k and $h_{\underline{w}}$ when normalised in L_1 is a sum of at most two atoms, we have to show

$$\sum_{\underline{w}:|\underline{w}|<|\underline{v}|}\left|\int f_{\underline{v}}h_{\underline{w}}\right|+\sum_{k,j}\left|\int f_{\underline{v}}\chi_j^k\right| \leqq C|\underline{v}|^{1/2}.$$

We will estimate each sum separately. Since $f_{\underline{v}}$ is linear on each $\underline{w}$ with $|\underline{w}|<|\underline{v}|$ we have

$$\sum_{\underline{w}:|\underline{w}|<|\underline{v}|}\left|\int f_{\underline{v}}h_{\underline{w}}\right| \leqq \sum_{|\underline{w}|<|\underline{v}|}\sum_{\underline{z}:|\underline{z}|=|\underline{v}|}\sum_{\underline{w}\subset\underline{z}}\left|\int f_{\underline{v}}h_{\underline{w}}\right|$$
$$\leqq \sum_{|\underline{w}|<|\underline{v}|}\sum_{\underline{z}:|\underline{z}|=|\underline{v}|}\sum_{\underline{w}\subset\underline{z}}|\underline{w}|^2|\underline{v}|^{-3/2}q^{r(\underline{z},\underline{v})}$$
$$\leqq \sum_{|\underline{w}|<|\underline{v}|}\sum_{\underline{z}:|\underline{z}|=|\underline{v}|}q^{r(\underline{z},\underline{v})}|\underline{w}|\,|\underline{v}|^{-1/2}$$
$$\leqq C\sum_{|\underline{w}|<|\underline{v}|}|\underline{w}|\,|\underline{v}|^{-1/2} \leqq C|\underline{v}|^{1/2}.$$

The second sum we write as

$$\sum_k\left|\int f_{\underline{v}}\chi_{\underline{v}}^k\right|+\sum{}'\left|\int f_{\underline{v}}\chi_j^k\right|$$

To estimate it we note that from (1) follows

$$\int_{[0,1]\setminus\underline{w}}|f_{\underline{v}}| \leqq C|\underline{v}|^{1/2}q^{r([0,1]\setminus\underline{w},\underline{v})} \tag{7}$$

Since $f_{\underline{v}}$ is orthogonal to constant we obtain

$$\sum_k\left|\int f_{\underline{v}}\chi_{\underline{v}}^k\right| = \sum_k\left|\int f_{\underline{v}}(\chi_{\underline{v}}^k-\chi_{\underline{v}}^k(\underline{v}))\right|$$
$$\leqq \sum_k\int_{[0,1]\setminus\{\chi_{\underline{v}}^k=\chi_{\underline{v}}^k|\underline{v}\}}|f_{\underline{v}}| \leqq \sum_k|\underline{v}|^{1/2}q^{\left(\frac{k}{2}-2\right)} \leqq C|\underline{v}|^{1/2}.$$

The last term is estimated as follows

$$\sum{}'\left|\int f_{\underline{v}}\chi_j^k\right| \leqq \sum_k\int_{[0,1]\setminus\operatorname{supp}\chi_{\underline{v}}^k}|f_v|$$
$$\leqq C|\underline{v}|^{1/2}\sum_k q^{\left(\frac{k}{2}-2\right)} \leqq C|\underline{v}|^{1/2}.$$

Lemma 3. *There exists a constant K such that for $f\in BMO$, $\|f\|_{BMO}=1$, $f=\sum_{\underline{v}}a(\underline{v})f_{\underline{v}}$ we have*

$$\sum_{\underline{v}\subset\underline{w}}|a(\underline{v})|^2 \leqq K|\underline{w}| \text{ for all } \underline{w}.$$

Proof. For fixed $\underline{w}$ we write

$$f=\sum_{\underline{v}\subset\underline{w}}+\sum_{\substack{\underline{v}:|\underline{v}|\leqq|\underline{w}|\\ \underline{v}\not\subset\underline{w}}}+\sum_{\underline{v}:|\underline{v}|>|\underline{w}|}a(\underline{v})f_{\underline{v}}=\sum{}_1+\sum{}_2+\sum{}_3$$

We will establish the following estimates

$$\left|\int_{\underline{w}}\sum{}_1\right| \leqq C|\underline{w}| \tag{8}$$

$$\left(\frac{1}{|\underline{w}|}\int_{\underline{w}}\left|\sum{}_2\right|^2\right)^{1/2} \leqq C. \tag{9}$$

For x_0 and x in $\underline{w}$

$$\left|\sum\nolimits_3(x_0)-\sum\nolimits_3(x)\right| \leqq C. \tag{10}$$

To prove (8) we use (7) and Lemma 2 to obtain

$$\left|\int_{\underline{w}}\sum\nolimits_1\right| \leqq \sum\nolimits_{\underline{v}\subset\underline{w}}|a(\underline{v})|\left|\int_{\underline{w}}f_{\underline{v}}\right| = \sum\nolimits_{\underline{v}\subset\underline{w}}|a(\underline{v})|\int_{[0,1]\setminus\underline{w}}f_{\underline{v}}\Big|$$
$$\leqq C\sum\nolimits_{|\underline{v}|\leqq|\underline{w}|}|\underline{v}|^{1/2}\sum\nolimits_{\substack{\underline{v}\subset\underline{w}\\ |\underline{v}|=\text{const}}}|\underline{v}|^{1/2}q^{r([0,1]\setminus\underline{w},\underline{v})}$$
$$\leqq C\sum\nolimits_{|\underline{v}|<|\underline{w}|}|\underline{v}| \leqq C|\underline{w}|.$$

To prove (9) we observe that for $|\underline{v}|<|\underline{w}|$

$$\int_{w}|f_{\underline{v}}|^2 \leqq C\,q^{r(\underline{w},\underline{v})} \tag{11}$$

so

$$\left(\frac{1}{|\underline{w}|}\int_{\underline{w}}\left|\sum\nolimits_2\right|^2\right)^{1/2} \leqq C\sum\nolimits_{\substack{\underline{v}:|\underline{v}|\leqq|\underline{w}|\\ \underline{v}\not\subset\underline{w}}}|\underline{v}|^{1/2}\left(\frac{1}{|\underline{w}|}\int_{\underline{w}}|f_{\underline{v}}|^2\right)^{1/2}$$
$$\leqq C\sum\nolimits_{|\underline{v}|\leqq|\underline{w}|}|\underline{v}|^{1/2}|\underline{w}|^{-1/2}\sum\nolimits_{\substack{\underline{v}:|\underline{v}|=\text{const}\\ \underline{v}\not\subset\underline{w}}}q^{r(\underline{w},\underline{v})/2} \leqq C.$$

To prove (10) we use Lemma 2 and argue exactly as in (6). Using (8), (9) and (10) we infer that for fixed $x_0\in\underline{w}$

$$\left|\sum\nolimits_3(x_0)-\frac{1}{|\underline{w}|}\int_{\underline{w}}f\right| \leqq C.$$

This and the definition of the *BMO* norm yields

$$\left(\frac{1}{|\underline{w}|}\int_{\underline{w}}\left|f-\sum\nolimits_3(x_0)\right|^2\right)^{1/2} \leqq C. \tag{12}$$

From (9), (10) and (12) we obtain

$$\int_{\underline{w}}\left|\sum\nolimits_1\right|^2 \leqq C|\underline{w}|.$$

Since $\sum_{\underline{v}\subset\underline{w}}|a(\underline{v})|^2=\int_0^1\left|\sum_1\right|^2$ in order to finish the proof we have to show that $\int_{[0,1]\setminus\underline{w}}\left|\sum_1\right|^2\leqq C|\underline{w}|$. Using (11) we have

$$\left(\int_{[0,1]\setminus\underline{w}}\left|\sum\nolimits_1\right|^2\right)^{1/2} \leqq \sum\nolimits_{\underline{v}\subset\underline{w}}|a(\underline{v})|\left(\int_{[0,1]\setminus\underline{w}}|f_{\underline{v}}|^2\right)^{1/2}$$
$$\leqq C\sum\nolimits_{\underline{v}\subset\underline{w}}|\underline{v}|^{1/2}q^{r([0,1])\setminus\underline{w},\underline{v})} \leqq C|\underline{w}|^{1/2}.$$

This finishes the proof of the Lemma and of the Theorem.

Remark 1. Our theorem readily yields an unconditional basis in $H_1(D)$, the space of analytic functions in the unit disc, such that

$$\sup_{r<1}\frac{1}{2\pi}\int_0^{2\pi}|f(re^{it})|\,dt<\infty.$$

One has to take the system G_n defined by S. V. Bočkariov [1] as a basis for the disc algebra. We also obtain that the natural predual of $H_1(D)$, namely $C(T)/A$, has an unconditional basis. This space can be identified (cf. [12]) with the space of compact Hankel operators.

The following Proposition shows the non-isometric nature of this result (for relevant definitions and concepts cf. [8]).

Proposition 1. *There exists a constant* $c \geqq 1.15$ *such that if the space* $H_1(D)$ *is isometric to a subspace of a Banach space* X *with an unconditional basis* (e_n), *then* $\operatorname{ubc}(e_n) \geqq c$.

Proof. Since $(z^n + z^{2n})$ tends weakly to zero in $H_1(D)$, the standard gliding hump argument shows that $\operatorname{ubc}(e_n)$ is at least the ratio between $\|1+z^n+z^{2n}\|$ and $\|1-z^n-z^{2n}\|$. Those norms are routinely estimated and the numerical calculation yields the estimate for c.

The fact that there is an orthonormal system which is an unconditional basis for $H_1(D)$ allows us to conclude by interpolation that there exists a constant C such that $\operatorname{ubc}(H_p(D)) \leqq C$ for $1 \leqq p \leqq 2$. This contrasts the situation for L_p and allows us to give a partial answer to Problem 4.1 of [6].

Proposition 2. *The Banach—Mazur distance between any subspace of* $H_p(D)$ *and* L_p *is at least* $C/(p-1)$ *for* $1 < p \leqq 2$.

Proof. Use the fact that the Haar basis is precisely reproducible in L_p ([7] Th. 4.1) and the fact that the unconditionality constant of this basis in L_p is of order $1/(p-1)$ for $1 < p \leqq 2$.

Actually, in the above proof we need only that $H_p(D)$ $1 \leqq p \leqq 2$ embeds with uniform constants into a space with unconditionally monotone basis. This fact follows from Stein's theorem (cf. [13] or [5] Th. 1.20), Theorem 1.1 of [11] and Theorem 1.g.5 of [8].

Remark 2. Since we can interpolate between H_1 and L_2 obtaining L_p (cf. [5] Th. D) we infer that the Franklin system is an unconditional basis for L_p, $1 < p < \infty$. This fact was first proved by Bočkariov [1].

Let us now consider the space $H_1(\sigma)$, the martingale H_1-space connected with the canonical dyadic martingale (cf. [10]). The Haar system is obviously an unconditional basis in $H_1(\sigma)$ and if we express the norm of a function in BMO (σ) in terms of its Fourier—Haar coefficients we see that the Franklin system in H_1 is equivalent to the Haar system in $H_1(\sigma)$. The system constructed by Carleson [2] also enjoys this property. So we obtain an explicite isomorphism between $H_1(\sigma)$ and H_1.

Added in proof (Aug. 3, 1982). Much simpler proofs of Lemma 2 are known now. Some can be found in S-Y. A. Chang—Z. Ciesielski "Spline characterisations of H^1" and in the paper by the author "H_p-spaces, $p \leqq 1$, and spline systems". This paper by the author contains results for splines of higher order and $p \leqq 1$ as well as the references to further results along this lines by S. Sjölin, J-O. Stromberg and others.

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Received February 10, 1980

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