# On the spectral synthesis problem for points in the dual of a nilpotent Lie group

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## 1. Introduction

Let $A$ be a *-semi-simple Banach algebra with involution *. One of the main problems concerning the structure of $A$ is the determination of the space $\mathcal{I}$ of the twosided closed ideals of $A$. Let $\mathrm{Prim}_*(A)$ be the space of the kernels of the topologically irreducible unitary representations of $A$ equipped with the Jacobson topology. For $I$ in $\mathcal{I}$, let $h(I)=\{J\in\mathrm{Prim}_*(A)|J\supset I\}$; $(h(I)$ is a closed subset of $\mathrm{Prim}_*(A))$ and define for the closed subset $C$ of $\mathrm{Prim}_*(A)$ the subset $\mathcal{I}_C$ of $\mathcal{I}$ by $\mathcal{I}_C=\{I\in\mathcal{I}|h(I)=C\}$. The closed subset $C$ of $\mathrm{Prim}_*(A)$ is called a *set of spectral synthesis* if $\mathcal{I}_C$ consists only of one point, namely the ideal $\ker C=\bigcap_{\mathcal{J}\in C}J$. The spectral synthesis problem has been most intensively studied for the algebra $A=L^1(G)$, where $G$ is an abelian, locally compact group $G$. The first result was the famous theorem of N. Wiener who showed that the empty set is a set of synthesis in $\mathrm{Prim}_* L^1(\mathbf{R})$. The latest deep results are those of I. Domar. (see for instance [4]).

Almost nothing is known for the algebra $L^1(G)$ is $G$ is not abelian. If $G$ is a connected, simply connected nilpotent Lie group, the dual space $\hat{G}$ is well known and thus also the space $\mathrm{Prim}_*(L^1(G))$.

Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathrm{Ad}^*$ the coadjoint action of $G$ on $\mathfrak{g}^*$. By Kirillow's theorem and Brown's proof of the Kirillow conjecture ([7], [2]) $\hat{G}$ is homeomorphic with the orbit space $\mathfrak{g}^*/_{\mathrm{Ad}^*(G)}$ and [1] tells us that $\mathrm{Prim}_*(L^1(G))\cong\mathfrak{g}^*/_{\mathrm{Ad}^*(G)}$. Thus we may indentify the closed subsets $C$ of $\mathrm{Prim}_*(L^1(G))$ with the closed $G$-invariant subsets of $\mathfrak{g}^*$. $L^1(G)$ has a remarkable property: For every closed subset $C$ of $\hat{G}$ there exists a twosided ideal $j(C)$ in $L^1(G)$ with the properties:

1) $h(j(A))=A$; 2) $j(A)$ is contained in every closed, twosided ideal $I$ of $L^1(G)$ with $h(I)\subset A$ ([11]).

If $G$ is a group of step 1 and of step 2 every point in $\hat{G}$ is a set of spectral synthesis [9]. In this paper we show that in general a point is not a set of synthesis if $G$ is of step 3. Indeed, we are able to determine explicitly the spaces $\mathscr{I}_{\{T\}}$, for every $T \in \hat{G}$.

In general $\mathscr{I}_{\{T\}}$ contains an infinity of elements.

In [12] it has been shown that for every $T$ in $\hat{G}$, the algebra $\ker T/_{j\{(T)\}}$ is nilpotent. The results of this paper make it possible to compute the degree of nilpotency of $\ker T/_{j\{(T)\}}$ if $G$ is of step 3.

**2.** Let $G$ be a connected and simply connected nilpotent Lie group and let $\mathfrak{g}$ be the Lie algebra of $G$. The exponential mapping is a homeomorphism from $\mathfrak{g}$ onto $G$.

We can thus define the Schwartz space $S(G)$ to be the space of all functions $f$ on $G$ such that $f \circ \exp$ is contained in the ordinary Schwartz space $S(\mathfrak{g})$ of the rapidly decreasing smooth functions on the real vectorspace $\mathfrak{g}$.

$S(G)$ is a dense $*$-subalgebra of $L^1(G)$. If $I$ is any element of $\mathscr{I}, I \cap S(G)$ is a twosided closed ideal in $S(G)$.

**(2.1) Proposition.** *Let G be a connected, simply connected nilpotent Lie group. For every $\pi$ in $\hat{G}$,* $\ker \pi \cap S(G)$ *is dense in* $\ker \pi$.

*Proof.* We show first, that for every tempered distribution $\omega$ on $S(G)$ which annihilates $\ker \pi \cap S(G)$ and for every $f_1, f_2$ in $S(G)$, there exists a constant $C>0$ (depending on $f_1$ and $f_2$) such that

$$|\langle \omega, f_1 * f * f_2 \rangle| \leqq C|\pi(f)|; \quad \forall f \in S(G).$$

($|\pi(f)|$ denotes the operatornorm of $\pi(f)$).

(2.2) There exists $k \in \mathbf{N}$ and a realization of $\pi$ on $L^2(\mathbf{R}^k)$ such that:

a) For every $f$ in $S(G)$ the operator $\pi(f)$ on $L^2(\mathbf{R}^k)$ is described by a Schwartz-kernel $K_\pi(f)$; that means: there exists a function $K_\pi(f)$ in $S(\mathbf{R}^k \times \mathbf{R}^k)$ so that:

$$(\pi(f)\xi)(x) = \int_{\mathbf{R}^k} K_\pi(f)(x, y)\xi(y)\,dy; \quad \forall \xi \in L^2(\mathbf{R}^k) \quad \forall x \in \mathbf{R}^k.$$

b) The mapping $K_\pi$: $S(G) \to S(\mathbf{R}^k \times \mathbf{R}^k)$ is surjective.

c) If $d\pi$ denotes the representation of the envelopping algebra $U(\mathfrak{g})_{\mathbf{C}}$ corresponding to $\pi$ on $L^2(\mathbf{R}^k)$, then $d\pi(U(\mathfrak{g})_{\mathbf{C}})$ is the algebra of differential operators with polynomial coefficients on $\mathbf{R}^k$. ([15] and [7])

Thus $K_\pi$ defines an algebraical and topological isomorphism also denoted by $K_\pi$, of the Fréchet spaces $S(G)/_{S(G) \cap \ker \pi}$ and $S(\mathbf{R}^k \times \mathbf{R}^k)$. This allows us to define a tempered distribution $\tilde{\omega}$ on $S(\mathbf{R}^k \times \mathbf{R}^k)$ by:

$$\langle \tilde{\omega}, K_\pi(f) \rangle := \langle \omega, f \rangle; \; f \in S(G).$$

There exists a continuous and bounded function $w$ in $L^2(\mathbf{R}^k\times\mathbf{R}^k)$ and a differential operator $D$ with polynomial coefficients such that

$$\langle\tilde{\omega}, g\rangle = \int_{\mathbf{R}^k\times\mathbf{R}^k} w(x, y) Dg(x, y)\, dx\, dy;\quad g\in S(\mathbf{R}^k\times\mathbf{R}^k)$$

(see [16]).

Now if $f_1, f, f_2\in S(G)$, $x, y\in\mathbf{R}^k$:

$$K_\pi(f_1*f*f_2)(x, y) = \int_{\mathbf{R}^k\times\mathbf{R}^k} K_\pi f_1(x, u) K_\pi f(u, v) K_\pi(v, y)\, du\, dv.$$

Thus: $DK_\pi(f_1*f*f_2)(x, y) = \sum_{i,j}^N \int_{\mathbf{R}^k\times\mathbf{R}^k} F_i^1(x, u) K_\pi f(u, v) F_j^2(v, y)\, du\, dv$

for some $F_i^1, F_j^2\in S(\mathbf{R}^k\times\mathbf{R}^k)$.

Taking

$f_i^1, f_j^2$ $(i, j = 1, \dots, N)$ in $S(G)$ with $K_\pi(f_i^1) = F_i^1$; $K_\pi(f_j^2) = F_j^2 (i, j = 1, \dots, N)$

we get:

$$|\langle\omega, f_1*f*f_2\rangle| = \Big|\int_{\mathbf{R}^k\times\mathbf{R}^k} w(x, y) \big(\sum_{\substack{i=1\\ j=1}}^N K_\pi(f_i^1*f*f_j^2)\big)(x, y)\, dx\, dy\Big|$$

$$\leqq \sum_{i,j=1}^N |w|_2 |K_\pi(f_i^1*f*f_j^2)|_2.$$

As for any $F$ in $S(\mathbf{R}^k\times\mathbf{R}^k)$, $|F_2|$ is the Hilbert — Schmidt norm of the operator defined by $F$ on $L^2(\mathbf{R}^k)$ we have:

$$|\langle\omega, f_1*f*f_2\rangle| \leqq \sum_{i,j=1}^N |w|_2 |\pi(f_i^1*f*f_j^2)|_{\text{H.S.}} \leqq \underbrace{\Big\{\sum_{i,j}^N |\pi(f_i)|_{\text{H.S.}}\, \pi(f_j)\Big\}}_{C} |\pi(f)|.$$

Let now $\varphi\in L^\infty(G)$ with $\langle\varphi, \ker\pi\cap S(G)\rangle = 0$.

Then: $|\langle\varphi, f_1*f*f_2\rangle| \leqq C|\pi(f)|$; $\forall f\in S(G)$ ($C$ depending on $f_1$ and $f_2$).

Hence $\langle\varphi, f_1*\ker\pi*f\rangle = 0$ for all $f_1, f_2\in S(G)$ and so $\langle\varphi, \ker\pi\rangle = 0$.

This implies (by Hahn — Banach):

$$\ker\pi\cap S(G) \text{ is dense in } \ker\pi.$$

q.e.d.

**3.** The determination of $\mathcal{I}_{\{T\}}$ for a point $T$ in $\hat{G}$, if $G$ is of step 3.

From now on $G$ will denote a connected and simply connected nilpotent Lie group of step 3, that means: if $\mathfrak{g}$ is the Lie algebra of $G$,

$$[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \neq 0;\quad [\mathfrak{g}, [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]] = 0.$$

Let $T$ be a point in $\hat{G}$ and denote by 0 the corresponding orbit in $\mathfrak{g}^*$.

Let $\mathfrak{z}$ be the centre of $\mathfrak{g}$ and $\mathfrak{z}_0$ a subspace of $\mathfrak{z}$ contained in the kernel of an element $l$ of $O$.

The subset $\mathfrak{z}_0^\perp=\{\varrho\in\hat{G}|\varrho(\exp\mathfrak{z}_0)=Id_{\mathscr{H}_\varrho}\}$ is closed in $\hat{G}$ and a set of spectral synthesis in $\hat{G}$ ([11]). Hence, as $T\in\mathfrak{z}_0^\perp$:

(3.1) Every element $I$ of $\mathscr{I}_{\{T\}}$ contains $\ker(z_0^\perp)$.

Let $\tilde{\mathfrak{g}}=\mathfrak{g}/_{\mathfrak{z}_0}$, $\tilde{G}=G/_{\exp\mathfrak{z}_0}$. As $T(\exp\mathfrak{z}_0)=Id$, $T$ defines an element $\tilde{T}$ of $\tilde{G}$. If $p$ denotes the canonical projection from $G$ onto $\tilde{G}$

$$T=\tilde{T}\circ p.$$

As $L^1(\tilde{G})=L^1(G)/_{\ker(\mathfrak{z}_0^\perp)}$ it follows from 3.1 that.

(3.2) The map $I\to I\bmod(\ker\mathfrak{z}_0^\perp)$ is an inclusion preserving bijection from $\mathscr{I}_{\{T\}}$ onto $\mathscr{I}_{\{\tilde{T}\}}$.

If for $l\in O$; $l([\mathfrak{g},[\mathfrak{g},\mathfrak{g}]])=0$ and if we put $\mathfrak{z}_0=[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]$, then $\tilde{\mathfrak{g}}$ is an algebra of step 2 and so $[\tilde{T}]$ is a set of synthesis, thus:

$$\{\ker\tilde{T}\}=\mathscr{I}_{\tilde{T}} \text{ and hence}$$

$$\mathscr{I}_T=\{\ker T\}.$$

We suppose from now on that $\langle l,[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]\rangle\neq 0$. It follows also from (3.2) that we can suppose that $\dim\mathfrak{z}=1$.

Thus we have the following situation:

$$(3.3)\qquad [\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]=\mathfrak{z} \quad\text{and}\quad \dim\mathfrak{z}=1.$$

We give now a detailed description of a nilpotent Lie algebra of step 3 satisfying (3.3).

Let $z\in\mathfrak{z}\setminus(0)$. Let $y_1,\ldots,y_k$ be elements of $[\mathfrak{g},\mathfrak{g}]$ such that $\{y_1,y_2,\ldots,y_k,z\}$ is a basis of $[\mathfrak{g},\mathfrak{g}]$.

As $[\mathfrak{g},[\mathfrak{g},\mathfrak{g}]]=\mathbf{R}z$, there exist $\varphi_1,\ldots,\varphi_k\in\mathfrak{g}^*$ such that

$$[u,y_i]=\varphi_i(u)z;\ \forall u\in\mathfrak{g},\quad i=1,\ldots,k.$$

(3.4) The $\varphi_i$'s are linearly independent:

$$\text{if}\quad \textstyle\sum_{i=1}^k c_i\varphi_i=0\quad\text{for some}\quad c_1,\ldots,c_k\in\mathbf{R}\quad\text{then:}$$

$$[u,\textstyle\sum_i c_iy_i]=(\sum_{i=1}^k c_i\varphi_i(u))z=0\quad\text{for every}\quad u\in\mathfrak{g}.$$

Thus $\sum_{i=1}^k c_iy_i\in\mathfrak{z}$ and hence $c_1=c_2=\ldots=c_k=0$.

This implies:

(3.5) There exist $x_1,\ldots,x_k$ in $\mathfrak{g}$, such that

$$[x_i,y_j]=\delta_{ij}z;\quad i,j=1,\ldots,k.$$

(3.6) Let $\mathfrak{h}=\bigcap_{i=1}^k\ker\varphi_i$; then $\mathfrak{h}=\{u\in\mathfrak{g}|[u,[\mathfrak{g},\mathfrak{g}]]=0\}$

$$\text{For}\quad l\in O,\quad\text{let}\quad \mathfrak{g}(l)=\{v\in\mathfrak{g}|\langle l,[v,\mathfrak{g}]\rangle=0\}.$$

(3.7) Let $\mathfrak{g}_0=\mathfrak{g}(l)+[\mathfrak{g},\mathfrak{g}]$. ($\mathfrak{g}_0$ depends only on $O$).

We show now that:

(3.8) $\mathfrak{g}_0$ is the centre of $\mathfrak{h}$.

It is clear that $[\mathfrak{g},\mathfrak{g}]$ is in the centre of $\mathfrak{g}$.

As $[\mathfrak{g}(l),[\mathfrak{g},\mathfrak{g}]]\subset\ker l\cap z=0$, $\mathfrak{g}(l)$ is contained in $\mathfrak{h}$.

As $[\mathfrak{g},[\mathfrak{h},\mathfrak{h}]]\subset[\mathfrak{h},[\mathfrak{g},\mathfrak{h}]]\subset[\mathfrak{h}[\mathfrak{g},\mathfrak{g}]]=0$.

(3.9) $$[\mathfrak{h},\mathfrak{h}]\subset\mathbf{R}z$$

so $[\mathfrak{g}(l),\mathfrak{h}]\subset\ker l\cap\mathbf{R}z=0$, thus

$\mathfrak{g}(1)+[\mathfrak{g},\mathfrak{g}]\subset$centre of $\mathfrak{h}$.

There exists an element $l_1$ on $O$ such that

$$l_1(y_i)=o;\quad i=1,\dots,k.$$

Let $v\in$centre of $\mathfrak{h}$; put $[x_i,v]=\sum_{j=1}^k c_{ij}y_j+c_iz$.

Then $\langle l_1,[x_i,v-\sum_{j=1}^k c_jy_j]\rangle=\langle l_1,[x_i,v]\rangle-\langle l_1,c_iz\rangle=c_i\langle l_1,z\rangle-c_i\langle l_1,z\rangle=0$

as $[v-\sum_{y=1}^k c_jy_j,\mathfrak{h}]=0$ we see that

$$v-\sum_{j=1}^k c_jy_j\in\mathfrak{g}(l_1)\quad\text{and so}\quad v\in\mathfrak{g}(l_1)+[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}_0.$$

This proves (3.8).

As $[\mathfrak{h},\mathfrak{h}]\subset\mathbf{R}z$ (see 3.9):

(3.10) There exist $u_i, v_j$ in $\mathfrak{h}$ $(i,j=1,\dots,s)$ such that $\mathfrak{h}=\sum_{i=1}^s\mathbf{R}u_i+\sum_{j=1}^s\mathbf{R}v_j+\mathfrak{g}_0$ and such that

$$[u_i,v_j]=\delta_{ij}z;\quad i,j=1,\dots,s.$$

(3.11) Let now $O_0$ be the restriction of $O$ to $\mathfrak{g}_0$, $O_0=G(l/_{\mathfrak{g}_0})$ for any 1 in $O$. $O_0$ is a closed $G$-invariant subset of $\mathfrak{g}_0^*$.

Let $G_0=\exp\mathfrak{g}_0$. $G$ acts as a group of automorphisms on $G_0$ by restriction of the inner automorphisms to $G_0$, so $G$ acts on $L^1(G_0)$ too by the formula:

$$f^g(x)=f(g^{-1}xg);\ f\in L^1(G_0),\quad x\in G_0,\ g\in G.$$

(3.12) For a closed subset $C$ of $\hat{G}_0$ let $\mathcal{I}_C^G$ be the set of all twosided closed ideals $I$ of $L^1(G_0)$ with $h(I)=C$, which are $G$-invariant.

**(3.13) Proposition:** *Let $G$ be a connected and simply connected Lie group of step 3 satisfying* (3.3). *Let $T\in\hat{G}$. Let $O$ be the $G$-orbit of $T$ in $\mathfrak{g}^*$. If $T(\text{centre}(G))\neq Id_{\mathscr{H}(T)}$, there exists an inclusion preserving bijection between $\mathcal{I}_{\{T\}}$ and $\mathcal{I}_{(O_0)}^G$ ($O_0$ as in* (3.10)).

*Proof.* Let $l\in O$ satisfy: $l(y_j)=0$, $j=1,\dots,k$, $l(x_i)=0$, $i=1,\dots,k$.

We verify immediately that, using (3.5):

(3.14) The map: $[\mathfrak{g}, \mathfrak{g}] \to l + \mathfrak{h}^{\perp} \subset \mathfrak{g}^*$

$$v \to \langle l, [\cdot, v] \rangle \text{ is surjective.}$$

(3.15) Denote by $\mathfrak{h}^{\perp}$ the set of the unitary characters of $G$ which are trivial on $H = \exp \mathfrak{h}$. For every $\chi \in \mathfrak{h}^{\perp}$, zhere exists $v \in [\mathfrak{g}, \mathfrak{g}]$, such that $\chi(\exp x) = e^{-i\langle l, [x, v]\rangle} \forall$ $x \in \mathfrak{g}$ (this follows from 3.14). As $l + \mathfrak{h}^{\perp}$ is a closed $G$-invariant subset of $\mathfrak{g}^*$, it defines a closed subset, also denoted $l + \mathfrak{h}^{\perp}$, of $\hat{G}$.

(3.16) $l + \mathfrak{h}^{\perp}$ is a set of spectral synthesis by ([5], 5.3).

(3.17) Let $K = \ker(l + \mathfrak{h}^{\perp}) \lhd L^1(G)$. If $z \in \text{centre}(\mathfrak{g})$ with $\langle l, z \rangle = 1$, then one computes easily that:

(3.18) $K = \{f \in L^1(G) \mid \int_{\mathbf{R}} f(g(\exp rz)) e^{-ir} dr = 0$ for almost all $g \in G\}$ and that

(3.19) for $f \in L^1(G)$, $\chi \in \mathfrak{h}^{\perp}$ one has using (3.18) (3.15) (3.5):

$$\chi \cdot f - f^{\exp v} \in K \quad \text{if} \quad \chi = \chi(v) \quad \text{as in (3.15).}$$

Let now $O_1$ be the restriction of $O$ to $\mathfrak{h}^*$.

From (3.16) we see that $K$ is contained in every element $I$ of $\mathscr{I}_{\{T\}}$ as $T \in l + \mathfrak{h}^{\perp}$.

Thus (3.19) implies: $\chi \cdot I \subset I$ for every $\chi \in \mathfrak{h}^{\perp}$, $I$ in $\mathscr{I}_{\{T\}}$. [5] now implies that there exists an inclusive preserving bijection between $\mathscr{I}_{\{T\}}$ and $\mathscr{I}^G_{O_1}$.

Now again the map: $\mathfrak{g}_0 \to l|_{\mathfrak{g}_0} + \mathfrak{g}_0^{\perp} \subset \mathfrak{h}^*$, $u \to \langle l|_{\mathfrak{g}_0}, [\cdot, u] \rangle$ is surjective (by 3.10).

We can use similiar arguments as above, to get: there exists an inclusion preserving bijection between

$$\mathscr{I}^G_{O_1} \quad \text{and} \quad \mathscr{I}^G_{O_0}. \quad \text{q.e.d.}$$

## 4. The determination of $\mathscr{I}^G_{O_0}$

Let $\mathfrak{g}$ be as in (3.3) and $\mathfrak{g}_0$ as in (3.7).

(4.1) Let $D_i = \text{ad} x_i|_{\mathfrak{g}_0}$; $i = 1, \ldots, k$, ($x_i$ as in 3.5).

The $D_i's$ are linearly independent and commuting endomorphisms of $\mathfrak{g}_0$.

Let $\mathbf{D} = \sum_{i=1}^k \mathbf{R} D_i$ and let $\overline{\mathbf{D}} = \exp \mathbf{D} \subset Gl(\mathfrak{g}_0)$.

(4.2) We can realize the $2k+1$-dimensional Heisenberg group $H_k$ by defining: $H_k = \overline{\mathbf{D}} X [\mathfrak{g}, \mathfrak{g}]$ and defining the multiplication of $H_k$ by: $(D, u) \cdot (D', u') = (D \cdot D', u + D(u'))$; $D, D' \in \overline{\mathbf{D}}$, $u, u' \in [\mathfrak{g}, \mathfrak{g}]$. The group $H_k$ acts as a group of diffeomorphisms on $\mathfrak{g}_0$ by the formula:

$$(D, u)(x) = D(x) + u \tag{4.3}$$

(4.4) Now as $\mathfrak{g}_0$ is abelian, we may identify the additive group $\mathfrak{g}_0$ with $G_0$ and so $L^1(G_0) = L^1(\mathfrak{g}_0)$.

We define the (isometric) action of $H_k$ on $L^1(\mathfrak{g}_0)$ by:

$$((D, u) \cdot f)(x) = f((D, u)^{-1}(x)); \ (D, u) \in H_k; \ f \in L^1(\mathfrak{g}_0), x \in \mathfrak{g}_0. \tag{4.5}$$

(4.5) allows us to define a representation of $L^1(H_k)$ on $L^1(\mathscr{g}_0)$:

(4.6) $$\alpha \circ f = \int_{H_k} \alpha(h) h \cdot f dh; \ \alpha \in L^1(H_k), f \in L^1(\mathscr{g}_0).$$

(4.7) Let $K_0 = \ker(l_0 + z^\perp)$ $(l_0 = l|_{\mathscr{g}_0}, l \in 0)$

(4.3) tells us that $K_0$ is invariant under the action of $H_k$ (and of course of $G$ also)

(4.8) Let $L^1(\mathscr{g}_0)_\chi$ be the algebra of all measurable functions $f$ on $\mathscr{g}_0$ satisfying

1) $$f(x+rz) = e^{ir} f(x), \ \forall r \in \mathbf{R} \quad \text{for almost all} \quad x \in \mathscr{g}_0$$

2) $$|f|_1 = \int_{\mathscr{g}_0/\mathbf{R}z} |f(x)|\, dx < \infty$$

with the multiplication defined by:

$$f * g(x) = \int_{\mathscr{g}_0/\mathbf{R}z} f(y) g(-y+x)\, dy, \ f, g \in L^1(\mathscr{g}_0)_\chi; \ x \in \mathscr{g}_0.$$

The map $P_\chi$: $L^1(\mathscr{g}_0) \to L^1(\mathscr{g}_0)_\chi$

$P_\chi f(x) = \int_{\mathbf{R}} f(x+rz) e^{-ir} dr$ is a continuous surjective homomorphism. Thus:

(4.9) $L^1(\mathscr{g}_0)/_{K_0}$ is isometrically isomorphic with $L^1(\mathscr{g}_0)_\chi$.

The dual space of $L^1(\mathscr{g}_0)_\chi$ is of course homeomorphic with the subspace $l_0 + z^\perp$ of $\mathscr{g}_0^*$. Let $\tilde{O}_0$ denote the image of $O_0$ in $L^1(\mathscr{g}_0)_\chi^\wedge$.

(4.10) The map: $I \to I \bmod K_0$ is an inclusion preserving bijection between $\mathscr{I}_{O_0}^G$ and $\mathscr{I}_{\tilde{O}_0}^G$.

Let us return for one moment to $H_k$.

It is well known that there exists exactly one representation $\pi$ of $H_k^\wedge$ with $\pi(\exp rz) = e^{-ir}\, Id (r \in \mathbf{R})$.

Let $J = \ker \pi$. Then:

(4.11) $\ker \pi = \{\alpha \in L^1(H_k) | \int \alpha((D, u+rz)) e^{-ir} dr = 0$, for almost all $(D, u)\}$.

Using (4.11) and (4.5) one computes easily that:

(4.12) $\ker \pi \circ L^1(\mathscr{g}_0) \subset K_0$.

Thus we can define a representation of $L^1(H_k)_\chi = L^1(H_k)_{/J}$ on $L^1(\mathscr{g}_0)_\chi$ by the formula (4.6).

The algebra $L^1(H_k)_\chi$ has many projectors:

(4.13) Let $\psi$ be the character of $\mathbf{R}z + Y (Y = \sum_{i=1}^k \mathbf{R} y_i)$: $\psi(y+rz) = e^{-ir}$, $y \in Y$, $r \in \mathbf{R}$.

If $\pi = \operatorname{ind}_{[\mathscr{g}, \mathscr{g}]}^{H_k} \psi$, $\pi$ acts on $L^2(\mathbf{R}^k)$ and $\pi$ fulfils the conditions of (1.1).

For $f in S(H_k)$: $$K_\pi(f)(D, D') = \int_{Y+\mathbf{R}z} f(D'^{-1} \cdot D, u) e^{i\langle D', u\rangle} du; \ D, D' \in \overline{\mathbf{D}};$$

here $\langle D, u' \rangle = \Sigma d_i u'_i - u'_0$, if $D = \exp(\Sigma d_i D_i)$ and $u' = \sum_{i=1}^k u'_i y_i + u'_0 z$.

(4.14) For $\xi \in S(\overline{\mathbf{D}})$, $|\xi|_2=1$, let $\alpha_\xi$ be the (unique) element of $S(H_k)_\chi = S(H_k)/_{\ker \pi \cap S(H_k)}$ with $K_\pi(\alpha_\xi)=\xi \otimes \bar{\xi}$, that means: $\pi(\alpha_\xi)$ is the projector on $\mathbf{C}\xi$. Thus $\alpha_\xi$ is a projector in $L^1(H_k)_\chi$.

(4.15) Let $\mathscr{P}$ be the set of all $\alpha_\xi$ in $S(H_k)_\chi$, such that $\pi(\alpha_\xi)$ is a one dimensional projector (on the subspace $\mathbf{C}\xi$, $|\xi|_2=1$) As $\{\pi\}$ is a set of synthesis in $\hat{H}_k$([9]), for every $\alpha \in \mathscr{P}$, the ideal $L^1(H_k)_\chi * \alpha * L^1(H_k)_\chi$ is dense in $L^1(H_k)_\chi$.

(4.16) Let $L^1(\mathscr{g}_0)_{\tilde{\chi}}$ be the algebra of all the measurable functions $h$ on $\mathscr{g}_0$ satisfying:

1) $h(x+y+rz) = e^{ir}h(x)$; for all $y \in Y$, $r \in \mathbf{R}$ for almost all $x \in \mathscr{g}_0$.

2) $$\int_{\mathscr{g}_{0/\mathbf{R}z+Y}} |h(x)|\, dx = |h|_1 < \infty$$

(4.17) *Remark:* Let $W$ be a subspace of $\mathscr{g}(l)$ such that $W \cap (Y+\mathbf{R}z)=0$ and such that $\mathscr{g}_0 = W+(Y+\mathbf{R}z)$; then the restriction map $f \to f/_W$ is an isometric isomorphism of the algebra

$$L^1(\mathscr{g}_0)_{\tilde{\chi}} \quad \text{onto} \quad L^1(W)=L^1(\mathscr{g}(l)+[\mathscr{g},\mathscr{g}]/[\mathscr{g},\mathscr{g}]).$$

(4.18) Let $C=C(\overline{\mathbf{D}}, L^1(\mathscr{g}_0)_{\tilde{\chi}})$ be the Banach algebra of all bounded continuous functions from $\overline{\mathbf{D}}(\cong \mathbf{R}^k)$ into $L^1(\mathscr{g}_0)_{\tilde{\chi}}$ (with pointwise multiplication).

Let $C_\infty$ be the closed subalgebra of the functions vanishing at infinity.

(4.19) Let $p$ be the projection from $L^1(\mathscr{g}_0)_\chi$ onto $L^1(\mathscr{g}_0)_{\tilde{\chi}}$ defined by:

$$p(f)(x) = \int_Y f(x+y)\,dy$$

**(4.20) Proposition:** *The map* $K: L^1(\mathscr{g}_0)_\chi \to C(\overline{\mathbf{D}}, L^1(\mathscr{g}_0)_\chi)$

$$Kf(D) = p(D^{-1}\cdot f)$$

*is a continuous and injective homomorphism of*

$$L^1(\mathscr{g}_0)_\chi \quad \text{into} \quad C_\infty.$$

*Proof.* As for any $f \in L^1(\mathscr{g}_0)_\chi$, $D \in \overline{\mathbf{D}}$, $|Kf(D)|_1 = |p(D^{-1}\cdot f)|_1 \leqq |D^{-1}\cdot f|_1 = |f|_1$, $K$ is a bounded operator.

If $\{D_n\}$ is a sequence in $\overline{\mathbf{D}}$, converging to $D$, $D_n^{-1}\cdot f$ converges to $D^{-1}\cdot f$ in $L^1(\mathscr{g}_0)_\chi$, for any $f$, and so $K(f)(D_k)$ converges to $K(f)(D)$; thus $K(f)$ is continuous for any $f$. It is clear that $K$ is a homomorphism.

For $(D', u') \in H_k$, $f \in L^1(\mathscr{g}_0)_\chi$:

$$K((D', u')\cdot f)(D) = p((D^{-1}\cdot(D', u'))\cdot f) = p((D^{-1}\cdot D', D^{-1}\cdot u')f).$$

For $x$ in $\mathscr{g}_0$, we have:

$$p((D^{-1}\cdot D', D^{-1}u')\cdot f)(x) = \int_Y f((D'^{-1}\cdot D(x+y)-D'^{-1}(u'))\,dy$$

$$= \int_Y f(D'^{-1}\cdot D(x+y-u')+\langle D, u'\rangle z)\,dy$$

$$= e^{i\langle D,u'\rangle}\int_Y f(D'^{-1}\cdot D(x+y))\,dy = e^{i\langle D,u'\rangle}Kf(D'^{-1}\cdot D)(x)$$

if $\langle D, u'\rangle = \sum_{i=1}^k d_i u_i' - u_0'$, where $D=\exp\left(\sum_{i=1}^k d_i D_i\right)$ and

$$u' = \sum_{i=1}^k u_i' y_i + u_0' z.$$

Thus:

(4.21) $$K((D', u')\cdot f)(D) = e^{i\langle D,u'\rangle}Kf(D'^{-1}\cdot D);\ D, D'\in\overline{\mathbf{D}},\ u'\in Y+\mathbf{R}z.$$

For $\alpha\in L^1(H_k)_\chi$, we get:

$$K(\alpha\circ f)(D) = p\left(D^{-1}\int_{H^k}\alpha(D', u')(D', u')\cdot f\,du'\,dD'\right)$$

$$= \int_{\overline{\mathbf{D}}}\int_{\mathbf{R}z+Y}\alpha(D', u')e^{i\langle D,u'\rangle}\,du')Kf(D'^{-1}\cdot D)\,dD' = \int_{\overline{\mathbf{D}}}\tilde{\alpha}(D, D')Kf(D')\,dD'.$$

(4.22) if we write $\tilde{\alpha}(D, D') = \int_{Y+\mathbf{R}z}\alpha(D'^{-1}\cdot D, u')e^{i\langle D,u'\rangle}\,du'.$

Thus

(4.23) $$K(\alpha\cdot f) = \int_{\overline{\mathbf{D}}}(K_\pi\cdot\alpha)(D, D')Kf(D')\,dD' \quad (\text{see } (4.13)).$$

As $S(H_k)_\chi$ is dense in $L^1(H_k)_\chi$ and as $L^1(H_k)_\chi$ has bounded approximate units we get:

(4.24) $K(S(H_k)_\chi\cdot L^1(\mathscr{g}_0)_\chi)$ is dense in $K(L^1(\mathscr{g}_0)_\chi)$.

On the other hand, if $\alpha\in S(H_k)_\chi$, it is clear from (4.21) (4.22) that $K(\alpha\cdot f)\subset C_\infty$ for every $f\in L^1(\mathscr{g}_0)_\chi$. Thus (4.23) implies that $K(L^1(\mathscr{g}_0)_\chi)\subset C_\infty$.

We show now that $K$ is injective.

If $K(f)=0$ for some $f$ in $L^1(\mathscr{g}_0)_\chi$ then for almost all $x$ in $\mathscr{g}_0$, for all $D$ in $\overline{\mathbf{D}}$:

$$0 = (Kf(D))(D^{-1}(x)) = \int_Y f(x+D)(y)\,dy = \int_Y e^{i\langle D,y\rangle}f(x+y)\,dy.$$

But then $f(x)\equiv 0$ for almost all $x$ in $\mathscr{g}$.

Thus $K$ is injective q.e.d.

**(4.25) Proposition:** *There exists a subalgebra $\mathscr{A}(T)$ in $L^1(\mathscr{g}_0)_{\bar\chi}$, such that for every $\alpha=\alpha_\xi\in\mathscr{P}$:*

$$K(\alpha\circ L^1(\mathscr{g}_0)_\chi) = \xi\otimes\mathscr{A}(T).$$

*$\mathscr{A}(T)$ is a Banach algebra under the equivalent norms $|\ |_\alpha$:*

$$|h|_\alpha = |f|_1 \quad \text{if} \quad K(f) = \xi \otimes h \quad \text{and} \quad f \in \alpha \cdot (L^1(\mathscr{g}_0))_\chi \quad (\alpha = \alpha_\chi \in \mathscr{P}).$$

*Proof:* For $\alpha \in \mathscr{P}$, $I_\alpha = \alpha \cdot I$ is a closed subspace of $L^1(\mathscr{g}_0)_\chi$ for every twosided closed ideal in $L^1(\mathscr{g}_0)_\chi$ (as $\alpha * \alpha = \alpha$).

(4.26) Put $L^1_\alpha = (L^1(\mathscr{g}_0)_\chi)_\alpha$.

For $f \in L^1_\alpha$, $\alpha \cdot f = f$ and thus by (4.23)

$$K(f)(D) = \int K_\pi(\alpha)(D, D') K(f)(D')\, dD' = \xi(D) \cdot \int_{\mathbf{R}^k} \overline{\xi(D')} K(f)(D')\, dD', \text{ if } \alpha = \alpha_\xi.$$

Put $\mathscr{A}(T)_\alpha = \{h \in L^1(\mathscr{g}_0)_\chi |$ there exists $f$ in $L^1_\alpha$ with $h = \int_{\mathbf{R}^k} \overline{\xi(D')} Kf(D')\, dD'\}$.

Then $\xi \otimes \mathscr{A}(T)_\alpha \supset K(L^1_\alpha)$.

If on the other hand $h = \int_{\mathbf{R}^k} \overline{\xi(D')} Kf(D')\, dD' \in \mathscr{A}(T)_\alpha$, then for $f' = \alpha \cdot f \in L^1_\alpha$:

$$K(f') = \xi \otimes \int_{\mathbf{R}^k} \overline{\xi(D')} Kf(D')\, dD' = \xi \otimes h.$$

Thus $\xi \otimes \mathscr{A}(T)_\alpha = K(L^1_\alpha)$.

(4.26) If $\alpha'$ is another element of $\mathscr{P}$ and $\alpha' = \alpha'_{\xi'}$ ($|\xi'|_\alpha = 1$) then there exists $\beta \in S(H_k)_\chi$ such that

$$\pi(\beta)\xi = \xi'. \quad ([15]).$$

Let $h \in \mathscr{A}(T)_\alpha$. There exists $f \in L^1_\alpha$, such that $Kf = \xi \otimes h$.

Let $f' = \alpha' * \beta \circ f = \alpha' \cdot (\beta \circ f)$. Then:

(4.27) $$f' \in L^1_{\alpha'} \quad \text{and} \quad Kf'(D) = \int K_\pi(\alpha' * \beta)(D, D') Kf(D')\, dD'$$

$$= \left(\int K_\pi(\alpha' * \beta(D, D') \xi(D')\, dD'\right) \cdot h = (\pi(\alpha' * \beta) \xi(D) \cdot h = \xi'(D) \cdot h.$$

Thus $h \in \mathscr{A}(T)_{\alpha'}$.

We see that $\mathscr{A}(T)_\alpha$ is independent of $\alpha$ in $\mathscr{P}$; we write $\mathscr{A}(T)$ from now on. If $h, h'$ are in $\mathscr{A}(T)$ and $f, f'$ are in $L^1_\alpha$ with

$$K(f) = \xi \otimes h, \; K(f') = \xi \otimes h', \; (\alpha = \alpha_\xi),$$

then $K(f * f') = \xi^2 \otimes h * h' = \xi' \otimes |\xi^2|_2 h * h'$, if $\xi' = |\xi^2|_2^{-1} \cdot \xi^2$.

(4.28) As $\xi' \in S(\mathbf{R}^k)$, there exists $\alpha' \in \mathscr{P}$ with $\alpha' = \alpha'_{\xi'}$.

Thus $h * h' \in \mathscr{A}(T)_{\alpha'} = \mathscr{A}(T)$ and so $\mathscr{A}(T)$ is an *algebra.*

(4.29) The map $M_\alpha \colon \mathscr{A}(T) \to L^1_\alpha$; $(\alpha \in \mathscr{P} \cap S(H_k)_\chi)$

$$M_\alpha(h) = f, \quad \text{if} \quad f \in L^1_\alpha \quad \text{and} \quad K(f) = \xi \otimes h; \quad (\alpha_\xi = \alpha),$$

is well defined (as $K$ is injective).

As $L^1_\alpha$ is closed, if we provide $\mathscr{A}(T)$ with the norm $|\ |_\alpha$:

$$|h|_\alpha = |M_\alpha(h)|_1$$

$\mathscr{A}(T)$ becomes a *Banach space.*

Take another element $\alpha' = \alpha'_{\xi'}$ in $\mathscr{P}$ and let $\beta \in S(H_k)$ be such that:

$$\pi(\beta)\xi = \xi'.$$

Then for any $h \in \mathscr{A}(T)$:

$$M_{\alpha'}(h) = (\alpha' * \beta) \circ M_\alpha(h) \quad (4.27).$$

Thus $|h|_{\alpha'} \leqq |\alpha' * \beta|_1 \cdot |h|_\alpha$. This shows that the norms $|\ |_\alpha (\alpha \in \mathscr{P})$ are all *equivalent.*

If $\alpha'$ is as in (4.28) then for $h, h' \in \mathscr{A}(T)$:

$$|(h * h')|_\alpha \leqq C|h * h'|_\alpha = C|M_{\alpha'}(h * h')|_1$$

$$= C|M_\alpha(h) * M_\alpha(h')| \leqq C|M_\alpha(h)|_1 \cdot |M_\alpha(h')|_1 \leqq C|h|_\alpha \circ |h'|_\alpha$$

(for some $C > 0$, as $|\ |_\alpha$ is equivalent to $|\ |_{\alpha'}$).

Thus $\mathscr{A}(T)$ is a Banach algebra. q.e.d.

**(4.30) Proposition:** *There exists an inclusion preserving bijection between the set of the G-invariant closed ideals in $L^1(\mathscr{g}_0)_\chi$ and the set of the closed ideals in $\mathscr{A}(T)$.*

*Proof.* Let $\mathscr{I}^G$ denote the first set and $\mathscr{I}$ denote the second set. Define the map $b_\alpha \colon \mathscr{I}^G \to \mathscr{I}$ by

$$\xi \otimes b_\alpha(I) = K(I_\alpha)\,(\alpha = \alpha_\xi \in \mathscr{P}).$$

As $M_\alpha(b(I)) = I_\alpha$, $b_\alpha(I)$ is a closed subspace of $\mathscr{A}(T)$; If $\alpha' = \alpha'_{\xi'}$ is another element of $\mathscr{P}$ we have:

$$(\alpha' * \beta) \cdot (I_\alpha) \subset I \quad (\beta \text{ as in } 4.26) \text{ and so}$$

$$\alpha' \cdot (\beta \cdot I_\alpha) \subset I_{\alpha'}.$$

Thus

$$\xi' \otimes b_{\alpha'}(I) = K'(I_{\alpha'}) \supset K(\alpha' \circ (\beta \circ I_\alpha)) = \xi' \otimes b_\alpha(I).$$

(4.31) This shows that $b_\alpha(I)$ is in fact independent of $\alpha$. We write $b(I)$ from now on.

If $h \in \mathscr{A}(T)$ and $h' \in b(I)$, then for $\alpha, \alpha'$ as in (4.28) $L^1_{\alpha'} \supset M_{\alpha'}(h * h') = M_\alpha(h) * M_\alpha(h') \subset L^1(\mathscr{g}_0)_\chi * I \subset I$.

Thus $h * h' \in b_{\alpha'}(I) = b(I)$. This shows that $b(I)$ is an *ideal; b* is thus well *defined.* $b$ is injective: if $I$ and $I'$ are in $\mathscr{I}^G$ with $b(I) = b(I')$, then: for any $\alpha \in \mathscr{P}: \alpha * I = \alpha * I'$

thus

$$\alpha * (L^1(g_0)_\chi * I) = \alpha * (L^1(g_0) * I' \quad \text{and}$$

$$(L^1(g_0)_\chi * \alpha * L^1(g_0)_\chi) * I = (L^1(g_0)_\chi * \alpha * L^1(g_0)_\chi) * I'.$$

But $\overline{L^1(g_0)_\chi * \alpha * L^1(g_0)_\chi} = L^1(g_0)_\chi$ (4.15).

Thus $I=I'$ (as $L^1(g_0)_\chi$ has bounded approximate units). $b$ is surjective: Let $E$ be a closed ideal in $\mathscr{A}(T)$.

Let $I$ be the closure of the vectorspace generated by the spaces $M_\alpha(E)$; $(\alpha \in \mathscr{P})$.
As $K(L^1_\alpha * M_{\alpha'}(E)) = (\xi \otimes \mathscr{A}(T)) \cdot (\xi' \otimes E) = \xi \cdot \xi' \otimes A(T) * E \subset \xi \cdot \xi' \otimes E$

$$(\alpha = \alpha_\xi \quad \text{and} \quad \alpha' = \alpha'_{\xi'} \in \mathscr{P})$$

we see that $I$ is a (closed) ideal in $L^1(g_0)_\chi$.

(4.32) As $K(\alpha' \cdot M_\alpha(E)) = \langle \xi, \xi' \rangle_{L^2(\mathbf{R}^k)} \xi' \otimes E$

we see that $\alpha' \cdot I \subset I$ and so $I$ is also $G$-invariant. (4.23) too shows that $b(I)=E$.

Thus $b$ is surjective.

It is clear that $b$ is inclusion preserving. q.e.d.

**(4.33) Proposition:** *$S(g_0)_{\tilde{\chi}}$ is contained in $\mathscr{A}(T)$ and dense in $\mathscr{A}(T)$. Hence $\mathscr{A}(T)$ is dense in $L^1(g_0)_{\tilde{\chi}}$.*

*Proof.* From the equation:

$$(Kf)(D)(x) = \int_Y f(D(x+y))\,dy \quad \text{it is clear that:}$$

(4.34) $$K(S(g_0)_\chi) \subset S(\overline{\mathbf{D}}) \hat{\otimes} S(g_0)_\chi (\simeq S(\overline{\mathbf{D}} \times W);\ W \text{ as in } (4.17)).$$

Let now $F$ in $S(\overline{\mathbf{D}}) \hat{\otimes} S(g_0)_\chi$.

Define the function $M(F)$ on $g_0$ by:

(4.35) $$M(F)(x) = \int_{\overline{\mathbf{D}}} F(D, D^{-1}(x))\,dD.$$

Let $W$ be as in 4.17. $(g_0 \cong W \oplus Y \oplus \mathbf{R}z)$.

The formula:

(4.36) $$M(F)(w+y+rz) = \int_{\overline{\mathbf{D}}} F(D, D^{-1}(x)) e^{-i\langle D, y\rangle + ir}\,dD$$

proves that $M(F) \in S(g_0)_\chi \subset L^1(g_0)_\chi$.

Furthermore for $D\in\overline{\mathbf{D}}$, $x\in\mathscr{g}_0$:

$$(4.37)\qquad (K(M(F))(D))(x) = \int_Y MF(D(x+y))\,dy$$

$$= \int_Y M(F)(D^{-1}(x)+y)e^{i\langle D,y\rangle}\,dy = \int_Y\int_{\mathbf{D}} F(D', D'^{-1}(D(x)+y))\,dD e^{-i\langle D,y\rangle}\,dy$$

$$= \int_Y\Big(\int_{\mathbf{D}} F(D', D'^{-1}\cdot D(x))e^{-i\langle D',y\rangle}\,dD'\Big)e^{-i\langle D,y\rangle}\,dy$$

$$= F(D, x) \quad \text{(by the Fourier inversion formula)}$$

We see that $S(\overline{\mathbf{D}})\hat{\otimes} S(\mathscr{g}_0)_\chi \subset K(L^1(\mathscr{g}_0)_\chi)$.
From this it follows easily that $S(\mathscr{g}_0)_{\tilde{\chi}}$ is contained in $\mathscr{A}(T)$.
As $\alpha\circ S(\mathscr{g}_0)_\chi$ is dense in $L^1_\alpha$, $S(\mathscr{g}_0)_{\tilde{\chi}}$ is then dense in $\mathscr{A}(T)(\alpha\in\mathscr{P})$. q.e.d

## 5. The determination of $\mathscr{A}(T)$

We give now an explicit formula for the norm $|\ |_\alpha$ (4.25) for a special $\alpha$ in $\mathscr{P}$. For $h$ in $S(\mathscr{g}_0)_{\tilde{\chi}}\subset\mathscr{A}(T)$ (4.33), for $\alpha$ in $\mathscr{P}$, the norm $|h|_\alpha$ is given by the expression:

$$(5.1)\qquad |h|_\alpha = |M_\alpha(h)|_1 = \int_{W\times Y} |M_\alpha(h)(w+y)|\,dw\,dy \quad (W \text{ as in } 4.17)$$

$$= \int_{W\times Y} \Big|\int \xi(D)h(D^{-1}(w))e^{-i\langle D,y\rangle}dD\Big|\,dw\,dy.$$

Now $(\exp D)(w) = w+D(w)+\frac{1}{2}D^2(w)$; $w\in W$, $D\in\mathbf{D}$.
As $D(w)\in Y+\mathbf{R}z$, put $D(w)=\sum_{i=1}^k a_i(D,w)y_i+b(D,w)z$.

(5.2)

$$\text{Thus: } |h|_\alpha = \int_{W\times Y}\Big|\int_{\mathbf{D}} \xi(\exp D)h(w)e^{-i\langle D,y\rangle-ib(D,w)+\frac{i}{2}\langle 1, D^2(w)\rangle}\,dD\Big|\,dy\,dw$$

$$= \int_W |h(w)|\,|\beta(w,y)|\,dy\,dw$$

where $\beta(x,y)=\int_{\mathbf{D}}\xi(\exp D)h(w)e^{-i\langle D,y\rangle-ib(D,w)+\frac{i}{2}\langle 1,D^2(w)\rangle}\,dD.$

We choose the function $\xi(\exp D)=e^{-|D|^2}$ where $|D|^2=\sum_{i=1}^k d_i^2$, if $D=\sum_{i=1}^k d_iD_i$.

(5.3) For $w\in W$, let $A(w)$ be the $k\times k$ matrix $\{a_{ij}(w)\}_{i,j=1}^k$ where $a_{ij}(w)=\langle 1, D_iD_j(w)\rangle$.

As $D_iD_j=D_jD_i$ $1\leqq j, i\leqq k$, it follows that the matrix $A(w)$ is symmetric and can thus be diagonalized. Let $U(=U(w))$ be an orthogonal matrix, such that $U^{-1}AU=T=\{t_{ij}\}_{1\leqq i,j\leqq k}$ and $t_{ij}=\delta_{ij}c_j$.

Write $D=\sum_{i=1}^k d_iD_i$ and make the change of variables $D\to U(D)$ in $\beta(n,y)$. Then:

$$\beta(w,y) = \int_{\mathbf{D}} e^{-|D|^2}e^{-i\langle U(D),y\rangle-ib(U(D),w)+\frac{i}{2}\langle l,(U(D))^2(w)\rangle}\,dD.$$

But:

(5.4) $$\left\langle l, \tfrac{1}{2} U(D)^2(w)\right\rangle = \textstyle\sum_{j=1}^k d_j^2 c_j (if\, D = \sum_{j=1}^k d_j D_j).$$

Let us put:

(5.5) $$\langle D_j,\ U^*(y)\rangle + b(U(D_j, w)) = b_j.$$

Then:

$$\beta(w, y) = \Pi_{j=1}^k \beta_j(w, y) \quad \text{where}$$

$$\beta_j(w, y) = \int_{-\infty}^{\infty} e^{-d_j^2 + i(\frac{1}{2} c_j d_j^2 - b_j d_j)} d(dj). \text{ As:}$$

$$\beta_j(w, y) = \int_{-\infty}^{\infty} \exp\left\{\left(-1+\frac{1}{2} ic_j\right)\left(u - \frac{1}{2}\left(\frac{ib_j}{1-\frac{1}{2} ic_j}\right)\right)^2 + \frac{1}{4}\left(\frac{ib_j}{1-\frac{1}{2} ic_j}\right)^2\right\} d(dj)$$

$$= \left(1-\frac{1}{2} ic_j\right)^{-\frac{1}{2}} \exp\left\{\frac{1}{4}\left(\frac{ib_j}{1-\frac{1}{2} ic_j}\right)^2\right\},$$

$$|\beta_j(m, y)| = \exp\left\{-\frac{1}{4} b_j^2 \cdot \left(1+\frac{1}{4} c_j^2\right)^{-1}\right\}\left(1+\frac{1}{4} c_j^2\right)^{-\frac{1}{4}}.$$

Thus $$|h|_\alpha = \int_W |h(w)|\, \Pi_{j=1}^k \exp\left\{\left(-\frac{1}{4} b_j^2 \left(1+\frac{1}{4} c^2\right)^{-1}\right)\right\}\left(1+\frac{1}{4} c_j^2\right)^{-\frac{1}{4}} dy\, dw.$$

Make the changes of variables $y \to U(y)$ and $y_j \to y_j - b(U(D_y), w)$.
Then:

(5.6) $$|h|_\alpha = \int_W |h(w)|\, \Pi_{j=1}^k \int_{\mathbf{R}} \exp\left\{\left(-\frac{1}{4} y_j^2\right)\left(1+\frac{1}{4} c_y^2\right)\right\}\left(1+\frac{1}{4} c_j^2\right)^{-\frac{1}{4}} dy_j$$

$$= \int_W |h(w)|\, \Pi_{j=1}^k \left(1+\frac{1}{4} c_j^2\right)^{\frac{1}{4}} dw.$$

The numbers $\left(1+\frac{1}{4} c_j^2\right)$ are the eigenvalues of the matrix

$$1+\frac{1}{4} A^2(w)$$

Thus:

(5.7) $$|h|_\alpha = \int_W |h(w)| \left\{\det\left(1+\frac{1}{4} A(w)^2\right)\right\}^{\frac{1}{4}} dw$$

Let us write:

(5.8) $$\omega(w) = \det\left(1+\frac{1}{4} A(w)^2\right)^{\frac{1}{4}}$$

As $\mathscr{S}(g_0)_{\tilde{\chi}}$ is dense in $\mathscr{A}(T)$ we get:

(5.9) $$\mathscr{A}(T) = \left\{h \in L^1(g_0)_{\tilde{\chi}} \,|\, |h|_\alpha = \int_W |h(w)|\,\omega(w)\,dw < \infty\right\}$$
$$= \left\{h \in L^1(W) \,|\, |h|_\omega = \int_W |h(w)|\,\omega(w)\,dw < \infty\right\}$$

**(5.10) Theorem:** *Let $g$ be a nilpotent Lie group of step 3. Let $G = \exp g$ be simply connected. Let $T \in \hat{G}$ and let $0 = g^*$ be the G-orbit corresponding to T.*

*Let $g_0 = g(l) + [g, g]$ $(l \in O)$.*

*Let $d_1, \dots, d_k$ be a supplementary basis of $g$ to $g_0$.*

*For $w \in g_0$, define the $k \times k$ matrix $A(w)$ by*

$$A(w) = \{a_{ij}(w)\}_{ij} = \{\langle l, [d_i, [d_j, w]]\rangle\}_{i,j}.$$

*Let* $$\omega(w) = \left[\det\left(1 + \frac{1}{4} A(w)^2\right)\right]^{\frac{1}{4}}.$$

*Let $Q_\omega$ be the set of polynomials $q$ on $g_0$ such that $q \cdot \omega^{-1}$ is bounded on $g_0$.*

*There exists an inclusion reversing bijection between $\mathscr{I}\{T\}$ and the space $Q_\omega(\mathrm{inv})$ of the translation invariant subspaces of $Q_\omega$ different from* (0).

*Proof.* If $T([[G, G], G] = \mathrm{Id}_{\chi_\pi}$, $A(w)$ is the $O$-matrix and $\mathscr{I}_{\{T\}} = \{\ker \pi\}$.

The theorem is then obvious.

We may thus suppose that $T$ is not trivial on $[[G, G], G]$. By (3.12) $\mathscr{I}_{\{T\}}$ is isomorphic with $\mathscr{I}^G_{\tilde{O}_0}$.

Under the canonical isomorphism from $L^1(g_0)_{\tilde{\chi}} \to L^1(W)$ (4.17) the dual vectorspace of $L^1(g_0)_{\tilde{\chi}}$ is $L^\infty_\omega(W) = \{\varphi: W \to \mathbf{C} | \varphi$ measurable $\varphi \cdot \omega^{-1}$ bounded$\}$

Let $\tilde{l} \in W^*$ be the restriction of $l$ to $W$.

If $I \in \mathscr{I}^G_{\{\tilde{O}_0\}}$ then $b(I) \subset \mathscr{I}_{\{\tilde{l}\}}$: (see 4.31 for the definition of b) because for any $\alpha = \alpha_\xi \in \mathscr{P}$, $f \in I_\alpha$,

$$\widehat{K(f)(D)}(\tilde{l}) = \xi(D) \int \bar{\xi}(D) \widehat{P(D^{-1} \cdot f)}(\tilde{l})\, dD'$$
$$= \xi(D) \int \overline{\xi(D')} \widehat{D'^{-1} \cdot f}(l)\, dD' = \xi(D) \int \overline{\xi(D)'} \hat{f} \cdot (D' \cdot l)\, dD' = 0$$

From (4.36) we see also that $b^{-1}(\mathscr{I}_{\{\tilde{l}\}}) \subset \mathscr{I}^G_{\{\tilde{O}_0\}}$. Thus:

(5.11) $$b(\mathscr{I}^G_{\{\tilde{O}_0\}}) = \mathscr{I}_{\{\tilde{l}\}}$$

(5.12) The smallest ideal $j(\tilde{l})$ contained in $\mathscr{I}_{\{\tilde{l}\}}$ is the ideal generated by the elements $h$ in $\mathscr{S}(W)$ whose Fourier transforms $\hat{h}$ have compact support disjoint from the point $\{\tilde{l}\}$.

As $j(\tilde{l})$ is contained in every element of $\mathscr{I}_{\{\tilde{l}\}}$, by Hahn — Banach:

(5.13) there exists an inclusion reversing bijection between the set $\mathscr{I}_{\{l\}}$ and the space of the translation invariant weak $*$ closed subspaces of $L^\infty_\omega(W)$ contained in $\{j(\bar{l})\}^\perp$ different from (0).

Let us denote this space by $\mathscr{I}^\infty_{\{l\}}$.

If $\varphi \in I^\perp$ for some $I \in \mathscr{I}_{\{l\}}$, then $\varphi \in j(\bar{l})^\perp$ and the restriction $\varphi_r$ of $\varphi$ to $\mathscr{S}(W)$ is a temperate distribution. The Fourier transform $\hat{\varphi}_r$ of $\varphi_r$ is a temperate distribution of $\mathscr{S}(W^*)$ which annihilates every element $k$ of $\mathscr{D}(W^*)$ with $k\,((\bar{l}))=0$ (5.12). Thus

(5.14) $\tilde{\varphi}=\sum_j c_j \delta^{(j)}_{\{l\}}$, where the $c_j$'s are constants and $\delta^{(j)}_{\{\ \}}$ denotes the $j$-th derivative of the Dirac measure at the point $\{\bar{l}\}$ ([16]).

Thus:

(5.15) $\varphi(w)=e^{-i\langle \bar{l}, w\rangle}(p(w)$ where $p$ denotes a polynomial on $\mathscr{g}_0)$.

As $\varphi \in L^\infty_\omega(\mathscr{g}_0)$, $p$ must be an element of $Q_\omega$.

On the other hand, every $p'$ in $Q_\omega$ defines an element $\varphi$ of $j(\bar{l})^\perp$ by (5.15). Thus there exists a bijection between $j(\bar{l})^\perp$ and $Q_\omega$ and the theorem follows from this. q.e.d

(5.16) *Examples:* Let $\mathscr{g}_{r,k}$ be the Lie algebra with the basis elements

$$d_1, \ldots, d_k,\ w_1, \ldots, w_r,\ y_1, \ldots, y_k,\ z. \quad (r \leqq k)$$

Let $\xi_{r+1}, \ldots, \xi_k$ be elements of $W^*$ different from 0.

Let $\xi_j (1 \leqq j \leqq r)$ be defined by $\xi_j(w_s)=\delta_{j,s}$, $s=1, \ldots, r$.

The Lie multiplication of $\mathscr{g}_{r,k}$ is given by:

$$[d_i, w_p] = \xi_i(w_p)y_i;\ 1 \leqq i \leqq k,\ 1 \leqq s \leqq r;$$

$$[d_i, y_j] = \delta_{ij} z \quad 1 \leqq i, j \leqq k.$$

$\mathscr{g}$ is a step 3 nilpotent Lie algebra.

Let $l \in \mathscr{g}^*$, such that $l(z)=1$. Then:

$$\mathscr{g}_0 = \mathscr{g}(l)+[\mathscr{g}, \mathscr{g}] = W+Y+\mathbf{R}z \quad \left(Y = \textstyle\sum_{i=1}^k \mathbf{R}y_i\right)$$

For $w \in W=\sum_{i=1}^r \mathbf{R}w_i$

$$a_{ij}(w) = \langle l, [d_i[d_j, w]]\rangle = \delta_{ij}\xi_j(w).$$

Thus $\omega(w)=\det\left(1+\frac{1}{2}A(w)^2\right)=\prod_{y=1}^k \left(1+\frac{1}{2}\xi_j^2(w)^{1/4}\right)$.

If $r = k$, $w = \sum_{i=1}^k t_i w_i$

$$\omega(w) = \prod_{j=1}^k \left(1+\frac{t_j^2}{2}\right)^{\frac{1}{4}}.$$

Then $Q_\omega=\mathbf{R}1$ and then $T$ corresponding to 1 is a point of synthesis in $\hat{G}_{r,k}$.
If $r<k$, $\xi_{r+1}=\sum_{j=1}^r a_j\xi_j$ and not all the $a_j$'s are zero.

So

$$|\xi_{r+1}(w)| \leqq \sum_{j=1}^{r} |a_y||\xi_j(w)| \leqq C(\sum_{j=1}^{r} |\xi_j(w)|^2)^{\frac{1}{2}} \leqq C''\left(\Pi_{j=1}^{r}\left(1+\frac{1}{4}|\xi_j(w)|^2\right)^{\frac{1}{2}}\right)$$

for some constants $C, C' > 0$.

And

$$|\xi_{r+1}(w)| = |\xi_{r+1}(w)|^{\frac{1}{2}}|\xi_{r+1}(w)|^{\frac{1}{2}} \leqq C''\left(\Pi_{j=1}^{r}\left(1+\frac{1}{4}\xi_j(w)|^2\right)^{\frac{1}{2}}\right)\left(1+\frac{1}{4}|\xi_{r+1}(w)|\right)^{\frac{1}{2}}$$

$$\leqq C'' \Pi_{j=1}^{k'}\left(1+\frac{1}{4}|\xi_j(w)|^2\right)^{\frac{1}{2}} = C''\omega(w) \quad \text{for some constant } C'' > 0.$$

Thus $Q_\omega$ contains an element, namely $\xi_{r+1}$, which is not a constant thus $T\in\hat{G}_{r,k}$ corresponding to 1 is not a set of synthesis.

If $r=1$

$$\omega(tw_1) = \Pi_{j=1}^{k}(1+C_k t^2)^{\frac{1}{2}} \quad \text{for some } C_1, \ldots, C_k > 0.$$

Thus $w(t)=0\left(t^{\frac{k}{2}}\right)$ and thus $\dim Q_\omega = \left[\frac{k}{2}\right]+1$.

Furthermore $\ker T \supsetneqq (\ker T)^2 \supseteqq \ldots \supseteqq (\ker T)^{\left[\frac{k}{2}\right]+1}$ are the only elements of $\mathscr{I}_{\{T\}}$.

If $r=2$, $k=4$ and $\xi_3=\xi_1$, $\xi_4=\xi_2$ then:

$$\omega(t_1 w_1 + t_2 w_2) = \left(1+\frac{1}{2}t_1^2\right)^{\frac{1}{2}}\left(1+\frac{1}{2}t_2^2\right)^{\frac{1}{2}} = \left(1+\frac{1}{2}(t_1^2+t_2^2)+\frac{1}{4}t_1^2 t_2^2\right)^{\frac{1}{2}}$$

$Q_\omega$ has the following basis: $\{1, t_1, t_2, t_1 t_2\}$ and the elements of $Q_\omega(\text{inv})$ are: $\{\mathbf{R}_1, \mathbf{R}(t_1+ct_2)+\mathbf{R}_1, \mathbf{R}t_2, Q_w | c\neq 0\}$.

Thus $Q_w(\text{inv})$ has an infinity of elements.

## 6. Final remarks

(6.1) The computations become much more difficult if $G$ is no longer of step 3. No general results are known.

(6.2) In [12], it has been shown that for any point $T$ in the dual of nilpotent connected Lie group, the algebra $\ker (T)/_{j(T)}$ is always nilpotent. The exact degree of nilpotency of this algebra is unknown (in general). It can be estimated by the degree of growth of $G$ is $T$ is in general position. (see [12]). Suppose now that there exists an ideal $\mathscr{h}$ in $\mathscr{g}$, such that $\langle l, [\mathscr{h}, \mathscr{h}]\rangle = 0$ ($l$ in the orbit $O$ of $T$) and such that $l+\mathscr{h}^\perp \subset O$.

Let $l_0 = l_{/\mathscr{h}}$ and $O_0 = G\cdot l_0 \subset \mathscr{h}^*$.

Let $H=\exp h$. Using theorem 2.4 of [5], it can be shown that the degrees of nilpotency of $\ker T/_{j(T)}$ and $\ker O_0/_{j(O_0)}$ coincide.

As $[\mathscr{h}, \mathscr{h}]$ is an ideal in $\mathscr{g}$ on which $l$ disappears, we may as well suppose that $[\mathscr{h}, \mathscr{h}]=0$, that means that $\mathscr{h}$ is abelian.

The determination of the degree of nilpotency is thus reduced to the study of the $G$-orbit $O_0$ of the element $l_0$ in the dual of the abelian group $\mathscr{h}$. It follows from [8] that the degree of nilpotency of $\ker O_0/_{j(O_0)}$ is less than $\dim\left[\frac{O_0}{2}\right]+1$.

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