Basis properties of Hardy spaces

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1. Introduction

Set $I=[0,1]$ and let $(\chi_n)_1^\infty$ denote the Haar orthogonal system. If $f\in L^1(I)$ we write $Gf(t)=\int_0^t f(u)\,du$, $t\in I$. Let m be an integer, $m\geqq 0$, and let $(f_n^{(m)})_{n=-m}^{\infty}$ denote the system of functions which is obtained when we apply the Gram-Schmidt orthonormalization procedure to the sequence of functions $1, t, t^2, \ldots, t^{m+1}, G^{m+1}\chi_2$, $G^{m+1}\chi_3$, $G^{m+1}\chi_4, \ldots$ on I. We use here the usual scalar product in $L^2(I)$. The systems $(f_n^{(m)})$ are called spline systems and in particular $(f_n^{(0)})$ is called the Franklin system. These systems are complete in $L^2(I)$ and have been studied by e.g. Z. Ciesielski and J. Domsta [6]. We shall write f_n instead of $f_n^{(m)}$ and set $f_n(t)=0$ for $t\in\mathbf{R}\setminus I$.

For $n\geqq 2$ we have $n=2^j+l$ where $j\geqq 0$, $1\leqq l\leqq 2^j$, and set $t_n=(l-1/2)2^{-j}$. Then $D^m f_n$ is absolutely continuous on I and it is known that

$$|D^k f_n(t)| \leqq M n^{k+1/2} r^{n|t-t_n|}, \quad 0\leqq k\leqq m+1, \quad n\geqq 2, \quad t\in I, \tag{1}$$

where M and r are constants depeding only on m and $0<r<1$ (see [6], p. 316).

Assume that ψ belongs to the Schwartz class of functions $S(\mathbf{R})$ and that $\int_{\mathbf{R}}\psi(x)\,dx\neq 0$. Set $\psi_t(x)=t^{-1}\psi(x/t)$, $t>0$, $x\in\mathbf{R}$, and for $f\in S'(\mathbf{R})$

$$f^*(x) = \sup_{t>0} |f*\psi_t(x)|, \quad x\in\mathbf{R}.$$

The Hardy space $H^p(\mathbf{R})$, $0<p<\infty$, is then defined to be the space of all f such that $\|f\|_{H^p}=\|f^*\|_p<\infty$, where $\|g\|_p$ is defined as $\left(\int |g(x)|^p\,dx\right)^{1/p}$.

For $\alpha>0$ we set $N=[\alpha]$, where [] denotes the integral part, and $\delta=\alpha-N$. If α is not an integer set

$$\dot\Lambda_\alpha = \left\{\varphi\in C^N(\mathbf{R});\ \sup_{h\neq 0} \|\Delta_h D^N\varphi\|_\infty/|h|^\delta < \infty\right\}$$

(here $\Delta_h F(x)=F(x+h)-F(x)$) and if α is an integer set

$$\dot\Lambda_\alpha = \left\{\varphi\in C^{N-1}(\mathbf{R});\ \sup_{h\neq 0} \|\Delta_h^2 D^{N-1}\varphi\|_\infty/|h| < \infty\right\}.$$

Also set $\tilde{\Lambda}_\alpha = \dot{\Lambda}_\alpha / P^N$, where P^N denotes the class of polynomials of degree $\leqq N$. The projection from $\dot{\Lambda}_\alpha$ to $\tilde{\Lambda}_\alpha$ is denoted Π. For $0<p\leqq 1$ set $\alpha = 1/p-1$. It is then well-known that for $0<p<1$ $\tilde{\Lambda}_\alpha$ is the dual space of H^p (see e.g. P. Sjögren [10]). If $f\in H^p$, $0<p<1$, and $\psi\in\tilde{\Lambda}_\alpha$ then

$$\psi(f) = \sum_1^\infty \lambda_j \int b_j \varphi \, dx,$$

where $\varphi\in\Pi^{-1}(\psi)\subset\dot{\Lambda}_\alpha$ and $\sum_1^\infty \lambda_j b_j$ is an atomic decomposition of f (here $\lambda_j\in\mathbf{C}$ and each b_j is a p-atom). If $\varphi\in\dot{\Lambda}_\alpha$ set $\varphi(f) = (\Pi(\varphi))(f)$ for $f\in H^p$. Also set

$$H^p(I) = \{f\in H^p(\mathbf{R});\ \operatorname{supp} f\subset I \ \text{ and } \ \varphi(f)\in\mathbf{R} \ \text{ for every real-valued } \ \varphi\in\dot{\Lambda}_\alpha\},$$

$0<p<1$.
It is also well-known that $(H^1)^* = \mathrm{BMO}$ and we set

$$H^1(I) = \{f\in H^1(\mathbf{R});\ \operatorname{supp} f\subset I \ \text{ and } f \text{ real-valued}\}.$$

Now assume that $1/(m+2)\leqq p\leqq 1$. It follows that $\alpha\leqq m+1$ and hence it is a consequence of (1) that we can find $g_n\in\dot{\Lambda}_\alpha$ ($g_n\in\mathrm{BMO}$ in the case $p=1$) such that $g_n = f_n$ on I. If $f\in H^p(\mathbf{R})$ we then set $a_n = a_n(f) = g_n(f)$, $n=-m, -m+1, \ldots$. If $f\in H^p(I)$ then a_n does not depend on the choice of g_n. This is a consequence of Lemma 3 below. We shall prove the following theorem.

Theorem. *Assume that* $m\geqq 0$ *and* $1/(m+2)<p\leqq 1$. *If* $f\in H^p(I)$ *then the following holds:*

$$C_p^{-1}\|f\|_{H^p} \leqq \Big\|\big(\textstyle\sum_{-m}^\infty a_n^2 f_n^2\big)^{1/2}\Big\|_p \leqq C_p\|f\|_{H^p}. \tag{2}$$

$$f = \textstyle\sum_{-m}^\infty a_n f_n \quad \textit{with convergence in } H^p. \tag{3}$$

If $(n_n)_{k=-m}^\infty$ *is an enumeration of* $-m, -m+1, -m+2, \ldots$, *then also*

$$f = \textstyle\sum_{-m}^\infty a_{n_k} f_{n_k} \quad \textit{with convergence in } H^p. \tag{4}$$

If $f = \sum_{-m+N+1}^\infty c_n f_n$ *with convergence in* H^p, *then* $c_n = a_n(f)$

(here $N = [1/p-1]$*).* (5)

Remark 1. It is easy to see that $f_n\in H^p(I)$ if $n\geqq -m+N+1$ and that $a_n=0$ for $n\leqq -m+N$ if $f\in H^p(I)$. The theorem implies that $(f_n)_{-m+N+1}^\infty$ is an unconditional basis for $H^p(I)$ if $1/(m+2)<p\leqq 1$. We shall also prove that these bases are equivalent.

Remark 2. The inequalities in the theorem hold as well with $(\sum_{-m}^\infty a_n^2 f_n^2)^{1/2}$ replaced by $(\sum_{-m}^\infty a_n^2 \chi_n^2)^{1/2}$ (χ_n is defined as the characteristic function χ_I whenever $n\leqq 1$).

For analogous results in the case $p>1$ see S. V. Bockarev [1], Z. Ciesielski, P. Simon and P. Sjölin [7] and Z. Ciesielski [4]. The case $p=1, m=0$ has been studied by P. Wojtaszczyk [11], Z. Ciesielski [5], F. Schipp and P. Simon [9] and A. Chang. The first explicit construction of an unconditional basis for H^1 was carried out by L. Carleson [3]. Earlier B. Maurey [8] had proved the existence of an unconditinal basis in H^1. In this paper C and r denote constants, which satisfy $C>0$ and $0<r<1$ and may vary from line to line.

2. Proof of the theorem

We shall first make a special choice of the functions g_n mentioned in the introduction. If $-m\leqq n\leqq 1$ then $f_n(t)=\sum_0^{m+1} c_k t^k$, $t\in I$, for some constants c_k. We then set

$$g_n(t)=\left(\sum_0^{m+1} c_k t^k\right)\psi(t), \quad t\in \mathbf{R},$$

where $\psi\in C_0^\infty(\mathbf{R})$ and $\psi(t)=1$, $-1/2\leqq t\leqq 3/2$, and $\psi(t)=0$ if $t\leqq -1$ or $t\geqq 2$.

We then construct g_n in the case $n\geqq 2$. First set $c_k=D^k f_n(1)$, $k=0, 1, \dots, m+1$. Then (1) yields $|c_k|\leqq M n^{k+1/2} r^{n(1-t_n)}$. We set

$$P(x)=\sum_{j=0}^{m+1}\frac{c_j}{j!}x^j$$

and $h_n(x)=P(x)\psi_n(x)$, $x\geqq 0$, where $\psi_n(x)=\psi(2nx)$. It follows that $h_n(x)=0$ for $x\geqq 1/n$ and $h_n^{(k)}(0)=c_k$, $k=0, 1, \dots, m+1$. We have

$$|P^{(k)}(x)|\leqq \sum_{j=k}^{m+1}|c_j|\left(\frac{1}{n}\right)^{j-k}\leqq C\sum_{j=k}^{m+1} n^{j+1/2} r^{n(1-t_n)} n^{k-j}=C n^{k+1/2} r^{n(1-t_n)},$$

$$0\leqq x\leqq 1/n, \quad k=0, 1, \dots, m+1.$$

It follows that

$$|h_n^{(k)}(x)|\leqq C n^{k+1/2} r^{n(1-t_n)}, \quad 0\leqq x\leqq 1/n, \quad k=0, 1, \dots, m+1.$$

We set $g_n(x)=h_n(x-1)$, $x>1$, and define $g_n(x)$ in an analogous way for $x<0$. Then $D^m g_n$ is absolutely continuous on $\mathbf{R}$, $g_n(t)=0$ if dist $(t, I)>1/n$ and

$$|D^k g_n(t)|\leqq M n^{k+1/2} r^{n|t-t_n|}, \quad 0\leqq k\leqq m+1, \quad n\geqq 2, \quad t\in\mathbf{R}, \tag{6}$$

where $0<r<1$.

Lemma 1. *If* $m\geqq 0$ *and* $1/(m+2)<p\leqq 1$ *then*

$$\left\|\left(\sum_{-m}^{\infty} a_n^2 f_n^2\right)^{1/2}\right\|_p\leqq C_p\|f\|_{H^p}, \quad f\in H^p(I).$$

Proof. The condition on p implies that $\alpha=1/p-1<m+1$ and hence $N=[\alpha]\leqq m$. The functions $f_{-m+N+1}, f_{-m+N+2}, f_{-m+N+3}, \dots$ are orthogonal to $f_{-m}, \dots, f_{-m+N}$ and hence orthogonal to $1, t, \dots, t^N$. It follows that f_n, $n\geqq -m+N+1$, are multiples of p-atoms and hence belong to $H^p(\mathbf{R})$ and $H^p(I)$.

Assume $\varphi\in C_0^\infty$, φ real, $\int\varphi\,dx=1$, $\varphi(x)=0$ for $|x|>1$, $\varphi_\varepsilon(x)=\varepsilon^{-1}\varphi(x/\varepsilon)$. For $f\in H^p(I)$ and $-m\leqq n\leqq -m+N$ we have

$$a_n(f) = g_n(f) = \lim_{\varepsilon\to 0} g_n(f*\varphi_\varepsilon) = \lim_{\varepsilon\to 0}\int g_n f*\varphi_\varepsilon\,dx = 0,$$

since g_n is a polynomial of degree $\leqq N$ in a neighbourhood of I and $f*\varphi_\varepsilon\in H^p\cap C_0^\infty$.

We fix a positive integer M and set

$$T_\varepsilon f(t) = \sum_{-m+N+1}^{M} \varepsilon_n a_n f_n(t),$$

where $\varepsilon_n=\pm 1$, $a_n=a_n(f)$, $f\in H^p(\mathbf{R})$ and $\varepsilon=(\varepsilon_n)$.

We shall first prove that

$$\|T_\varepsilon b\|_p \leqq C_p \tag{7}$$

if b is a p-atom. We may assume that b is real-valued. Then there exists an interval $J=[c,d]$ such that $\operatorname{supp} b\subset J$, $\|b\|_\infty\leqq |J|^{-1/p}$ and

$$\int b(t)t^k\,dt = 0, \quad k = 0, 1, \dots, N. \tag{8}$$

Set $B_1(s)=\int_{-\infty}^s b(t)\,dt$ and $B_k(s)=\int_{-\infty}^s B_{k-1}(t)\,dt$, $k=2, \dots, N+1$.

It follows from (8) that $\operatorname{supp} B_k\subset J$, $k=1, 2, \dots, N+1$, and it also follows that

$$\|B_k\|_\infty \leqq |J|^{k-1/p}, \quad k = 1, 2, \dots, N+1. \tag{9}$$

We have $T_\varepsilon b(t)=\sum_{-m+N+1}^{M}\varepsilon_n a_n f_n(t)$, where $a_n=a_n(b)$, and integrating by parts we obtain

$$a_n(b) = \int g_n(s)b(s)\,ds = (-1)^{N+1}\int D^{N+1}g_n(s)B_{N+1}(s)\,ds.$$

For $-m+N+1\leqq n\leqq 1$ it is clear that

$$\|\varepsilon_n a_n f_n\|_p \leqq C_p|a_n| \leqq C_p\|b\|_{H^p} \leqq C_p.$$

Setting $S_\varepsilon b(t)=\sum_2^M \varepsilon_n a_n f_n(t)$ it is therefore enough to prove

$$\|S_\varepsilon b\|_p \leqq C_p. \tag{10}$$

An application of the Hölder inequality shows that

$$\int_{I\cap 2J}|S_\varepsilon b|^p\,dt \leqq \Big(\int_I |S_\varepsilon b|^2 dt\Big)^{p/2}\Big(\int_{2J} dt\Big)^{1-p/2} = C\,\big(\textstyle\sum_2^M a_n^2\big)^{p/2}|J|^{1-p/2}. \tag{11}$$

We claim that

$$\sum_2^M a_n^2 \leqq C|J|^{1-2/p}. \tag{12}$$

Setting $h_n=g_n-f_n$ we have

$$a_n = \int g_n b\,ds = \int f_n b\,ds+\int h_n b\,ds = c_n+d_n.$$

(f_n) is an orthonormal system and hence

$$\sum_2^M c_n^2 \leqq \int_I b^2 dt \leqq \int b^2 dt \leqq |J|^{1-2/p}.$$

We have

$$\sum_2^M d_n^2 = \sum_2^M \Big(\int h_n(t) b(t)\,dt\Big)\Big(\int h_n(s)\,b(s)\,ds\Big) = \iint G(t,s)\,b(t)\,b(s)\,dt\,ds,$$

where $G(t,s)= \sum_2^M h_n(t)h_n(s)$.

Setting $Q=I\times I$ and $Q_1=(1,1)+Q$, $Q_2=(1,-1)+Q$, $Q_3=(-1,1)+Q$ and $Q_4=(-1,-1)+Q$ one finds that

$$\sum_2^M d_n^2 = \sum_1^4 I_i, \quad \text{where} \quad I_i = \iint_{Q_i} G(t,s)b(t)b(s)\,dt\,ds,\ i=1,2,3,4.$$

For $(t,s)\in Q_1$ we have $\left|\sum_{2^j+1}^{2^{j+1}} h_n(t)h_n(s)\right| \leqq C\,2^j\chi_j(t,s)$, where χ_j is the characteristic function of the square $[1,1+2^{-j}]\times[1,1+2^{-j}]$. It follows that $|G(t,s)|\leqq C\big((t-1)^2+(s-1)^2\big)^{-1/2}$, $(t,s)\in Q_1$, and hence

$$|I_1| \leqq C|J|^{-2/p}\iint_{(J\times J)\cap Q_1}\big((t-1)^2+(s-1)^2\big)^{-1/2}dt\,ds$$

$$\leqq C|J|^{-2/p}\iint_{(u^2+v^2)^{1/2}\leqq\sqrt{2}|J|}(u^2+v^2)^{-1/2}du\,dv = C|J|^{1-2/p}.$$

We have the same estimates for I_2, I_3 and I_4 and conclude that $\sum_2^M d_n^2 \leqq C|J|^{1-2/p}$. We have proved (12) and it then follows from (11) that

$$\int_{I\cap 2J}|S_\varepsilon b|^p dt \leqq C. \tag{13}$$

We shall now prove that

$$\int_{I\setminus 2J}|S_\varepsilon b|^p dt \leqq C. \tag{14}$$

We have

$$|S_\varepsilon b(t)| \leqq \sum_2^M |a_n f_n(t)| = \sum_2^M \Big|\int D^{N+1}g_n(s)f_n(t)B_{N+1}(s)\,ds\Big|$$

and invoking (1), (6) and (9) we obtain

$$|S_\varepsilon b(t)| \leqq C\sum_2^M |J|^{N+1-1/p}\int_J n^{N+2}r^{n|s-t_n|}r^{n|t-t_n|}ds$$

$$\leqq C|J|^{N+1-1/p}\sum_{j=0}^\infty 2^{j(N+2)}\int_J\Big(\sum_{2^j+1}^{2^{j+1}} r^{n|s-t_n|}r^{n|t-t_n|}\Big)ds$$

$$\leqq C|J|^{N+1-1/p}\int_J\Big(\sum_0^\infty 2^{j(N+2)}q^{2^j|t-s|}\Big)ds,$$

where $0<q<1$. We observe that

$$\sum_0^\infty 2^{j(N+2)} q^{2^j \gamma} \leqq C\int_0^\infty x^{N+1} q^{\gamma x}\, dx = C\int_0^\infty y^{N+1} q^y\, dy\, \gamma^{-(N+2)} = C\gamma^{-(N+2)}, \quad \gamma > 0,$$

and hence

$$|S_\varepsilon b(t)| \leqq C|J|^{N+1-1/p} \int_J |t-s|^{-N-2}\, ds \leqq C|J|^{N+2-1/p} |t-t_0|^{-N-2}, \quad t\in I\setminus 2J,$$

where t_0 denotes the center of J. It follows that

$$\int_{I\setminus 2J} |S_\varepsilon b|^p\, dt \leqq C|J|^{(N+2)p-1} \int_{I\setminus 2J} |t-t_0|^{-(N+2)p}\, dt$$

$$\leqq C|J|^{(N+2)p-1} \int_{|J|}^\infty t^{-(N+2)p}\, dt = C,$$

since $(N+2)p>1$.

We have proved (14) and the proof of (10) and (7) is complete.

Now let $f\in H^p(I)$ and let $\sum_1^\infty \lambda_i b_i$ be an atomic decomposition of f with

$$\left(\sum_1^\infty |\lambda_i|^p\right)^{1/p} \leqq C_p \|f\|_{H^p}.$$

It follows that $a_n(f)=\sum_1^\infty \lambda_i a_n(b_i)$ and hence $T_\varepsilon f(t)=\sum_1^\infty \lambda_i T_\varepsilon b_i(t)$.

Thus

$$|T_\varepsilon f(t)|^p \leqq \sum_1^\infty |\lambda_i|^p |T_\varepsilon b_i(t)|^p$$

and

$$\int_I |T_\varepsilon f(t)|^p\, dt \leqq C_p \sum_1^\infty |\lambda_i|^p \leqq C_p \|f\|_{H^p}^p.$$

Using a property of the Rademacher functions (see A. Zygmund [12], p. 213) we then conclude that

$$\int_I \left(\sum_{-m}^M a_n^2 f_n^2\right)^{p/2} dt \leqq C_p \|f\|_{H^p}^p$$

and the lemma follows when we let M tend to infinity.

Lemma 2. *Assume that* $m\geqq 0$ *and* $1/(m+5/2)<p\leqq 1$. *If* $c_n\in \mathbf{R}$, $n\geqq -m+N+1$, *and* $\left\|\left(\sum_{-m+N+1}^\infty c_n^2 f_n^2\right)^{1/2}\right\|_p < \infty$, *then* $\sum_{-m+N+1}^\infty c_n f_n$ *converges in* H^p *and*

$$\left\|\sum_{-m+N+1}^\infty c_n f_n\right\|_{H^p} \leqq C_p \left\|\left(\sum_{-m+N+1}^\infty c_n^2 f_n^2\right)^{1/2}\right\|_p.$$

Proof. It is sufficient to prove the lemma in the case when only finitely many c_n are non-vanishing. The general case then follows from a limiting argument if we use the fact that H^p is complete.

Since $f_n\in H^p$, $n\geqq -m+N+1$, we have for $-m+N+1\leqq n\leqq 1$

$$\|c_n f_n\|_{H^p}^p \leqq C|c_n|^p = C|c_n|^p \int f_n^2\, dx \leqq C\int (c_n^2 f_n^2)^{p/2}\, dx.$$

It is therefore enough to prove

$$\left\|\sum_2^\infty c_n f_n\right\|_{H^p} \leqq C\left\|\left(\sum_2^\infty c_n^2 f_n^2\right)^{1/2}\right\|_p.$$

Since all f_n, $n\geqq 2$, are orthogonal to $1, t, \ldots, t^{m+1}$ the iterated primitive functions $G^k f_n$, $1\leqq k\leqq m+2$, will be supported in I and satisfy the estimate

$$|G^k f_n(t)| \leqq Mn^{-k+1/2} r^{n|t-t_n|}.$$

Let $\psi\in C_0^\infty(-1, 1)$ with $\int \psi\, dx=0$ such that $\sup_{t>0}|\hat{\psi}(t\xi)|\geqq c>0$ for $\xi=\pm 1$ and let $\psi_t(x)=\psi(x/t)/t$.

Then by A. P. Calderón and A. Torchinsky [2], Theorem 6.9, p. 56,

$$\left\|\sum_2^\infty c_n f_n\right\|_{H^p} \leqq C_p\left\|A_\psi\left(\sum_2^\infty c_n f_n\right)\right\|_p, \quad p>0,$$

where

$$A_\psi(f)(x) = \left\{\iint_{|y-x|<t} |f*\psi_t(y)|^2 \frac{dy\, dt}{t^2}\right\}^{1/2}, \quad f\in S'.$$

We will show that

$$\left\|A_\psi\left(\sum_2^\infty c_n f_n\right)\right\|_p \leqq C_p\left\|\left(\sum_2^\infty c_n f_n\right)^{1/2}\right\|_p, \quad \frac{1}{m+5/2}<p\leqq 1.$$

To do this we shall define an auxiliary function in the upper half plane and for this need that there for each $n=2^j+l$, $j\geqq 0$, $1\leqq l\leqq 2^j$, exists a subinteravl I_n' of the interval $I_n=[(l-1)2^{-j}, l2^{-j}]$ such that

$$\begin{cases} |I_n'| \geqq cn^{-1} \\ |f_n(x)| \geqq cn^{1/2}, \quad x\in I_n', \end{cases} \tag{15}$$

for some constant $c>0$.

Proof of (15): The function $D^{m+1}f_n$ makes a jump at t_n of magnitude, say $A\geqq 0$. Thus at least one of the left and the right limit at t_n has magnitude $\geqq A/2$.

I_n is divided by t_n into two intervals of length 2^{-j-1} and on at least one of them f_n can be written in the form $A_0\prod_1^{m+1}(x-\alpha_i)$ where α_i are complex numbers and $|A_0|\geqq A/(2(m+1)!)$. Now we can find a subinterval I_n' of length $\delta=2^{-j-1}/(3(m+2))$ such that $\operatorname{dist}(I_n', \operatorname{Re}\alpha_i)\geqq\delta$ for every i. It follows easily that $|f_n(x)|\geqq cA2^{-j(m+1)}$ on I_n'.

In order to estimate A we define $\Delta_h f(x)=f(x+h)-f(x)$ and $x_+=\max(x, 0)$ and set

$$g(x) = x_I(x)(\Delta_{2^{-j-1}})^{m+2}(x-t_n)_+^{m+1}.$$

Then g is supported on $[t_n-(m+2)2^{-j-1}, t_n]$, $\|g\|_\infty\leqq C2^{-j(m+1)}$ and consequently $\|g\|_2\leqq C2^{-j(m+3/2)}$.

Looking at the discontinuities of $D^{m+1}g$ we see that we can write $g=\sum_{i=-m}^n b_i f_i$ where $b_i=\int g f_i\, dx$. In particular $|b_n|=\left|\int g f_n\, dx\right|\leqq\|g\|_2\leqq C2^{-j(m+3/2)}$. Since

$D^{m+1}f_i$, $i<n$, are continuous at t_n we find that $D^{m+1}g$ makes a jump of magnitude $|b_n|A\leqq CA2^{-j(m+3/2)}$. On the other hand we check directly that $D^{m+1}g$ makes a jump of magnitude $(m+1)!$ at t_n. Thus $A\geqq c2^{j(m+3/2)}$.

From this inequality and the estimate above we conclude that (15) holds.

Now we define the function F on $\mathbf{R}_2^+=\{(x, t);\ x\in\mathbf{R},\ t>0\}$ by

$$F(x, t)=\begin{cases}|c_n|n^{1/2} & \text{when } (x, t)\in I_n^+=\frac{1}{2}I_n'\times[2^{-j-1}, 2^{-j}[\quad n\geqq 2,\\ 0 & \text{when } (x, t)\in\mathbf{R}_+^2\setminus\bigcup_2^\infty I_n^+.\end{cases}$$

Note that all I_n^+, $n\geqq 2$, are disjoint. Furthermore, if we define

$$AF(x)=\left\{\iint_{|y-x|<\gamma t}|F(y, t)|^2\frac{dy\,dt}{t^2}\right\}^{1/2}, \quad x\in\mathbf{R},$$

with $\gamma>0$ small enough and

$$g_\lambda^*(F)(x)=\left\{\iint_{\mathbf{R}_+^2}|F(y, t)|^2(1+|x-y|/t)^{-2\lambda}\frac{dy\,dt}{t^2}\right\}^{1/2}, \quad x\in\mathbf{R}, \quad \lambda>0,$$

then

$$\frac{1}{C}AF(x)\leqq\left(\sum_2^\infty c_n^2(f_n(x))^2\right)^{1/2}\leqq Cg_\lambda^*(F)(x)$$

and also

$$\frac{1}{C}AF(x)\leqq\left(\sum_2^\infty c_n^2(x_n(x))^2\right)^{1/2}\leqq Cg_\lambda^*(F)(x)$$

for all $\lambda>0$. But if $p>1/\lambda$ we also have

$$\|g_\lambda^*(F)\|_p\leqq C_p\|AF\|_p$$

(see [2], Theorem 3.5, p. 20). This gives the equivalence between the norms

$$\left\|\left(\sum_{-m}^\infty c_n^2f_n^2\right)^{1/2}\right\|_p \quad\text{and}\quad \left\|\left(\sum_{-m}^\infty c_n^2\chi_n^2\right)^{1/2}\right\|_p \quad\text{for}\quad p>0.$$

To prove Lemma 2 it is enough to prove

$$\left\|A_\psi\left(\sum_2^\infty c_nf_n\right)\right\|_p\leqq C_p\|g_\lambda^*(F)\|_p \tag{16}$$

for all λ with $0<\lambda<m+5/2$. We need to estimate $\psi_t*f_n(y)$.

Case 1: $tn\geqq 1$. By integration by parts we get

$$|\psi_t*f_n(y)|=|(D^{m+2}\psi_t)*(G^{m+2}f_n)(y)|\leqq\|D^{m+2}\psi_t\|_\infty\int_{|z-y|<t}|G^{m+2}f_n(z)|\,dz$$

$$\leqq Ct^{-m-3}\int_{|z-y|<t}n^{-m-3/2}r^{n|t_n-z|}\,dz\leqq C(nt)^{-m-3}n^{1/2}r^{\max\{n(|t_n-y|-t),\,0\}}.$$

Case 2: $tn<1,\ t<y<1-t$. Integrating by parts we obtain

$$|\psi_t*f_n(y)| = |(G\psi_t)*(Df_n)(y)| \leqq \|G\psi_t\|_1 \sup_{|z-y|<t} |Df_n(z)| \leqq Ctn^{3/2}r^{n|t_n-y|}.$$

Case 3: $tn<1,\ |y|<t$ *or* $|y-1|<t$. We have

$$|\psi_t*f_n(y)| \leqq \|\psi_t\|_1 \sup_{|z-y|<t} |f_n(z)| \leqq Cn^{1/2}r^{n|t_n-y|}.$$

In the remaining case $tn<1,\ y<-t$ or $y>1+t$, it is clear that

$$\psi_t*f_n(y) = 0.$$

From the definition of F we get

$$\Big|\psi_t*\Big(\sum_{n\geqq 1/t} c_n f_n\Big)(y)\Big| \leqq C\sum_{n\geqq 1/t}(tn)^{-3-m}r^{\max\{n(|t_n-y|-t),0\}}|c_n|n^{1/2}$$

$$\leqq C\iint_{\mathbf{R}\times\{s;\, s<2t\}} (s/t)^{3+m}r^{\max\{(|z-y|-t)/s,\,0\}}F(z,s)\frac{dz\,ds}{s^2}$$

$$\leqq C\left(\iint_{\mathbf{R}\times\{s;\, s<2t\}} (s/t)^{2m+5-\varepsilon}r^{\max\{(|z-y|-t)/s,\,0\}}|F(z,s)|^2\frac{dz\,ds}{s^2}\right)^{1/2}$$

for all $\varepsilon>0$. Here we have used the Cauchy — Schwarz inequality and the fact that

$$\iint_{\mathbf{R}\times\{s;\, s<2t\}} (s/t)^{1+\varepsilon}r^{\max\{(|z-y|-t)/s,\,0\}}\frac{dz\,ds}{s^2} \leqq C.$$

Now set

$$A^{(\mathrm{I})}(x) = \left(\iint_{|y-x|<t} \Big|\psi_t*\Big(\sum_{n\geqq 1/t} c_n f_n\Big)(y)\Big|^2\frac{dy\,dt}{t^2}\right)^{1/2}.$$

Then we obtain

$$(A^{(\mathrm{I})}(x))^2$$

$$\leqq C\iint_{|y-x|<t}\left(\iint_{\mathbf{R}\times\{s;\, s<2t\}} (s/t)^{2m+5-\varepsilon}r^{\max\{(|z-y|-t)/s,\,0\}}|F(z,s)|^2\frac{dz\,ds}{s^2}\right)\frac{dy\,dt}{t^2}$$

$$\leqq C\iint_{\mathbf{R}^2_+}|F(z,s)|^2\left(\iint_{\{|y-x|<t\}\cap\{t>s/2\}} (s/t)^{2m+5-\varepsilon}r^{\max\{(|z-y|-t)/s,\,0\}}\frac{dy\,dt}{t^2}\right)\frac{dz\,ds}{s^2}$$

and since the inner integral is less than

$$\int_{s/2}^{\infty}(s/t)^{2m+5-\varepsilon}r^{\max\{(|z-x|-2t)/s,\,0\}}\frac{dt}{t} \leqq C\left(1+\frac{|z-x|}{s}\right)^{-2m-5+2\varepsilon}$$

it follows that

$$A^{(\mathrm{I})}(x) \leqq Cg^*_{m+5/2-\varepsilon}(F)(x)$$

for all $\varepsilon>0$.

By the estimate in Case 2 we get when $t<y<1-t$

$$\left|\psi_t*\left(\sum_{n<1/t}c_nf_n\right)(y)\right| \leqq C\sum_{n<1/t}tnr^{n|t_n-y|}|c_n|n^{1/2}$$

$$\leqq C\iint_{\mathbf{R}\times\{s;\, s>t/2\}}(t/s)r^{|z-y|/s}F(z,s)\frac{dz\,ds}{s^2}$$

$$\leqq C\left(\iint_{\mathbf{R}\times\{s;\, s>t/2\}}(t/s)r^{|z-y|/s}|F(z,s|^2\frac{dz\,ds}{s^2}\right)^{1/2},$$

where we have used the Cauchy — Schwarz inequality.

Set

$$A^{(\mathrm{II})}(x)=\left(\iint_{\{|y-x|<t\}\cap\{t<y<1-t\}}\left|\psi_t*\left(\sum_{n<1/t}c_nf_n\right)(y)\right|^2\frac{dy\,dt}{t^2}\right)^{1/2}.$$

We get

$$(A^{(\mathrm{II})}(x))^2\leqq C\iint_{|y-x|<t}\left(\iint_{\mathbf{R}\times\{s;\, s>t/2\}}(t/s)r^{|z-y|/s}|F(z,s)|^2\frac{dz\,ds}{s^2}\right)\frac{dy\,dt}{t^2}$$

$$\leqq C\iint_{\mathbf{R}^2_+}|F(z,s)|^2\left(\iint_{\{|y-x|<t\}\cap\{t;\, t<2s\}}(t/s)r^{|z-y|/s}\frac{dy\,dt}{t^2}\right)\frac{dz\,ds}{s^2}$$

and since the inner integral is less than

$$Cr^{|z-x|/s}\int_0^{2s}(t/s)\frac{dt}{t}\leqq Cr^{|z-x|/s}$$

we obtain

$$A^{(\mathrm{II})}(x)\leqq Cg^*_\lambda(F)(x)$$

for all $\lambda>0$.

We then set

$$A^{(\mathrm{III})}(x)=\left(\iint_{\{|y-x|<t\}\cap(\{|y|<t\}\cup\{|y-1|<t\})}\left|\psi_t*\left(\sum_{n<1/t}c_nf_n\right)(y)\right|^2\frac{dy\,dt}{t^2}\right)^{1/2}.$$

For $A^{(\mathrm{III})}$ we can get no pointwise estimate but we shall prove that

$$\|A^{(\mathrm{III})}\|_p\leqq C_p\|g^*_\lambda(F)\|_p$$

for all $\lambda>0$.

We have

$$(A^{(\mathrm{III})}(x))^2\leqq\iint_{\{|y-x|<t\}\cap\{|y|<t\}}\left|\psi_t*\left(\sum_{n<1/t}c_nf_n\right)(y)\right|^2\frac{dy\,dt}{t^2}$$

$$+\iint_{\{|y-x|<t\}\cap\{|y-1|<t\}}\left|\psi_t*\left(\sum_{n<1/t}c_nf_n\right)(y)\right|^2\frac{dy\,dt}{t^2}=(A^{(\mathrm{III}_0)}(x))^2+(A^{(\mathrm{III}_1)}(x))^2.$$

Since $A^{(\mathrm{III}_0)}$ and $A^{(\mathrm{III}_1)}$ can be treated in the same way we shall only consider $A^{(\mathrm{III}_0)}$ and prove that

$$\|A^{(\mathrm{III}_0)}\|_p\leqq C_p\|g^*_\lambda(F)\|_p,\quad \lambda>0. \tag{17}$$

We set

$$A_j^{(\mathrm{III})}(x)=\left(\iint_{\{|y-x|<t\}\cap\{|y|<t\}}\left|\psi_t*\left(\sum_{\substack{n<1/t\\2^j<n\leqq 2^{j+1}}}c_nf_n\right)(y)\right|^2\frac{dy\,dt}{t^2}\right)^{1/2},\quad j\geqq 0,$$

and it follows that

$$A^{(\mathrm{III}_0)}(x)\leqq\sum_{j=0}^{\infty}A_j^{(\mathrm{III})}(x).$$

We shall prove that

$$\|A_j^{(\mathrm{III})}\|_p^p\leqq C\int_{2^{-j-1}}^{2^{-j}}|g_\lambda^*(F)(x)|^p\,dx \tag{18}$$

and since

$$\|A^{(\mathrm{III}_0)}\|_p^p\leqq\sum_{j=0}^{\infty}\|A_j^{(\mathrm{III})}\|_p^p$$

we get (17) from (18) by summation.

We have

$$\psi_t*\left(\sum_{\substack{n<1/t\\2^j<n\leqq 2^{j+1}}}c_nf_n\right)(y)=0$$

if $t\geqq 2^{-j}$ and by the estimates in Case 3 we have when $|y|<t<2^{-j}$

$$\left|\psi_t*\left(\sum_{\substack{n<1/t\\2^j<n\leqq 2^{j+1}}}c_nf_n\right)(y)\right|\leqq C\sum_{2^j+1}^{2^{j+1}}r^{n|t_n-y|}|c_n|n^{1/2}$$

$$\leqq C\iint_{\mathbf{R}\times\{2^{-j-1}<s<2^{-j}\}}r^{|r-y|/s}F(z,s)\frac{dz\,ds}{s^2}$$

$$\leqq C\left(\iint_{\mathbf{R}\times\{2^{-j-1}<s<2^{-j}\}}r^{|z-y|/s}|F(z,s)|^2\frac{dz\,ds}{s^2}\right)^{1/2}\leqq Cg_\lambda^*(F)(w)$$

for any w with $|w|<2^{-j}$ and all $\lambda>0$. Here we used the fact that

$$\iint_{\mathbf{R}\times\{2^{-j-1}<s<2^{-j}\}}r^{|z-y|/s}\frac{dz\,ds}{s^2}\leqq C.$$

Thus if $|w|<2^{-j}$ we have

$$A_j^{(\mathrm{III})}(x)\leqq Cg_\lambda^*(F)(w)\left(\iint_{\{|y-x|<t\}\cap\{|y|<t<2^{-j}\}}\frac{dy\,dt}{t^2}\right)^{1/2}$$

and since $\{|y-x|<t\}\cap\{|y|<t<2^{-j}\}\subset\{|y|<t,\ |x|/2<t<2^{-j}\}$ the integral is majorized by

$$2\int_{|x|/2}^{2^{-j}}\frac{dt}{t}=2\log\frac{2^{-j+1}}{|x|},\quad |x|<2^{-j+1},$$

and

$$A_j^{(\mathrm{III})}(x)\leqq Cg_\lambda^*(F)(w)\left(\log^+\frac{2^{-j+1}}{|x|}\right)^{1/2}.$$

Hence

$$\|A_j^{(\mathrm{III})}\|_p^p \leqq C|g_\lambda^*(F)(w)|^p \int_{-2^{-j+1}}^{2^{-j+1}} \left(\log\frac{2^{-j+1}}{|x|}\right)^{p/2} dx$$

$$\leqq C2^{-j}|g_\lambda^*(F)(w)|^p \int_{-1}^{1} \left(\log\frac{1}{|x|}\right)^{p/2} dx \leqq C2^{-j}|g_\lambda^*(F)(w)|^p$$

and since this holds for all w with $|w|<2^{-j}$ we get (18) by integration over w.

The above estimates for $A^{(\mathrm{I})}$, $A^{(\mathrm{II})}$ and $A^{(\mathrm{III})}$ and the inequality

$$A_\psi\left(\sum_2^\infty c_n f_n\right) \leqq A^{(\mathrm{I})}+A^{(\mathrm{II})}+A^{(\mathrm{III})}$$

now yield (16) for $\lambda<m+5/2$.

This completes the proof of Lemma 2.

Lemma 3. *For* $f\in H^p(\mathbf{R})$, $0<p\leqq 1$, set $f_c(x)=f(x/c)$, $c>0$ (f_c is well-defined if f is a function and the definition is easily extended to distributions). Then $f_c\to f$ in H^p as $c\to 1$.

Proof. First let $b\in C_0^\infty\cap H^p$. We have

$$\|b-b_c\|_{H^p} \leqq C_p\left(\|b-b_c\|_p+\|H(b-b_c)\|_p\right),$$

where H denotes the Hilbert transform. Since $b\in C_0^\infty$ it is clear that $\lim_{c\to 1}\|b-b_c\|_p=0$.

We set $g(t)=g_x(t)=\pi^{-1}(x-t)^{-1}$ for $t\in\operatorname{supp} b$ and $|x|$ large. Then $g^{(n)}(t)=c_n(x-t)^{-n-1}$ for some constants c_n and hence

$$Hb(x) = \int g(t)b(t)\,dt = \int\left(g(t)-\sum_{n=0}^N \frac{g^{(n)}(0)}{n!}t^n\right)b(t)\,dt$$

and

$$|Hb(x)| \leqq C|x|^{-N-2}\int |t|^{N+1}|b(t)|\,dt = C|x|^{-N-2}$$

for large values of $|x|$. It is also easy to see that $Hb_c=(Hb)_c$ and that Hb is continuous. Using these facts, the above estimate of $Hb(x)$ and the inequality $(N+2)p>1$, we apply the Lebesgue convergence theorem to conclude that

$$\lim_{c\to 1}\|H(b-b_c)\|_p = \lim_{c\to 1}\|Hb-(Hb)_c\|_p = 0.$$

It follows that $\lim_{c\to 1}\|b-b_c\|_{H^p}=0$.

It is well-known that $C_0^\infty\cap H^p$ is dense in H^p and the lemma follows if we also invoke this fact.

Lemma 4. *Assume* $m\geqq 0$ *and* $1/(m+2)<p\leqq 1$. *Let* $\mathscr{P}$ *denote the set of all finite linear combinations with real coefficients of the functions* f_n, $n\geqq -m+N+1$. *Then* $\mathscr{P}$ *is dense in* $H^p(I)$.

Proof. Set $H_0^p(I)=\{f\in H^p(I);\ \text{supp}\, f\subset I^\circ\}$, where I° denotes the interior of I. We first observe that $H_0^p(I)$ is dense in $H_0^p(I)$. In fact, if $f\in H^p(I)$ set $h(x)=f(x+1/2)$. Then $h_c(x-1/2)$ approximates f as c tends to 1 and $\text{supp}\, h_c(x-1/2)\subset I^\circ$ if $c<1$.

By convolution with an approximate identity we then conclude that $H_0^p(I)\cap C_0^\infty$ is dense in $H^p(I)$.

Now let $f\in H_0^p(I)\cap C_0^\infty$ and thus $\int f(x)x^k\,dx=0$, $k=0, 1, \ldots, N$. Set

$$S_n f=\sum_{-m}^{n} a_k f_k, \quad \text{where}$$

$$a_k=a_k(f)=\int f_k f\,dx.$$

Since (f_n) is a complete orthonormal system we have $\lim_{n\to\infty}\|S_n f-f\|_2=0$. We shall use the estimate

$$\|f-S_n f\|_{H^p}\leqq C\|f-S_n f\|_p+C\|H(f-S_n f)\|_p$$

and the first term on the right hand side clearly tends to zero as $n\to\infty$.

We write

$$\|H(f-S_n f)\|_p^p=\int_{|x|\leqq 2}|H(f-S_n f)|^p\,dx+\int_{|x|>2}|H(f-S_n f)|\,dx=A_n+B_n.$$

Using the Hölder inequality and the boundedness of H on $L^2(\mathbf{R})$ we conclude that

$$A_n\leqq C\Big(\int_{-2}^{2}|H(f-S_n f)|^2\,dx\Big)^{p/2}\leqq C\Big(\int|f-S_n f|^2\,dx\Big)^{p/2}$$

and hence $\lim_{n\to\infty}A_n=0$.

Estimating $H(f-S_n f)$ in the same way as we estimated Hb in the proof of Lemma 3 we obtain

$$|H(f-S_n f)(x)|\leqq C\|f-S_n f\|_2|x|^{-N-2}, \quad |x|>2.$$

It follows that $\lim_{n\to\infty}B_n=0$ and hence $S_n f$ tends to f in H^p and the proof of the lemma is complete.

Proof of the Theorem. We first prove (3). Assume $f\in H^p(I)$ and set $S_n f=\sum_{-m}^{n} a_k f_k$, where $a_k=a_k(f)$. It then follows from Lemma 2 and Lemma 1 that

$$\|S_n f\|_{H^p}\leqq C_p\Big\|\Big(\sum_{-m}^{n} a_k^2 f_k^2\Big)^{1/2}\Big\|_p\leqq C_p\|f\|_{H^p}.$$

Assume $\varepsilon>0$, let $\mathscr{P}$ be defined as in Lemma 4 and choose $P\in\mathscr{P}$ such that $\|f-P\|_{H^p}<\varepsilon$. Then $S_nP=P$ if n is large enough and hence $S_n f-f=S_n f-S_n P+P-f$.

It follows that

$$\|S_n f-f\|_{H^p}^p \leqq \|S_n(f-P)\|_{H^p}^p+\|P-f\|_{H^p}^p \leqq C\|f-P\|_{H^p}^p \leqq C\varepsilon^p,$$

if n is large enough, and thus (3) is proved.

The second inequality in (2) follows from Lemma 1 and the first inequality is a consequence of Lemma 2 and (3).

To prove (4) we use (2) to conclude that

$$\Big\|f-\sum_{-m}^{n} a_{n_k} f_{n_k}\Big\|_{H^p} \leqq C_p\Big\|\Big(\sum_{k=n+1}^{\infty} a_{n_k}^2 f_{n_k}^2\Big)^{1/2}\Big\|_p$$

and the right hand side tends to zero since $\|(\sum_{-m}^{\infty} a_{n_k}^2 f_{n_k}^2)^{1/2}\|_p$ is finite.

To prove (5) we observe that if $f=\sum_{-m+N+1}^{\infty} c_k f_k$ then

$$a_n(f) = g_n(f) = \Sigma c_k g_n(f_k) = \Sigma c_k \int g_n f_k\, dx = c_n.$$

The proof of the theorem is complete.

Remark. If we observe that

$$\Big(\sum_2^{\infty} c_n^2(\chi_{n+k}(x))^2\Big)^{1/2} \leqq C_k(g_\lambda^*(F)(x)+g_\lambda^*(F)(1-x))$$

for any $k\in Z$, we obtain equivalence between the norms

$$\Big\|\sum_0^{\infty} b_n f_{n-m+N+1}^{(m)}\Big\|_{H^p}$$

and

$$\Big\|\sum_0^{\infty} b_n f_{n-m'+N+1}^{(m')}\Big\|_{H^p}$$

for $m>1/p-2$, $m'>1/p-2$ and $0<p\leqq 1$.

In fact, if $b_n\in\mathbf{R}$, then

$$\Big\|\sum_0^{\infty} b_n f_{n-m+N+1}^{(m)}\Big\|_{H^p} \leqq C_p\Big\|\Big(\sum_0^{\infty} b_n^2 f_{n-m+N+1}^{(m)2}\Big)^{1/2}\Big\|_p$$

$$\leqq C_p\Big\|\Big(\sum_0^{\infty} b_n^2 \chi_{n-m+N+1}^2\Big)^{1/2}\Big\|_p = C_p\Big\|\Big(\sum_{-m'+N+1}^{\infty} b_{\ell+m'-N-1}^2 \chi_{\ell+m'-m}^2\Big)^{1/2}\Big\|_p$$

$$\leqq C_p(|b_0|+\ldots+|b_{m'-N}|+\|g_\lambda^*(F)\|_p)$$

$$\leqq C_p\Big\|\Big(\sum_{-m'+N+1}^{\infty} b_{\ell+m'-N-1}^2 f_\ell^{(m')2}\Big)^{1/2}\Big\|_p = C_p\Big\|\Big(\sum_0^{\infty} b_n^2 f_{n-m'+N+1}^{(m')2}\Big)^{1/2}\Big\|_p$$

$$\leqq C_p\Big\|\sum_0^{\infty} b_n f_{n-m'+N+1}^{(m')}\Big\|_{H^p}$$

(here F is defined with c_n replaced by $b_{n+m'-N-1}$ and $\lambda>1/p$). It follows that $(f_n^{(m)})_{-m+N+1}^{\infty}$ and $(f_n^{(m')})_{-m'+N+1}^{\infty}$ are equivalent bases for $H^p(I)$ under the above conditions on m and m'.

Remark. During the preparation of this paper we have learnt from P. Wojtaszczyk that he has used the theory of molecules to study basis properties of the Franklin system. We remark that the theory of molecules can be used also for $m\geqq 1$. In fact, using the notation and estimates in the proof of Lemma 1, we can prove that

$$\|S_\varepsilon b\|_2^{1-\theta}\| |t-t_0|^\gamma S_\varepsilon b\|_2^\theta \leqq C,$$

where $1/p-1/2<\gamma<N+3/2$ and $\theta=(1/p-1/2)/\gamma$.

This estimate and Theorem 7.1 in [10] can then be used to give an alternative proof of (3) and (4) in our theorem.

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