K-functionals and moduli of continuity in weighted polynomial approximation

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1. Introduction

The concept of K-functional was introduced and studied by Peetre ([10], [11]). If A_0 and A_1 are normed linear spaces, both contained in a topological vector space A, then the K-functional is defined by

$$K(A_0, A_1, f, t) = \inf\{\|f_0\|_{A_0} + t\|f_1\|_{A_1} \colon f = f_0 + f_1, f_0 \in A_0, f_1 \in A_1\} \tag{1}$$

Let $A_0 = C_{2\pi}$ = space of all 2π-periodic continuous functions with $\|f\|_C = \max_{x \in [-\pi, \pi]} |f(x)|$ and $A_1 = C'_{2\pi}$ = space of all 2π-periodic functions vanishing at 0 and with derivatives in $C_{2\pi}$ with $\|f\|_{C'} = \max_{x \in [-\pi, \pi]} |f'(x)|$. Peetre obtained ([12]) an explicit expression for the K-functional in this case as follows.

$$K(C_{2\pi}, C'_{2\pi}, f, t) = \tfrac{1}{2}\, \omega^*(f, 2t) \tag{2}$$

where ω^* is the least concave majorant of the modulus of continuity of f. It is well-known that this majorant is equivalent to (of the same order of magnitude as) the modulus of continuity of the function. (See, for example, [8]). Such an equivalence can also be obtained between the modulus of continuity of r^{th} order and the K-functional between $C_{2\pi}$ and the space of all 2π-periodic r-times differentiable functions vanishing at 0 along with the first $(r-1)$ derivatives. ([13], [2]). The relation between the K-functionals and the trigonometric approximation is now evident.

For weighted approximation on the whole real line by polynomials, we have obtained in [7], the direct and converse theorems entirely in terms of the K-functionals. Earlier, Freud had introduced a first order modulus of continuity in $L^p(\mathbf{R})$ and proved that this is equivalent to a suitable K-functional ([5]). He considers weights of the form $w_Q(x) = \exp(-Q(x))$ where $Q(x)$ is an even, convex, $C^2(0, \infty)$ function with

$Q'(x) \to \infty$ as $x \to \infty$. Define, for $w_Q f \in L^p(\mathbf{R})$,

$$\omega_1(L^p, w_Q, f, \delta) = \sup_{|h| \leqq \delta} \|w_Q(x+h)f(x+h) - w_Q(x)f(x)\|_p + \delta \left\|\min(\delta^{-1}, |Q'(x)|) w_Q(x) f(x)\right\|_p \tag{3}$$

$$\Omega_1(L^p, w_Q, f, \delta) = \inf_{A \in \mathbf{R}} \omega_1(L^p, w_Q, f-A, \delta) \tag{4}$$

$$K_1(L^p, w_Q, f, \delta) = \inf\{\|w_Q f_1\|_p + \delta \|w_Q f_2'\|_p\} \tag{5}$$

where inf is over all f_1 and f_2 with $f = f_1 + f_2$, $w_Q f_1 \in L^p(\mathbf{R})$ f_2 is an integral of a locally integrable functon f_2' such that $w_Q f_2' \in L^p(\mathbf{R})$. (We say that f_2 is differentiable). Freud's theorem then states that under the condition

$$\limsup_{x \to \infty} \frac{Q''(x)}{Q'(x)^2} < 1 \tag{6}$$

there exist positive constants C_1 and C_2 depending on Q and p only such that

$$C_1 \Omega_1(L^p, w_Q, f, \delta) \leqq K_1(L^p, w_Q, f, \delta) \leqq C_2 \Omega_1(L^p, w_Q, f, \delta) \tag{7}$$

In the following paper, we reverse this order of thoughts. We shall evaluate the order of magnitude of the second order K-functional which plays the role of Zygmund modulus of smoothness in our paper ([7]). It is then natural to call the resulting expression as the second order modulus of continuity in weighted approximation. During the proof, we shall also show that with a slight modification in the definitions (3) and (4), the same method also gives the result (7) of Freud. All our results are valid for arbitrary rearrangement invariant Banach function spaces on $\mathbf{R}$; thus giving an extension of (7) even for the first order modulus of continuity. A discussion of these spaces as well as the version of Calderón's interpolation theorem which we shall be using is given in ([1]).

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2. Main results

Let $\mathfrak{X}$ be a rearrangement invariant Banach function space on $\mathbf{R}$. (an r.i. space). We denote $\| \ \|_{\mathfrak{X}}$ by $\| \ \|$. Let w be a weight function and $wf \in \mathfrak{X}$. Define, for $r \geqq 1$, (r integer)

$$K_r(\mathfrak{X}, w, f, \delta) = \inf\{\|wf_1\| + \delta \|wf_2^{(r)}\|\} \tag{8}$$

where inf is over all f_1 and f_2 such that $f=f_1+f_2$, $wf_1\in\mathfrak{X}$, f_2 is r-times differentiable i.e. f_2 is an r-times iterated integral of a locally integrable function $f_2^{(r)}$ such that $wf_2^{(r)}\in\mathfrak{X}$. K_r is the r^{th} order K-functional.

We consider weight functions of the form $w_Q(x)=\exp\left(-Q(x)\right)$ where Q satisfies:

(*) Q is even, convex, $C^2(0,\infty)$ function with $Q'(x)\to\infty$ as $x\to\infty$. Let

$$Q'_\delta=\min\left\{\delta^{-1},(1+Q'^2)^{\frac{1}{2}}\right\} \tag{9}$$

Define, for $w_Q f\in\mathfrak{X}$

$$\omega_1(\mathfrak{X},w_Q,f,\delta)=\sup_{|h|\leqq\delta}\|w_Q(x+h)f(x+h)-w_Q(x)f(x)\|+\delta\|Q'_\delta w_Q f\| \tag{10}$$

$$\Omega_1(\mathfrak{X},w_Q,f,\delta)=\inf_{a\in\mathbf{R}}\omega_1(\mathfrak{X},w_Q,f-a,\delta) \tag{11}$$

$$T_hf(x)=f(x+h),\quad \Delta_hf(x)=f(x+h)-f(x),\quad \Delta_h^r=\Delta_h^{r-1}\Delta_h \tag{12}$$

$$\omega_2(\mathfrak{X},w_Q,f,\delta)=\sup_{|h|\leqq\delta}\|\Delta_h^2(w_Qf)\|+\delta\sup_{|h|\leqq\delta}\|Q'_\delta\Delta_h(w_Qf)\|+\delta^2\|Q'^2_\delta w_Qf\| \tag{13}$$

$$\Omega_2(\mathfrak{X},w_Q,f,\delta)=\inf_{a.b\in\mathbf{R}}\omega_2(\mathfrak{X},w_Q,f-a-bx,\delta) \tag{14}$$

We call Ω_1 and Ω_2 the first and second order modulus of continuity respectively.

Theorem 1: *Let Q satisfy (*). Suppose any one of the following conditions holds:*

$$\limsup_{x\to\infty}\frac{Q''(x)}{Q'(x)^2}<1 \tag{6bis}$$

$$\limsup_{x\to\infty}\frac{Q'(2x)}{Q'(x)}<\infty \tag{15}$$

Then there exist positive constants C_3 and C_4 depending only on $\mathfrak{X}$ and Q such that for every f with $w_Qf\in\mathfrak{X}$,

$$C_3\Omega_1(\mathfrak{X},w_Q,f,\delta)\leqq K_1(\mathfrak{X},w_Q,f,\delta)\leqq C_4\Omega_1(\mathfrak{X},w_Q,f,\delta),\quad 0\leqq\delta\leqq1 \tag{16}$$

Theorem 2(a): *Suppose Q satisfies (*). In addition, let*

i) *Q'' be continuous at* 0

ii) $$\limsup_{x\to\infty}\sup_{|u|\leqq1}\left|\frac{Q'(x+u)}{Q'(x)}\right|<\infty \tag{17}$$

iii) $$\limsup_{x\to\infty}\frac{Q''(x)}{Q'(x)^2}<\frac{1}{2} \tag{18}$$

Then there exist positive constants C_5 *and* C_6 *depending only on* $\mathfrak{X}$ *and* Q *such that for each* f *with* $w_Q f \in \mathfrak{X}$,

$$C_5 \Omega_2(\mathfrak{X}, w_Q, f, \delta) \leqq K_2(\mathfrak{X}, w_Q, f, \delta^2) \leqq C_6 \Omega_2(\mathfrak{X}, w_Q, f, \delta), \quad 0 < \delta \leqq 1 \tag{19}$$

(b) Suppose Q *satisfies* (*), *(ii) and (iii) above. Then there exists a function* $\bar{Q}$ *satisfying* (*), *(i), (ii), (iii) above such that*

$$C_7 \exp(-\bar{Q}(x)) \leqq \exp(-Q(x)) \leqq C_8 \exp(-\bar{Q}(x)) \tag{20}$$

for some positive constants C_7 *and* C_8 *and for all* x. *We can choose* $\bar{Q}(x) = Q(x)$ *if* $|x| \geqq a$, *for some* $a > 0$ *depending upon* Q.

Remarks: (1) The operator T_h defined in (12) is an isometry on $L^1(\mathbf{R})$ and on $L^\infty(\mathbf{R})$. Thus by the version of Calderón's theorem given in [1], it is also an isometry on $\mathfrak{X}$; i.e. every *r.i.* space is also translation invariant. So, formulae (10) and (13) are meaningful. It can be shown that under the condition (*), $x^n w_Q(x) \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R}) \subset \mathfrak{X}$. (See [4] for the first relation and [1] for the second.) Thus, formulae (11) and (14) are meaningful.

(2) It is easy to construct examples where $w_Q f \in \mathfrak{X}$ but $w_Q T_h f \notin \mathfrak{X}$. Thus, we have to consider $\Delta_h(w_Q f)$ and $\Delta_h^2(w_Q f)$ in (10) and (13) instead of $w_Q \Delta_h f$ and $w_Q \Delta_h^2 f$, which perhaps, would have been more natural.

(3) It is clear that the order of magnitude of the K-functionals is unaltered if we replace w by an equivalent weight function. Hence, in view of Theorem 2(b), we can evaluate the order of magnitude of $K_2(\mathfrak{X}, w_Q, f, \delta)$ even if Q'' is not continuous at 0; simply by considering $\Omega_2(\mathfrak{X}, w_{\bar{Q}}, f, \delta)$ in such cases.

(4) All conditions on Q are satisfied if $Q(x) = |x|^\alpha$, $\alpha \geqq 2$. If $1 < \alpha < 2$, then Q'' is not continuous at 0, but all other conditions are satisfied. The K-functional is then evaluated as we remarked above.

3. Preliminary lemmas

In what follows, we assume that Q is even, convex, $C^2(0, \infty)$ and $Q'(x) \to \infty$ as $x \to \infty$. By $A \ll B$ we mean that $A \leqq cB$ for some constant $c > 0$ depending only on $\mathfrak{X}$ and Q.

Lemma 1: *(a) Suppose for some* $r \geqq 1$

$$\limsup_{x \to \infty} \frac{Q''(x)}{Q'(x)^2} = \theta < \theta_1 < \frac{1}{r} \tag{21}$$

Then $|Q'(x)|^r e^{-Q(x)} \ll 1$.

(b) If (15) holds then

$$|Q'(x)|^r e^{-Q(x)} \ll e^{-Q\left(\frac{x}{2}\right)} \ll 1 \quad \text{for all} \quad r \tag{22}$$

Proof: (a)

$$\frac{d}{dx}\left(Q'(x)^r e^{-Q(x)}\right) = rQ'(x)^{r+1}e^{-Q(x)}\left[\frac{Q''(x)}{Q'(x)^2}-\frac{1}{r}\right].$$

Hence $Q'(x)^r e^{-Q(x)}$ is eventually decreasing and then the claim follows for $x \geqq 0$ by boundedness of Q' near zero and then for all $x \in \mathbf{R}$ by evenness of Q.

(b) Let $\limsup_{x\to\infty} \frac{Q'(2x)}{Q'(x)} < K$. Choose $M \geqq 1$ such that $x \geqq M$ implies $Q'(2x) < KQ'(x)$. We have, for $x \geqq 2M$ and an integer r,

$$Q'(x)^r e^{-Q(x)+Q\left(\frac{x}{2}\right)} \leqq r! e^{Q'(x)} e^{-\int_{x/2}^{x} Q'(t)\,dt}$$

$$\leqq r! e^{Q'(x)-\frac{x}{2}Q'\left(\frac{x}{2}\right)} \leqq r! \exp\left\{\left(1-\frac{x}{2K}\right)Q'(x)\right\} \leqq r! \quad \text{if} \quad x \geqq \max(2M, 2K).$$

If $0 \leqq x \leqq \max(2M, 2K)$, the claim is clear by boundedness of Q' and and continuity of Q. The result is now proved since Q is even. ∎

Remark: In view of the fact that $Q'(x) \to \infty$ as $x \to \infty$, an application of Lemma 1 to a number s slightly larger than r yields that

$$|Q'(x)|^r e^{-Q(x)} \to 0 \quad \text{as} \quad |x| \to \infty.$$

Lemma 2: *Suppose (21) holds, and Q'' is bounded on compact sets in* $[0, \infty)$. *Then, we have*

(a) $Q'^2 \pm Q'' \ll 1 + Q'^2$ (23)

(b) There exists K such that $x \geqq K$ implies

$$1+Q'^2(x) \ll Q'^2(x) - Q''(x) \tag{24}$$

Proof: (a)

$$Q'^2(x)+Q''(x) = Q'(x)^2\left[1+\frac{Q''(x)}{Q'(x)^2}\right] \leqq (1+\theta_1)Q'(x)^2 \leqq (1+\theta_1)(1+Q'(x)^2)$$

if $x \geqq K$ where K is so chosen that $x \geqq K \Rightarrow \frac{Q''(x)}{Q'(x)^2} < \theta_1$.

For $x \geqq K$, the claim follows by the boundedness of Q'' on compact sets.

(b) Note that $Q'(x) \to \infty$ as $x \to \infty$. Let $x \geqq K$ imply $Q'(x) > 1$ and $\frac{Q''(x)}{Q'(x)^2} < \theta_1$. For $x \geqq K$, we have

$$1+Q'(x)^2 \leqq 2Q'(x)^2 \leqq \frac{2}{1-\theta_1}(Q'(x)^2 - Q''(x)). \quad ∎$$

Corollary 3: *If (21) holds, then*

$$\frac{e^{Q(t)}}{(1+Q'(t)^2)^{1/2}}\int_{\max(t,K)}^{\infty}[1+Q'^2(x)]e^{-Q(x)}\,dx \quad \textit{is bounded}$$

where K is prescribed in Lemma 2(b).

Proof. Let $\max(t,K)=\bar{t}$. By Lemma 2(b),

$$\int_{\bar{t}}^{\infty}(1+Q'(x)^2)e^{-Q(x)}\,dx \ll \int_{\bar{t}}^{\infty}[Q'(x)-Q''(x)]e^{-Q(x)}\,dx = Q'(\bar{t})e^{-Q(\bar{t})}$$

(By Lemma 1(a))

$$\leqq \begin{cases}(1+Q'(K)^2)^{1/2}e^{-Q(K)} & \text{if} \quad t\leqq K\\ (1+Q'(t)^2)^{1/2}e^{-Q(t)} & \text{if} \quad t\geqq K.\end{cases}$$

This completes the proof. ∎

Lemma 4: *(a) Let f be a differentiable function (i.e. let f be the indefinite integral of a locally integrable function),* $f(0)=0$ *and* $w_Q f'\in L^1$. *Then*

$$\|(1+Q'^2)^{1/2}w_Q f\|_1 \ll \|w_Q f'\|_1 \tag{25}$$

(b) Let Q satisfy (21). Let f be a differentiable function (in the above sense), $f(0)=0$ *and* $(1+Q'^2)^{1/2}w_Q f'\in L^1$.

$$\|(1+Q'^2)w_Q f\|_1 \ll \|(1+Q'^2)^{1/2}w_Q f'\|_1 \tag{26}$$

Thus, if Q satisfies (21) and f is twice differentiable (i.e. f is a twice iterated integral of a locally integrable function), $f(0)=f'(0)=0$; *and* $w_Q f''\in L^1$, *then*

$$\|(1+Q'^2)w_Q f\|_1 \ll \|w_Q f''\|_1 \tag{27}$$

Proof. (a) Let $\psi=w_Q f'$. We have:

$$\|w_Q f'\|_1 = \int_0^{\infty}\{|\psi(t)|+|\psi(-t)|\}\,dt \tag{28}$$

Further, since $f(0)=0$,

$$f(x)=\begin{cases}\int_0^{x} e^{Q(t)}\psi(t)\,dt & \text{if} \quad x\geqq 0\\ \int_0^{-x} e^{Q(t)}\psi(-t)\,dt & \text{if} \quad x\leqq 0.\end{cases}$$

Let $|\psi(t)|+|\psi(-t)|=g(t)$. We have,

$$\|(1+Q'^2)^{1/2}w_Q f\|_1 = \int_0^{\infty}[1+Q'^2(x)]^{1/2}w_Q(x)\{|f(x)|+|f(-x)|\}\,dx$$

$$\leqq \int_0^{\infty}(1+Q'^2(x))^{1/2}w_Q(x)\int_0^{x} e^{Q(x)}g(t)\,dt\,dx. \tag{29}$$

Now, clearly, since Q' is bounded near 0,

$$\int_0^K (1+Q'(x)^2)^{1/2} w_Q(x) \int_0^x e^{Q(t)} g(t)\, dt\, dx \ll \int_0^K g(t)\, dt \tag{30}$$

where K is so large that $Q'(x) > l$ if $x \geqq K$

$$\int_K^\infty (1+Q'(x)^2)^{1/2} w_Q(x) \int_0^x e^{Q(t)} g(t)\, dt\, dx$$

$$= \int_0^\infty e^{Q(t)} g(t) \int_{\max(t,K)}^\infty (1+Q'(x)^2)^{1/2} w_Q(x)\, dx\, dt$$

$$\ll \int_0^\infty e^{Q(t)} e^{-Q(\bar{t})} g(t)\, dt \quad \text{where} \quad \bar{t} = \max(t, K) \ll \int_0^\infty g(t)\, dt. \tag{31}$$

The result follows from (28), (29), (30), (31).

(b) Let
$$\psi(t) = (1+Q'^2(t))^{1/2} w_Q(t) f'(t).$$
Then
$$\|(1+Q'^2)^{1/2} w_Q f'\|_1 = \int_0^\infty \{|\psi(t)| + |\psi(-t)|\}\, dt \tag{32}$$

$$\|(1+Q'^2) w_Q f\|_1 = \int_0^\infty [1+Q'^2(x)] w_Q(x) \{|f(x)| + |f(-x)|\}\, dx$$

$$\leqq \int_0^\infty (1+Q'(x)^2) w_Q(x) \int_0^x \frac{e^{Q(t)}}{(1+Q'(t)^2)^{1/2}} g(t)\, dt \tag{33}$$

where
$$g(t) = |\psi(t)| + |\psi(-t)|.$$
Now, as before,

$$\int_0^K (1+Q'(x)^2 w_Q(x) \int_0^x \frac{e^{Q(t)}}{(1+Q'(t)^2)^{1/2}} g(t)\, dt\, dx \ll \int_0^K g(t)\, dt \tag{34}$$

where we choose K so large that $Q'(x) > 1$ and (24) holds for $x \geqq K$.

$$\int_K^\infty (1+Q'(x)^2) w_Q(x) \int_0^x \frac{e^{Q(t)}}{(1+Q'(t)^2)^{1/2}} g(t)\, dt\, dx$$

$$= \int_0^\infty \frac{e^{Q(t)} g(t)}{(1+Q'(t)^2)^{1/2}} \int_{\max(t,K)} (1+Q'(x)^2) w_Q(x)\, dx\, dt \ll \int_0^\infty g(t)\, dt \tag{35}$$

by Corollary 3.

The proof is now complete in view (32), (33), (34), (35). ∎

Out next task is to obtain the analogue of the above lemma for L^∞.

The following lemma will play a role similar to that played by Corollary 3 in the proof of Lemma 4.

Lemma 5: *(a) Let (21) hold. Then*

$$[1+Q'(x)^2]^{r/2}w_Q(x)\int_0^x \frac{e^{Q(t)}}{(1+Q'(t)^2)^{\frac{r-1}{2}}}\,dt$$

is bounded.

(b) Let (15) hold. Then the conclusion above is valid for all r (r integer, ≧1).

Proof. (a) Clearly, it suffices to show the boundedness if $x \geqq K$ for a suitably chosen large K. We choose K so that $x \geqq K \Rightarrow \frac{Q''(x)}{Q'(x)^2} < \theta_1 < \frac{1}{r}$ and $Q'(x) > 1$. Now, by Lemma 1(*a*), it suffices to show that

$$(1+Q'(x)^2)^{r/2}w_Q(x)\int_K^x \frac{e^{Q(t)}}{(1+Q'(t)^2)^{\frac{r-1}{2}}}\,dt$$

is bounded for $x \geqq K$;

hence to show that $Q'(x)^r w_Q(x)\int_K^x \frac{e^{Q(t)}}{Q'(t)^{r-1}}\,dt$ is bounded. But

$$I = \int_K^x \frac{e^{Q(t)}}{Q'(t)^{r-1}}\,dt = \int_K^x \frac{Q'(t)\,e^{Q(t)}}{Q'(t)^r}\,dt = \frac{e^{Q(x)}}{Q'(x)^r} - \frac{e^{Q(K)}}{Q'(K)^2} + \int_K^x \frac{re^{Q(t)}Q''(t)}{Q'(t)^{r+1}}\,dt$$

$$\leqq \frac{e^{Q(x)}}{Q'(x)^r} + \int_K^x re^{Q(t)}\frac{Q''(t)}{Q'(t)^{r+1}}\,dt \leqq \frac{e^{Q(x)}}{Q'(x)^r} + r\theta_1 I.$$

Thus, the claim is proved since $r_1^\theta < 1$.

(b) Let $\limsup_{x\to\infty} \frac{Q'(2x)}{Q'(x)} < M$. Choose K so large that

$$x \geqq K \Rightarrow Q'(x) < Q'\left(\frac{x}{2}\right)M \quad \text{and} \quad Q'\left(\frac{x}{2}\right) > 1.$$

Again it suffices to show that

$$[Q'(x)]^r e^{-Q(x)}\int_0^x \frac{e^{Q(t)}}{(1+Q'(t)^2)^{\frac{r-1}{2}}}\,dt$$

is bounded for $x \geqq K$ (in view of Lemma 1(b)). We have for $x \geqq K$,

$$Q'(x)^r e^{-Q(x)}\int_{x/2}^x \frac{e^{Q(t)}}{[1+Q'(t)^2]^{\frac{r-1}{2}}}\,dt \leqq \frac{Q'(x)^r e^{-Q(x)}}{}\int_{x/2}^x e^{Q(t)}Q'(t)\,dt$$

$$= \frac{Q'(x)^r e^{-Q(x)}}{Q'\left(\frac{x}{2}\right)^r}\left[e^{Q(x)} - e^{Q\left(\frac{x}{2}\right)}\right] \leqq M^r\left(1 - e^{Q\left(\frac{x}{2}\right)-Q(x)}\right) \leqq M^r. \tag{36}$$

By Lemma 1(b),

$$Q'(x)^r e^{-Q(x)} \int_0^1 \frac{e^{Q(t)}}{[1+Q'(t)^2]^{\frac{r-1}{2}}} dt \tag{37}$$

is bounded.

Further

$$Q'(x)^r e^{-Q(x)} \int_1^{x/2} \frac{e^{Q(t)}}{[1+Q'(t)^2]^{\frac{r-1}{2}}} dt$$

$$\ll Q'(x)^r e^{-Q(x)} \int_1^{x/2} \frac{e^{Q(t)}}{Q'(t)^{r-1}} dt \leqq \frac{Q'(x)^r e^{-Q(x)}}{Q'(1)^r} \int_1^{x/2} Q'(t) e^{Q(t)} dt$$

$$\ll Q'(x)^r e^{-Q(x)} \left[e^{Q\left(\frac{x}{2}\right)} - e^{Q(1)}\right] \ll Q'(x)^r e^{-Q(x)+Q\left(\frac{x}{2}\right)} \ll 1 \tag{38}$$

by Lemma 1(b).

The claim is proved by (36), (37), (38), ▮

Lemma 6: *(a) Let either (6) or (15) (hypothesis of Theorem 1) hold. Let f be differentiable, $f(0)=0$ and $w_Q f' \in L^\infty(\mathbf{R})$. Then*

$$\|(1+Q'(x)^2)^{1/2} w_Q(x) f(x)\|_\infty \ll \|w_Q(x) f'(x)\|_\infty \tag{39}$$

(b) Let (18) in the hypothesis of Theorem 2 or (15) hold. Let f be differentiable, $f(0)=0$ and $w_Q f' \in L^\infty(\mathbf{R})$. Then

$$\|(1+Q'(x)^2) w_Q(x) f(x)\|_\infty \ll \|(1+Q'^2)^{1/2} w_Q f'\|_\infty \tag{40}$$

In particular, if f is twice differentiable, $f(0)=f'(0)=0$ and $w_Q f'' \in L^\infty(\mathbf{R})$ then

$$\|(1+Q'^2) w_Q f\|_\infty \ll \|w_Q f''\|_\infty \tag{41}$$

Proof. (a) Let $\psi(x)=w_Q(x)f'(x)$. If $x>0$

$$(1+Q'(x)^2)^{1/2} w_Q(x)|f(x)| \leqq (1+Q'(x)^2)^{1/2} w_Q(x) \int_0^x e^{Q(t)} |\psi(t)|\, dt$$

$$\leqq \|\psi\|_\infty (1+Q'(x)^2)^{1/2} w_Q(x) \int_0^x e^{Q(t)}\, dt \tag{42}$$

Clearly (42) also holds if $x \leqq 0$. Thus (39) follows from Lemma 5. (5a if (6) holds, 5b if (15) holds)

(b) Let $\psi(x) = ((1+Q'^2)^{1/2}) w_Q f'(x)$. As before,

$$(1+Q'(x)^2) w_Q(x)|f(x)| \leqq \|\psi\|_\infty (1+Q'(x)^2) w_Q(x) \int_0^x \frac{e^{Q(t)}}{(1+Q'(t))^{1/2}} dt.$$

The conclusion then follows from Lemma 5(a) if (18) holds and 5(b) if (15) holds. ▮

Applying Lemmas 4(a) and 6(a) to the operator

$$T\colon\ g \to (1+Q'(x)^2)^{1/2} w_Q(x) \int_0^x e^{Q(t)} g(t)\, dt$$

we see by the version of Calderón's theorem given in [1] that under the hypothesis of Lemma 6(a), (39) holds in an arbitrary r.i. space $\mathfrak{X}$ for differentiable functions f with $f(0)=0$ and $w_Q f' \in \mathfrak{X}$. Similarly, if (18) is satisfied, and f is a twice differentiable function with $f(0)=f'(0)=0$ and $w_Q f'' \in \mathfrak{X}$ we have inequality (41) even in the norm of $\mathfrak{X}$.

For small enough δ, we can solve the equation $Q'(x)=\delta^{-1}$. We call the greatest such solution x_δ.

Lemma 7: *(a) Let (21) hold. Then $\delta Q'(x_\delta+r\delta)$ and consequently*

$$\delta\left[1+(Q'(x_\delta+r\delta))^2\right]^{1/2}$$

is bounded as $\delta \to 0$.

(b) If (15) holds, then the above conclusion holds for all r.

Proof. (a)

$$\delta - \frac{1}{Q'(x_\delta+r\delta)} = \frac{1}{Q'(x_\delta)} - \frac{1}{Q'(x_\delta+r\delta)} = \int_{x_\delta}^{x_\delta+r\delta} \frac{Q''(t)}{Q'(t)^2}\, dt < r\theta_1\delta$$

if δ is small enough.

Hence $\delta Q'(x_\delta+r\delta) \leqq \dfrac{1}{1-r\theta_1} < +\infty$.

(b) If δ is so small that $x_\delta+r\delta \leqq 2x_\delta$ and $\dfrac{Q'(2x_\delta)}{Q'(x_\delta)} \leqq M$ (say), we have

$$\delta Q'(x_\delta+r\delta) = \frac{Q'(x_\delta+r\delta)}{Q'(x_\delta)} \leqq \frac{Q'(2x_\delta)}{Q'(x_\delta)} \leqq M. \quad \blacksquare$$

Let us now summarize the results obtained so far.

Proposition 8:

(a) Lemma 7.

(b) Lemma 2(a): Under the assumptions (i) and (iii) of Theorem 1,

$$Q''(x) \pm Q'^2(x) \ll 1+Q'^2(x), \quad x \in \mathbf{R}.$$

(c) Let the hypothesis of Theorem 1 hold. Let f be differentiable, $f(0)=0$ and $w_Q f' \in \mathfrak{X}$.

Then

$$\|(1+Q'^2)^{1/2} w_Q f\|_{\mathfrak{X}} \ll \|w_Q f'\|_{\mathfrak{X}} \tag{43}$$

(d) Let (18) (hypothesis (iii) of Theorem 2) hold. Let f be twice differentiable, $f(0)=f'(0)=0$ and $w_Q f''\in\mathfrak{X}$. Then

$$\|(Q''+Q'^2)w_Q f\|_{\mathfrak{X}} \ll \|(1+Q'^2)w_Q f\|_{\mathfrak{X}} \ll \|w_Q f''\|_{\mathfrak{X}} \tag{44}$$

Lemma 9: *Let for each $t\in[a,b]$, $g(\cdot,t)\in\mathfrak{X}$. Let $g(x,t)$ be jointly measurable. Then $\|g(\cdot,t)\|_{\mathfrak{X}}$ is measurable and*

$$\left\|\int_a^b g(x,t)\,dt\right\|_{\mathfrak{X}} \leqq \int_a^b \|g(\cdot,t)\|_{\mathfrak{X}}\,dt \tag{45}$$

Proof. The measurability assertion is found in [9]. Let $\mathfrak{X}'$ be the associate space of $\mathfrak{X}$. Since $\mathfrak{X}$ has the Fatou property,

$$\left\|\int_a^b g(x,t)\,dt\right\|_{\mathfrak{X}} = \sup_{\|h\|_{\mathfrak{X}'}=1}\int_{\mathbf{R}}|h(x)|\int_a^b|g(x,t)|\,dt\,dx \quad ([14])$$

$$= \sup_{\|h\|_{\mathfrak{X}'}=1}\int_a^b\int_{\mathbf{R}}|h(x)||g(x,t)|\,dx\,dt \leqq \sup_{\|h\|_{\mathfrak{X}'}=1}\int_a^b\|g(\cdot,t)\|_{\mathfrak{X}}\,dt = \int_a^b\|g(\cdot,t)\|_{\mathfrak{X}}. \quad \blacksquare$$

4. Lower estimates

Observe that, by triangle inequality,

$$\omega_r(\mathfrak{X}, w_Q, f_1+f_2, \delta) \leqq \omega_r(\mathfrak{X}, w_Q, f_1, \delta)+\omega_r(\mathfrak{X}, w_Q, f_2, \delta) \quad r = 1, 2. \tag{46}$$

Further, since $\delta Q'_\delta\leqq 1$, we have for $w_Q f\in\mathfrak{X}$,

$$\omega_r(\mathfrak{X}, w_Q, f, \delta) \ll \|w_Q f\|_{\mathfrak{X}}. \tag{47}$$

From here onwards, $\mathfrak{X}$ and Q are fixed. Their mention will be suppressed; for example, $\omega_r(\mathfrak{X}, w_Q, f, \delta)\equiv\omega_r(f,\delta)$, $\|\cdot\|_{\mathfrak{X}}\equiv\|\cdot\|$.

Lower estimate in Theorem 1

Let $f=f_1+f_2$ be an arbitrary decomposition with $w_Q f_1\in\mathfrak{X}$, f_2 differentiable, $w_Q f_2'\in\mathfrak{X}$. Let

$$f_2^*(x) = f_2(x)-f_2(0); \quad f^* = f_1+f_2^*.$$

Then

$$\Omega_1(f,\delta) \leqq \omega_1(f^*,\delta) \leqq \omega_1(f_1,\delta)+\omega_1(f_2^*,\delta) \ll \|w_Q f_1\|+\omega_1(f_2^*,\delta) \quad \text{(By (47))}.$$

We have

$$\delta\|Q'_\delta w_Q f_2^*\| \leqq \delta\|(1+Q'^2)^{1/2}w_Q f_2^*\| \ll \delta\|w_Q f_2^{*\prime}\| = \delta\|w_Q f_2'\| \tag{49}$$

by Proposition 8(c).

Let $|h|\leqq\delta$. Using Lemma 9, translation invariance of $\|\cdot\|$ and then Proposition 8(c), we have

$$\|\Delta_h(w_Q f_2^*)\| = \left\|\int_0^h (w_Q f_2^*)'(x+u)\,du\right\| = \left\|\int_0^h [-Q' w_Q f_2^* + w_Q f_2'](x+u)\,du\right\|$$
$$\leqq |h|\{\|Q' w_Q f_2^*\| + \|w_Q f_2'\|\} \ll \delta\|w_Q f_2'\| \tag{50}$$

The estimate

$$\Omega_1(f,\delta) \ll K_1(f,\delta) \tag{51}$$

follows from (48), (49), (50) if we observe that $f=f_1+f_2$ was an arbitrary decomposition.

Lower estimate in Theorem 2a

Let $f=f_1+f_2$, $w_Q f_1\in\mathfrak{X}$, f_2 be twice differentiable, $w_Q f_2''\in\mathfrak{X}$. Let $f_2^*(x)=f_2(x)-f_2(0)-f_2'(0)x$ and $f^*=f_1+f_2^*$. We have

$$\Omega_2(f,\delta) \leqq \omega_2(f^*,\delta) \leqq \omega_2(f_1,\delta)+\omega_2(f_2^*,\delta) \ll \|w_Q f_1\| + \omega_2(f_2^*,\delta). \tag{52}$$

Using Lemma 9, translation invariance of $\|\cdot\|$, convexity of Q and Proposition 8(c) and 8(d) we get, for $|h|\leqq\delta$,

$$\|\Delta_h^2(w_Q f_2^*)\| = \left\|\int_0^h\int_0^h (w_Q f_2^*)''(x+u_1+u_2)\,du_1\,du_2\right\|$$
$$= \left\|\int_0^h\int_0^h [(Q'^2-Q'')w_Q f_2^* - 2Q' w_Q f_2^{*\prime} + w_Q f_2^{*\prime\prime}](x+u_1+u_2)\,du_1\,du_2\right\|$$
$$\ll |h|^2\{\|(Q''+Q'^2)w_Q f_2^*\| + \|Q' w_Q f_2^{*\prime}\| + \|w_Q f_2''\|\} \ll \delta^2\|w_Q f_2''\|. \tag{53}$$

By Proposition 8(d)

$$\delta^2\|Q_\delta'^2 w_Q f_2^*\| \leqq \delta^2\|(1+Q'^2)w_Q f_2^*\| \ll \delta^2\|w_Q f_2''\|. \tag{54}$$

Further, using assumption (ii) (inequality (17)) translation invariance and Proposition 8(d), 8(c) we get for $|h|\leqq\delta\leqq 1$,

$$\delta\|Q_\delta'\Delta_h(w_Q f_2^*)\| \leqq \delta\left\|Q_\delta'(x)\int_0^h (w_Q f_2^*)'(x+u)\,du\right\|$$
$$= \delta\left\|Q_\delta'(x)\int_0^h (-Q' w_Q f_2^* + w_Q f_2^*)'(x+u)\,du\right\|$$
$$\leqq \delta\left\|Q_\delta'(x)\int_0^{|h|} (|Q' w_Q f_2^*| + |w_Q f_2^{*\prime}|)(x+u)\,du\right\|$$
$$\ll \delta\left\|\int_0^{|h|} [(1+Q'^2)|w_Q f_2^*| + (1+Q'^2)^{\frac{1}{2}}|w_Q f_2^{*\prime}|](x+u)\,du\right\|$$
$$\leqq \delta^2\{\|(1+Q'^2)w_Q f_2^*\| + \|(1+Q'^2)^{\frac{1}{2}} w_Q f_2^{*\prime}\|\} \ll \delta^2\|w_Q f_2''\|. \tag{55}$$

Observe that $f=f_1+f_2$ was an arbitrary decomposition; so that (52) (53), (54) and (55) imply

$$\Omega_2(f,\delta) \ll K_2(f,\delta^2) \tag{56}$$

5. Upper estimate

Let x'_δ be the greatest positive solution of the equation $1+Q'(x'_\delta)^2=\delta^{-2}$. Note $x'_\delta \leqq x_\delta$ where $Q'(x_\delta)=\delta^{-1}$. Put

$$\psi(x)=\begin{cases} w_Q(x)f(x) & \text{if } |x|\leqq x'_\delta \\ 0 & \text{otherwise} \end{cases} \tag{57}$$

Then

$$w_Q(x)f(x)-\psi(x)=\begin{cases} 0 & \text{if } |x|\leqq x'_\delta \\ w_Q(x)f(x) & \text{if } |x|\geqq x'_\delta. \end{cases}$$

But if $|x|>x'_\delta$, $(1+Q'(x)^2))^{1/2}\geqq\delta^{-1}$. Thus, $Q'_\delta(x)=\delta^{-1}$. Hence

$$w_Q(x)f(x)-\psi(x)=\begin{cases} 0 & \text{if } |x|\leqq x'_\delta \\ \delta^r Q'_\delta(x)^r w_Q(x)f(x) & \text{if } |x|> x'_\delta \end{cases}$$

$$r\geqq 1 \text{ any integer.} \tag{58}$$

Hence

$$\|w_Qf-\psi\|\leqq\delta^r\|Q_\delta'^r w_Q f\|,\quad r\geqq 1. \tag{59}$$

Put

$$\varphi_1(x)=\delta^{-1}w_Q^{-1}(x)\int_0^\delta \psi(x+t)\,dt \tag{60}$$

$$\varphi_2(x)=\delta^{-2}w_Q^{-1}(x)\int_0^\delta\int_0^\delta\left[2\psi\left(x+\frac{t_1+t_2}{2}\right)-\psi(x+t_1+t_2)\right]dt_1\,dt_2. \tag{61}$$

Clearly, using Lemma 9 and (59), for $r=1,2$

$$\|w_Qf-w_Q\varphi_r\|\leqq\|w_Qf-\psi\|+\|w_Q\varphi_r-\psi\|\leqq\delta^r\|Q_\delta'^r w_Qf\|+\sup_{|h|\leqq\delta}\|\Delta_h^r\psi\|$$

$$\leqq\delta^r\|Q_\delta'^r w_Qf\|+\sup_{|h|\leqq\delta}\|\Delta_h^r(w_Qf)\|+2^r\|w_Qf-\psi\|$$

$$\ll\sup_{|h|\leqq\delta}\|\Delta_h^r(w_Qf\|+\delta^r\|Q_\delta'^r w_Qf\|\leqq\omega_r(f,\delta)\quad r=1,2 \tag{62}$$

Upper estimate in Theorem 1

$$|\delta\varphi_1'(x)|=\left|Q'(x)w_Q^{-1}(x)\int_0^\delta\psi(x+t)\,dt+w_Q^{-1}(x)\Delta_\delta\psi(x)\right|$$

$$\leqq\left|Q'(x)w_Q^{-1}(x)\int_0^\delta\Delta_t\psi(x)\,dt\right|+|Q'(x)w_Q^{-1}(x)\psi(x)|+w_Q^{-1}(x)|\Delta_\delta\psi(x)|. \tag{63}$$

Therefore

$$\delta\|w_Q\varphi_1'\|\leqq\left\|Q'\int_0^\delta\Delta_t\psi\,dt\right\|+\delta\|Q'\psi\|+\|\Delta_\delta\psi\|. \tag{64}$$

We shall estimate each term on the right hand side of (64) separately. Note that $\Delta_t\psi=0$ if $|x|\geqq x'_\delta+\delta$. Otherwise

$$|Q'(x)|\leqq(1+Q'(x)^2)^{1/2}\leqq(1+Q'(x'_\delta+\delta)^2)^{1/2}\leqq(1+Q'(x_\delta+\delta)^2)^{1/2}$$
$$\ll\delta^{-1}\quad\text{(By lemma 6).}$$

Then by Lemma 8, (59),

$$\left\|Q'\int_0^\delta\Delta_t\psi\,dt\right\|\ll\sup_{|t|\leqq\delta}\|\Delta_t\psi\|\ll\sup_{|h|\leqq\delta}\|\Delta_h(w_Qf)\|+\|w_Qf-\psi\|$$
$$\ll\sup_{|h|\leqq\delta}\|\Delta_h(w_Qf\|+\delta\|Q'_\delta w_Qf\|\leqq\omega_1(f,\delta).\tag{65}$$

Also if $|x|>x'_\delta$ then $\psi=0$. Otherwise

$$|Q'(x)|\leqq(1+Q'(x)^2)^{1/2}\leqq(1+Q'(x'_\delta)^2)^{1/2}\leqq\delta^{-1}.$$

So, $Q'_\delta(x)=(1+Q'(x)^2)^{1/2}\geqq|Q'(x)|$.

Hence

$$\delta\|Q'\psi\|\leqq\delta\|Q'_\delta\psi\|\leqq\delta\|Q'_\delta w_Qf\|+\|w_Qf-\psi\|\ll\delta\|Q'_\delta w_Qf\|\leqq\omega_1(f,\delta)\tag{66}$$

$$\|\Delta_\delta\psi\|\leqq\sup_{|h|\leqq\delta}\|\Delta_h(w_Qf)\|+2\|w_Qf-\psi\|\ll\sup_{|h|\leqq\delta}\|\Delta_h(w_Qf)\|+\delta\|Q'_\delta w_Qf\|\leqq\omega_1(f,\delta).\tag{67}$$

Inequalities (64), (65), (66), (67) imply

$$\delta\|w_Q\psi'_1\|\ll\omega_1(f,\delta).$$

Hence from (62)

$$K_1(f,\delta)\leqq\|w_Q(f-\varphi_1)\|+\delta\|w_Q\varphi'_1\|\ll\omega_1(f,\delta).\tag{68}$$

Observe now that $K_1(f,\delta)=K_1(f-a,\delta)$ for all $a\in\mathbf{R}$. This completes the proof of Theorem 1. ∎

Upper estimates in Theorem 2a

We have,

$$w_Q\varphi''_2=w_Q(w_Q^{-1}w_Q\varphi_2)''=(w_Q\varphi_2)''+2Q'(w_Q\varphi_2)'+(Q'^2+Q'')w_Q\varphi_2\tag{69}$$

where φ_2 is defined in (61).

We shall estimate $\|w_Q\varphi''_2\|$ by estimating the norm of each of the terms on the right hand side separately. Using Proposition 8(b):

$$\delta^2\|(Q''+Q'^2)w_Q\varphi_2\|\ll\delta^2\|(1+Q'^2)w_Q\varphi_2\|$$
$$\leqq\left\|(1+Q'^2)\int_0^\delta\int_0^\delta\Delta^2_{\frac{t_1+t_2}{2}}\psi(x)\,dt_1\,dt_2\right\|+\delta^2\|(1+Q'^2)\psi\|.\tag{70}$$

If $|x| \geqq x'_\delta + 2\delta$, integrand in the first term is zero. Otherwise, $1+Q'^2(x) \leqq Q'^2(x'_\delta+2\delta) \ll \delta^{-2}$ (by Lemma 7(a)). So,

$$\left\|(1+Q'^2)\int_0^\delta\int_0^\delta \Delta^2_{\frac{t_1+t_2}{2}}\psi(x)\,dt_1\,dt_2\right\|$$

$$\ll \delta^{-2}\left\|\int_0^\delta\int_0^\delta \Delta^2_{\frac{t_1+t_2}{2}}\psi(x)\,dt_1\,dt_2\right\| \ll \sup_{|t|\leqq\delta}\|\Delta_t^2\psi\| \quad \text{(Lemma 9)}$$

$$\leqq \sup_{|t|\leqq\delta}\|\Delta_t^2(w_Q f)\| + 4\|w_Q f - \psi\|$$

$$\ll \sup_{|t|\leqq\delta}\|\Delta_t^2(w_Q f)\| + \delta^2\|Q'^2_\delta w_Q f\| \quad ((59) \text{ with } r=2) \quad \leqq \omega_2(f,\delta). \tag{71}$$

If $|x| > x'_\delta$, $\psi = 0$. Otherwise $Q'_\delta(x) = (1+Q'^2(x))^{1/2}$. So,

$$\delta^2\|(1+Q'^2)\psi\| \leqq \delta^2\|Q'^2_\delta\psi\| \leqq \delta^2\|Q'^2_\delta w_Q f\| + \|w_Q f - \psi\|$$

$$\ll \delta^2\|Q'^2_\delta w_Q f\| \leqq \omega_2(f,\delta). \tag{72}$$

Hence from (70) and (71),

$$\delta^2\|(Q''+Q'^2)w_Q\varphi_2\| \ll \omega_2(f,\delta) \tag{73}$$

$$\delta^2(w_Q\varphi_2)' = \int_0^\delta\left[4\Delta_{\frac{\delta}{2}}\psi\left(x+\frac{t}{2}\right) - \Delta_\delta\psi(x+t)\right]dt$$

$$= \int_0^\delta[\Delta^2_{\frac{t}{2}}\psi - \Delta^2_{\frac{\delta+t}{2}}\psi + 2\Delta_{\frac{\delta+t}{2}}\psi - 2\Delta_{\frac{t}{2}}\psi](x)\,dt,$$

Observe, again, that $\delta^2(w_Q\varphi_2)'(x)$ is zero if $|x| > x'_\delta + 2\delta$ and otherwise, by Lemma 7, $[1+Q'^2(x)]^{1/2} \ll \delta^{-1}$. Thus, $[1+Q'^2(x)]^{1/2} \ll Q'_\delta(x)$ if $|x| \leqq x'_\delta + 2\delta$. Then

$$\delta^2\|(w_Q\varphi_2)'Q'\| \leqq \delta^2\|(w_Q\varphi_2)'(1+Q'^2)^{1/2}\| \ll \left\|(1+Q'^2)^{1/2}\int_0^\delta[\Delta_{\frac{t+\delta}{2}}\psi - \Delta_{\frac{t}{2}}\psi]\,dt\right\|$$

$$+\left\|(1+Q'^2)^{1/2}\int_0^\delta[\Delta^2_{\frac{t}{2}}\psi - \Delta^2_{\frac{\delta+t}{2}}\psi]\,dt\right\|.$$

By Lemma 9, and our observation above, we now get

$$\delta^2\|(w_Q\varphi_2)'Q'\| \ll \sup_{|t|\leqq\delta}\delta\|Q'_\delta\Delta_t\psi\| + \sup_{|t|\leqq\delta}\|\Delta_t^2\psi\|$$

$$\ll \delta\sup_{|t|\leqq\delta}\|Q'_\delta\Delta_t(w_Q f)\| + \sup_{|t|\leqq\delta}\|\Delta_t^2(w_Q f)\| + \|w_Q f - \psi\| \ll \omega_2(f,\delta). \tag{74}$$

(Using (59) to estimate the last term)

$$\delta^2\|(w_Q\varphi_2)''\| = \|8\Delta^2_{\frac{\delta}{2}}\psi - \Delta^2_\delta\psi\| \ll \sup_{|t|\leqq\delta}\|\Delta_t^2\psi\|$$

$$\ll \sup_{|t|\leqq\delta}\|\Delta_t^2(w_Q f)\| + \|w_Q f - \psi\| \ll \omega_2(f,\delta). \tag{75}$$

(Using (59) with $r=2$). From (73), (74), (75) and (69) it follows that

$$\delta^2 \|w_Q \varphi_2''\| \ll \omega_2(f, \delta). \tag{76}$$

Finally, (76) and (62) with $r=2$ imply that

$$K_2(f, \delta) \ll \omega_2(f, \delta).$$

To complete the proof, observe that for all $a, b \in \mathbf{R}$, $K_2(f, \delta) = K_2(f-a-bx, \delta)$, so that the above inequality proves Theorem 2a.

There are many ways in which the function $\bar{Q}$ in Theorem 2b can be constructed. We give one construction. Observe that since $Q'(x) \to \infty$ as $x \to \infty$, there exists $a>0$ such that $Q''(a)>0$ (Q is convex). We distinguish three cases; in each case, $\bar{Q}(x) = Q(x)$ if $x \geqq a$ and $\bar{Q}(x) = \bar{Q}(|x|)$ if $x \leqq 0$. We define $\bar{Q}$ on $[0, a]$ as follows:

Case I: $aQ''(a) \leqq Q'(a)$

$$\bar{Q}(x) = Q(a) + Q'(a)(x-a) + \frac{Q''(a)}{2}(x-a)^2 + \frac{1}{4a^3}(Q'(a) - aQ''(a))(x-a)^4.$$

Case II: $Q'(a) < aQ''(a) < 2Q'(a)$

Let $$A = Q'(a) - \frac{1}{2} aQ''(a), \quad c = \sqrt{\frac{2AQ''(a)}{a}}, \quad d = \sqrt{\frac{2Aa}{Q''(a)}}$$

$$k = Q(a) - aQ'(a) + \frac{a^2}{3} Q''(a). \quad \text{Put}$$

$$\bar{Q}(x) = \begin{cases} \dfrac{cx^2}{2} + \dfrac{Ad}{3} + k & \text{if } 0 \leqq x \leqq d \\ Ax + \dfrac{Q''(a)}{6a} x^3 + k & \text{if } d \leqq x \leqq a. \end{cases}$$

Case III: $2Q'(a) \leqq aQ''(a)$

$$\bar{Q}(x) = \begin{cases} Q(a) - \dfrac{2}{3} \dfrac{Q'(a)^2}{Q''(a)}, & 0 \leqq x \leqq \dfrac{aQ''(a) - 2Q'(a)}{Q''(a)} \\ \dfrac{Q''(a)^2}{4Q'(a)} \left[x - \dfrac{aQ''(a) - 2Q'(a)}{Q''(a)} \right]^2 & \\ + Q(a) - \dfrac{2}{3} \dfrac{Q'(a)^2}{Q''(a)} & \text{if } \dfrac{aQ''(a) - 2Q'(a)}{Q''(a)} \leqq x \leqq a. \end{cases}$$

The remaining assertions are now easy to verify. (For the verification of (20), observe that $|Q(x) - \bar{Q}(x)| \leqq M$ for some $M>0$ and all $x \in \mathbf{R}$ because of continuity of Q and $\bar{Q}$.) ∎

Remark: A careful examination of the proof shows that both Theorem 1 and Theorem 2 are valid if we define the K-functionals by taking inf over all f_1, f_2 such that $f=f_1+f_2$, f_2 has compact support and is once (resp. twice) differentiable, $w_Q f_2'$ (resp. $w_Q f_2''$)$\in \mathfrak{X}$, $w_Q f_1 \in \mathfrak{X}$.

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