Criteria for absolute convergence of multiple Fourier series

Z. Nowak

1. Introduction

The classical theorem of Bernstein can be generalized to the form (Bochner [1, p. 376] and Wainger [3, Theorem 15, p. 78]):

(i) *If a function* $f(t_1, \ldots, t_n)$ *is periodic in each variable and belongs to* $\mathrm{Lip}(\alpha)$ *with* $\alpha > n/2$ *then its Fourier series converges absolutely (if* α *is an integer then* $\mathrm{Lip}(\alpha)$ *means* C^α*; otherwise it means functions whose partial derivatives of order* $[\alpha]$ *are in* $\mathrm{Lip}(\alpha-[\alpha])$ *in the ordinary sense).*

(ii) *There exists a periodic function* $f(t_1, \ldots, t_n) \in \mathrm{Lip}(n/2)$ *whose Fourier series does not converge absolutely.*

In this paper we present certain estimates for the absolute sums of Fourier series (Theorem 1 below) and derive criteria for the absolute convergence (Corollary) which are more precise than (i). In analogy with (ii) we show that our criteria, and thus also the underlying estimates cannot be very much improved (Theorem 2).

2. Main results

Let $m=(m_1, m_2, \ldots, m_n)$, where $m_1, \ldots, m_n$ are integers, $t=(t_1, \ldots, t_n) \in R^n$ and $e^{imt}=e^{i(m_1t_1+\ldots+m_nt_n)}$. Let $\sum_m f_m e^{imt}$ be the Fourier series of a function $f(t)$, integrable on $T^n=\{t\colon 0 \leqq t_k \leqq 2\pi;\ k=1, \ldots, n\}$ and 2π-periodic in each variable. We denote $\|f\|_A=\sum_m |f_m|$ and $\|f\|_2=\|f\|_{L_2(T^n)}$. If $\partial^q f/\partial t_k^q \in L_2(T^n)$ for some $q=0, 1, 2, \ldots$ (as usual, $\partial^0 f/\partial t_k^0 \equiv f$) then we put

$$\omega_{j,k}^{(q)}(f, y) = \left\| \frac{\partial^q f}{\partial t_k^q}(t_1, \ldots, t_j+y, \ldots, t_n) - \frac{\partial^q f}{\partial t_k^q}(t_1, \ldots, t_n) \right\|_2.$$

In Section 3 we prove

Theorem 1. *Let $f(t)$ be a periodic function such that*

(a) $$\frac{\partial^j f}{\partial t_k^j}, \quad k = 1, 2, \dots, n; \quad j = 0, 1, \dots, q-1; \quad q = [n/2],$$

are integrable functions, essentially absolutely continuous in t_k (if $n=1$ then (a) *should be dropped),*

(b) $$\frac{\partial^q f}{\partial t_k^q} \in L_2(T^n) \quad \textit{for} \quad k = 1, 2, \dots, n.$$

Let $j_1, j_2, \dots, j_n$ be positive integers not larger than n. If n is even then for some $c = c(n)$ we have

(1) $$\|f\|_A \leqq |f_{0,\dots,0}| + c \sum_{k=1}^n \left[\left\| \frac{\partial^q f}{\partial t_k^q} \right\|_2 + \int_0^{1/2} \frac{\omega_{j_k,k}^{(q)}(f, y)}{y |\ln y|^{1/2}} \, dy \right].$$

Moreover, let $j_1', j_2', \dots, j_n'$ and $j_1'', j_2'', \dots, j_n''$ be positive integers not larger than n and such that each pair j_k', j_k'' satisfies one of the conditions: $j_k' = j_k'' = k$ or $j_k' \neq j_k''$. If n is odd then for some $c = c(n)$ we have

(2) $$\|f\|_A \leqq |f_{0,\dots,0}| + c \sum_{k=1}^n \left[\left\| \frac{\partial^q f}{\partial t_k^q} \right\|_2 + \int_0^{1/2} \frac{\omega_{j_k',k}^{(q)}(f, y) + \omega_{j_k'',k}^{(q)}(f, y)}{y^{3/2}} \, dy \right].$$

If we choose $j_k' = j_k'' = k$ then (2) takes the form

$$\|f\|_A \leqq |f_{0,\dots,0}| + c \sum_{k=1}^n \left[\left\| \frac{\partial^q f}{\partial t_k^q} \right\|_2 + \int_0^{1/2} \frac{\omega_{k,k}^{(q)}(f, y)}{y^{3/2}} \, dy \right].$$

For $n=1$, when $q=0$, we obtain Bernstein's theorem (this is essentially what Zygmund proves in [4, Theorem 3.1, p. 240]). We may also put $j_k = k$ into (1).

Let us denote $\ln(1, y) = \ln y$, $\ln(k, y) = \ln[\ln(k-1, y)]$ for $k = 2, 3, \dots$, and $\Pi_l(y) = \prod_{k=1}^l \ln(k, y)$ for $l = 1, 2, \dots$ Theorem 1 implies

Corollary. *Suppose that $f(t)$ satisfies the assumptions of Theorem* 1. *Suppose also that for sufficiently small $y > 0$ and for $k = 1, 2, \dots, n$ we have*

(3) $$\omega_{j_k,k}^{(q)}(f, y) \leqq c |\ln y|^{1/2} \left[\ln\left(l, \frac{1}{y}\right) \right]^{-\alpha} \Pi_l^{-1}\left(\frac{1}{y}\right) \quad \textit{if } n \textit{ is even},$$

(4) $$\omega_{j_k',k}^{(q)}(f, y) + \omega_{j_k'',k}^{(q)}(f, y) \leqq c y^{1/2} \left[\ln\left(l, \frac{1}{y}\right) \right]^{-\alpha} \Pi_l^{-1}\left(\frac{1}{y}\right) \quad \textit{if } n \textit{ is odd},$$

where $c, \alpha > 0$, l is a positive integer and j_k, j_k', j_k'' are integers subject to the restrictions stated in Theorem 1. Then $\|f\|_A < \infty$.

The proof follows from (1) and (2) by elementary integration.

We have derived the succession of criteria (3) and (4), $l=1, 2, \dots,$ of increasing generality, each of them more general than (i). In Section 4 we prove

Theorem 2. *The functions*

$$f_l(t) = \sum_{m_1=M}^{\infty} \sum_{m_2=m_1}^{\infty} \cdots \sum_{m_n=m_{n-1}}^{\infty} m_n^{-n} \Pi_l^{-1}(m_n) e^{imt}, \quad l = 1, 2, \dots, M = M(l), \tag{5}$$

satisfy the assumptions of Theorem 1. Moreover, for any integers $1 \leqq j_k, j_k', j_k'' \leqq n$ *they satisfy the inequalities* (3) *(when n is even) or* (4) *(when n is odd) with* $\alpha=0$, *but* $\|f_l\|_A=\infty$.

We see that the criteria (3) and (4) are exact in the sense that the restriction $\alpha>0$ can not be relaxed.

3. Proof of Theorem 1

Let $(i_1, i_2, \dots, i_n)$ be a permutation of the n-tuple $(1, 2, \dots, j, 0, \dots, 0)$, where $1 \leqq j \leqq n$ and let $N_{i_1, \dots, i_n}$ be the set of all $m=(m_1, \dots, m_n)$ such that $m_k=0$ if $i_k=0$, $m_k \geqq 1$ if $i_k=1$ and $m_k \geqq m_l$ if $i_k=i_l+1$. We shall put $|m|=(|m_1|, \dots, |m_n|)$.

Let $1 \leqq j < n$ and let k be such that $i_k=j$. If n is even, that is when $q=n/2$, then Hölder's inequality gives

$$\Big(\sum_{|m| \in N_{i_1, \dots, i_n}} |f_m|\Big)^2 \leqq \Big(\sum_{|m| \in N_{i_1, \dots, i_n}} m_k^{-2q}\Big)\Big(\sum_{|m| \in N_{i_1, \dots, i_n}} m_k^{2q}|f_m|^2\Big) \leqq c \sum_m m_k^{2q}|f_m|^2 = c\left\|\frac{\partial^q f}{\partial t_k^q}\right\|_2^2 \tag{6}$$

(c will denote various constants depending on n) where the last equality follows from (a) and (b).

Now let $(i_1, \dots, i_n)$ be a permutation of $(1, 2, \dots, n)$. For even n we shall prove that

$$\sum_{|m| \in N_{i_1, \dots, i_n}} |f_m| \leqq c \int_0^{1/2} \frac{\omega_{j,k}^{(q)}(f, y)}{y|\ln y|^{1/2}}\, dy \quad \text{for} \quad j = 1, 2, \dots, n, \tag{7}$$

where k is such that $i_k=n$. The proof will be given only for $i_1=1$, $i_2=2, \dots, i_n=n$. It applies obviously to other permutations $(i_1, \dots, i_n)$.

For brevity we put $N=N_{1, \dots, n}$, $\omega(y)=\omega_{j,n}^{(q)}(f, y)$ for j fixed and $\Omega(y)=\sup(\omega(\hat{y}), |\hat{y}|<y)$ for $y>0$. If $\Omega(y)\equiv 0$ then (7) is obvious. Otherwise we can define

$$\varphi(y) = \frac{1}{\Omega(y)\, y\, |\ln y|^{1/2}} \quad \text{and} \quad g_{m_j} = \int_{1/4|m_j|}^{1/2} \varphi(y)\, dy. \tag{8}$$

Hölder's inequalities for sums and integrals give

(9) $$\Big(\sum_{|m|\in N}|f_m|\Big)^2 \leqq \Big(\sum_{|m|\in N} m_n^{-2q} g_{m_j}^{-1}\Big)\Big(\sum_{|m|\in N} m_n^{2q} g_{m_j}|f_m|^2\Big) \equiv \sum\nolimits_1 \sum\nolimits_2,$$

$$g_{m_j}^{-1} \leqq \left(\int_{1/4|m_j|}^{1/2}\frac{dy}{y}\right)^{-2}\int_{1/4|m_j|}^{1/2}\frac{\Omega(y)|\ln y|^{1/2}}{y}\,dy$$

$$= \ln^{-2}(2|m_j|)\int_{1/4|m_j|}^{1/2}\frac{\Omega(y)\,|\ln y|^{1/2}}{y}\,dy.$$

Consequently

$$\sum\nolimits_1 \leqq c\sum_{m\in N} m_n^{-n}\ln^{-2}(2m_j)\int_{1/4m_j}^{1/2}\frac{\Omega(y)\,|\ln y|^{1/2}}{y}\,dy$$

$$= c\int_0^{1/2}\frac{\Omega(y)\,|\ln y|^{1/2}}{y}\Big[\sum_{m\in N,\, m_j\geqq 1/4y} m_n^{-n}\ln^{-2}(2m_j)\Big]\,dy$$

(10) $$\leqq c\int_0^{1/2}\frac{\Omega(y)\,|\ln y|^{1/2}}{y}\Big[\sum_{m\in N,\, m_n\geqq 1/4y} m_n^{-n}\ln^{-2}(2m_1)\Big]\,dy =$$

$$= c\int_0^{1/2}\frac{\Omega(y)\,|\ln y|^{1/2}}{y}\Big\{\sum_{m_n\geqq 1/4y} m_n^{-n}\Big[\sum_{m_{n-1}=1}^{m_n}\cdots\sum_{m_1=1}^{m_2}\ln^{-2}(2m_1)\Big]\Big\}\,dy$$

$$\leqq c\int_0^{1/2}\frac{\Omega(y)}{y|\ln y|^{1/2}}\,dy.$$

In order to estimate $\sum_2$ we first note that

(11) $$g_{m_j} = \frac{1}{4\pi}\int_{\pi/|m_j|}^{2\pi}\varphi\left(\frac{y}{4\pi}\right)dy$$

$$\leqq \frac{1}{4\pi}\int_0^{2\pi}\left[\varphi\left(\frac{y}{4\pi}\right)+\varphi\left(\frac{y}{4\pi}+\frac{1}{4|m_j|}\right)\right]\sin^2(m_jy/2)\,dy \leqq c\int_0^{2\pi}\varphi\left(\frac{y}{4\pi}\right)\sin^2(m_jy/2)\,dy.$$

From Parseval's formula for $\omega(y)$ and from the inequalities $\Omega(4\pi y)\leqq\Omega(13y)\leqq 13\Omega(y)$ it follows that

$$\sum\nolimits_2 \leqq c\int_0^{2\pi}\Big[\sum_{|m|\in N} m_n^{2q}|f_m|^2\sin^2(m_jy/2)\Big]\varphi\left(\frac{y}{4\pi}\right)dy$$

(12) $$\leqq c\int_0^{2\pi}\Omega^2(y)\varphi\left(\frac{y}{4\pi}\right)dy = c\int_0^{1/2}\Omega^2(4\pi y)\varphi(y)\,dy \leqq c\int_0^{1/2}\Omega^2(y)\varphi(y)\,dy$$

$$= c\int_0^{1/2}\frac{\Omega(y)}{y|\ln y|^{1/2}}\,dy.$$

Using (9), (10), (12) and the inequality

(13) $$\Omega(y) \leqq \frac{10}{y}\int_0^y \omega(y)\,dy$$

(Garsia [2, p. 91]) we obtain

$$\sum_{|m|\in N}|f_m| \leqq c\int_0^{1/2}\frac{\Omega(y)}{y|\ln y|^{1/2}}\,dy \leqq c\int_0^{1/2}\omega(y)\left(\int_y^{1/2}\frac{d\hat{y}}{\hat{y}^2|\ln \hat{y}|^{1/2}}\right)dy$$

$$\leqq c\int_0^{1/2}\frac{\omega(y)}{y|\ln y|^{1/2}}\,dy,$$

as required. Estimate (1) follows by combining all the estimates (6) and (7) with $j=j_k$.

Now, let n be odd. Instead of (7) we then have

$$\sum_{|m|\in N_{i_1,\dots,i_n}}|f_m| \leqq c\int_0^{1/2}\frac{\omega_{j,k}^{(q)}(f,y)}{y^{3/2}}\,dy, \tag{14}$$

where $q=(n-1)/2$ and k,j are such that $i_k=n$ and $i_j\geqq 2$, except when $j=n=1$. The proof will be given only for $i_1=1,\dots,i_n=n$. Assuming $\Omega(y)>0$ for $y>0$, we define $\varphi(y)=\Omega^{-1}(y)y^{-3/2}$. Let g_{m_j} be such as in (8). Hölder's inequality gives

$$g_{m_j}^{-1} \leqq \left(\int_{1/4|m_j|}^{1/2} y^{-n/2-5/4}\,dy\right)^{-2}\int_{1/4|m_j|}^{1/2}\Omega(y)y^{-n-1}\,dy$$

$$\leqq c|m_j|^{-n-1/2}\int_{1/4|m_j|}^{1/2}\Omega(y)y^{-n-1}\,dy.$$

Consequently

$$\sum\nolimits_1 \leqq c\int_0^{1/2}\Omega(y)y^{-n-1}s(y)\,dy,$$

where with obvious simplifications for $j=n$ we have

$$s(y) = \sum_{m_j\geqq 1/4y} m_j^{-n-1/2}\Big(\sum_{m_{j+1}=m_j}^{\infty}\cdots\sum_{m_n=m_{n-1}}^{\infty} m_n^{1-n}\Big)\Big(\sum_{m_{j-1}=1}^{m_j}\cdots\sum_{m_1=1}^{m_2} 1\Big)$$

$$\leqq c\sum_{m_j\geqq 1/4y} m_j^{-n-1/2}m_j^{1-j}m_j^{j-1} \leqq cy^{n-1/2},$$

provided that $j\geqq 2$ or $j=n=1$, as assumed (otherwise, a divergent series appears). Hence,

$$\sum\nolimits_1 \leqq c\int_0^{1/2}\frac{\Omega(y)}{y^{3/2}}\,dy. \tag{15}$$

As in (11) and (12) we prove that

$$\sum\nolimits_2 \leqq c\int_0^{1/2}\Omega^2(y)\varphi(y)\,dy = c\int_0^{1/2}\frac{\Omega(y)}{y^{3/2}}\,dy.$$

Combining the last estimate with (9), (15) and (13) in the same way as previously we obtain (14) for our choice of $i_1,\dots,i_n$.

We thus see that each sum appearing in (14) has a majorizing term on the right-hand side of (2). For $n=1$ the proof is complete. Let $n>1$. It is easy to see that the functions

$$\varphi_l(t_1,\dots,t_{l-1},t_{l+1},\dots,t_n) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt_l,\quad l=1,2,\dots,n$$

satisfy (a) and (b) in their variables. For each $k\neq l$, l fixed, let $j_k\neq l$ be one of the integers j_k', j_k'' appearing in (2). Applying (1) to φ_l we find

$$(16)\quad \sum_{m,m_l=0}|f_m| = \|\varphi_l\|_A \leqq |(\varphi_l)_{0,\dots,0}| + c\sum_{k\neq l}\left(\left\|\frac{\partial^q\varphi}{\partial t_k^q}\right\|_2 + \int_0^{1/2}\frac{\omega_{j_k,k}^{(q)}(\varphi,y)}{y|\ln y|^{1/2}}\,dy\right)$$

$$\leqq |f_{0,\dots,0}| + c\sum_{k\neq l}\left(\left\|\frac{\partial^q f}{\partial t_k^q}\right\|_2 + \int_0^{1/2}\frac{\omega_{j_k,k}^{(q)}(f,y)}{y^{3/2}}\,dy\right),$$

$$l = 1, 2, \dots, n.$$

Combining all the estimates (14) and (16) we obtain (2).

4. Proof of Theorem 2

We shall need the following propositions.

Proposition 1. *For sufficiently large* $M=M(l)$ *and* $0<y\leqq 1/M$ *we have*

$$(17)\quad S_k(y) \equiv \sum_{M\leqq j\leqq 1/y} j^k \Pi_l^{-2}(j) \leqq 2^{k+2}y^{-k-1}\Pi_l^{-2}\left(\frac{1}{y}\right) \quad \textit{for} \quad k=0,1,\dots,$$

$$(18)\quad S_{-1}(y) \equiv \sum_{j\geqq 1/y} j^{-1}\Pi_l^{-2}(j) \leqq 2\,|\ln y|\,\Pi_l^{-2}\left(\frac{1}{y}\right),$$

$$(19)\quad S_{-k}(y) \equiv \sum_{j\geqq 1/y} j^{-k}\Pi_l^{-2}(j) \leqq 2^k y^{k-1}\Pi_l^{-2}\left(\frac{1}{y}\right) \quad \textit{for} \quad k=2,3,\dots.$$

Proof. If $k=0, 1, \dots$ then for sufficiently large $x\geqq M$ we obtain

$$\frac{d}{dx}\left(x^{k+1}\Pi_l^{-2}(x)\right) = x^k\Pi_l^{-2}(x)\left[k+1-2\sum_{j=1}^{l}\Pi_j^{-1}(x)\right] \geqq \frac{1}{2}x^k\Pi_l^{-2}(x),$$

$$S_k(y) \leqq \int_M^{1/y+1} x^k\Pi_l^{-2}(x)\,dx \leqq 2\int_M^{1/y+1}\frac{d}{dx}\left(x^{k+1}\Pi_l^{-2}(x)\right)dx$$

$$\leqq 2^{k+2}y^{-k-1}\Pi_l\left(\frac{1}{y}\right).$$

Furthermore

$$-\frac{d}{dx}\left(\ln x\,\Pi_l^{-2}(x)\right) = \frac{1}{x}\Pi_l^{-2}(x)\left[1+2\ln x\sum_{j=2}^{l}\Pi_j^{-1}(x)\right] \geqq \frac{1}{x}\Pi_l^{-2}(x)$$

$$S_{-1}(y) \leqq \int_{1/y-1}^{\infty} x^{-1}\Pi_l^{-2}(x)\,dx \leqq -\int_{1/y-1}^{\infty}\frac{d}{dx}\left(\ln x\,\Pi_l^{-2}(x)\right)dx$$

$$= \ln\left(\frac{1}{y}-1\right)\Pi_l^{-2}\left(\frac{1}{y}-1\right) \leqq 2\,|\ln y|\,\Pi_l^{-2}\left(\frac{1}{y}\right).$$

Finally, if $k=2, 3, \ldots$ then

$$-\frac{d}{dx}(x^{1-k}\Pi_l^{-2}(x)) = x^{-k}\Pi_l^{-2}(x)[k-1+2\sum_{j=1}^{l}\Pi_j^{-1}(x)] \geqq x^{-k}\Pi_l^{-2}(x),$$

$$S_{-k}(y) \leqq \int_{1/y-1}^{\infty} x^{-k}\Pi_l^{-2}(x)\,dx \leqq -\int_{1/y-1}^{\infty}\frac{d}{dx}(x^{1-k}\Pi_l^{-2}(x))\,dx$$

$$= \left(\frac{1}{y}-1\right)^{1-k}\Pi_l^{-2}\left(\frac{1}{y}-1\right) \leqq 2^k y^{k-1}\Pi_l^{-2}\left(\frac{1}{y}\right).$$

Proposition 2. *Let $f=\sum_m f_m e^{imt}\in L_1(T^n)$ and let $\sum_m im_j f_m e^{imt}$ be the Fourier series of a function $\varphi\in L_1(T^n)$. Then f is essentially absolutely continuous in t_j and $\partial f/\partial t_j=\varphi$ a. e. on T^n.*

Proof. Let $\psi=\int_0^{t_j}\varphi(t)dt_j$. The function $\psi-f\sim\sum_{m,m_j=0}c_m e^{imt}$ is essentially independent of t_j. Hence, together with ψ the function f is essentially absolutely continuous in t_j and $\partial f/\partial t_j=\partial\psi/\partial t_j=\varphi$ a. e. on T^n.

We proceed to the proof of Theorem 2. Let M, appearing in (5), be so large that (17), (18) and (19) hold for $0<y\leqq 1/M$. Let $N(M)=\{m\colon M\leqq m_1\leqq\ldots\leqq m_n\}$. In view of Proposition 2 the properties (a) and (b) will be verified if we show that the term-by-term differentiations $\partial^j/\partial t_k^j$ $(j=0, 1, \ldots, q=[n/2];\ k=1, \ldots, n)$ of the series in (5) produce functions of class $L_2(T^n)$. For that purpose it is enough to note that

$$\sum_{m\in N(M)} m_n^{-2n}\Pi_l^{-2}(m_n)m_k^{2j} \leqq \sum_{m_n=M}^{\infty} m_n^{-2n+2q}\Pi_l^{-2}(m_n)\Big(\sum_{m_{n-1}=1}^{m_n}\cdots\sum_{m_1=1}^{m_2}1\Big)$$

$$\leqq c\sum_{m_n\geqq M} m_n^{-1}\Pi_l^{-2}(m_n) = cS_{-1}\left(\frac{1}{M}\right) < \infty.$$

By (a) and (b) the Parseval formula gives

$$(\omega_{j,k}^{(q)})^2 = 4(2\pi)^n\sum_{m\in N(M)} m_n^{-2n}\Pi_l^{-2}(m_n)m_k^{2q}\sin^2\left(\frac{m_j y}{2}\right)$$

$$\leqq cy^2\sum_{m\in N(M), m_j\leqq 1/y} m_n^{-2n+2q}\Pi_l^{-2}(m_n)m_j^2$$

$$+c\sum_{m\in N(M), m_j>1/y} m_n^{-2n+2q}\Pi_l^{-2}(m_n) \equiv \sigma_1+\sigma_2.$$

With obvious simplifications for $j=n$ and $n=1$ we can write

$$\sigma_1 = cy^2\sum_{m_1=M}^{[1/y]}\sum_{m_2=m_1}^{[1/y]}\cdots\sum_{m_j=m_{j-1}}^{[1/y]} m_j^2\Big[\sum_{m_{j+1}=m_j}^{\infty}\cdots\sum_{m_n=m_{n-1}}^{\infty} m_n^{-2n+2q}\Pi_l^{-2}(m_n)\Big].$$

The sum in the square bracket is not larger than $cm_j^{2q-n-j}\Pi_l^{-2}(m_j)$, as can be shown by successive application of (19), where we put $k=2n-2q$, $2n-2q-1, \ldots, n+j+1-2q\geqq 2$ and $y=1/\mathrm{m}_{n-1}$, $1/m_{n-2}, \ldots, 1/m_j\leqq 1/M$, respectively. Rearranging the remaining sums we find

$$\sigma_1 \leqq cy^2\sum_{m_j=M}^{[1/y]} m_j^{2q-n-j+2}\Pi_l^{-2}(m_j)\Big(\sum_{m_{j-1}=M}^{m_j}\cdots\sum_{m_1=M}^{m_2}1\Big)$$

$$\leqq cy^2\sum_{m_j=M}^{[1/y]} m_j^{2q-n+1}\Pi_l^{-2}(m_j).$$

If $y \leqq 1/M$ then for even and odd n ($q=n/2$ or $(n-1)/2$) the estimate (17) ($k=1, 0$) gives

$$\sigma_1 \leqq c\, \Pi_l^{-2}\left(\frac{1}{y}\right) \quad \text{and} \quad \sigma_1 \leqq cy\, \Pi_l^{-2}\left(\frac{1}{y}\right), \tag{20}$$

respectively. Furthermore

$$\sigma_2 \leqq c \sum_{m_n > 1/y} m_n^{-2n+2q}\, \Pi_l^{-2}(m_n) \left(\sum_{m_{n-1}=M}^{m_n} \cdots \sum_{m_1=M}^{m_2} 1\right)$$
$$\leqq c \sum_{m_n > 1/y} m_n^{-n+2q-1}\, \Pi_l^{-2}(m_n).$$

Now, for even and odd n the estimates (18) and (19) with $k=2$ imply

$$\sigma_2 \leqq c|\ln y|\, \Pi_l^{-2}\left(\frac{1}{y}\right) \quad \text{and} \quad \sigma_2 \leqq cy\, \Pi_l^{-2}\left(\frac{1}{y}\right), \tag{21}$$

respectively. Using (20) and (21) we obtain the inequalities (3) and (4) with $\alpha=0$ for any j_k, j_k' and j_k''.

Let us note, however, that for some $c=c(M, n)>0$ we have

$$\|f_l\|_A = \sum_{m_n=M}^{\infty} m_n^{-n}\, \Pi_l^{-1}(m_n) \left(\sum_{m_{n-1}=M}^{m_n} \cdots \sum_{m_1=M}^{m_2} 1\right)$$
$$\geqq c \sum_{m_n=M}^{\infty} m_n^{-1}\, \Pi_l^{-1}(m_n) \geqq c \int_M^{\infty} x^{-1}\, \Pi_l^{-1}(x)\, dx = c \ln(l+1, x)\Big]_{x=M}^{x\to\infty} = \infty.$$

The proof is complete.

The author wishes to thank Dr. Ingemar Wik for his valuable suggestions which made it possible to simplify the proofs and state the theorems in their present form.

References

1. Bochner, S., Review of "On absolute convergence of multiple Fouıier series", *Math. Rev.* **8** (1947).
2. Garsia, A. M., A remarkable inequality and the uniform convergence of Fourier series, *Indiana Univ. Math. J.* **25** (1976), 85—102.
3. Wainger, S., *Special trigonometric series in k-dimensions,* Mem. Amer. Math. Soc. **59**, 1965.
4. Zygmund, A., *Trigonometric series I,* Cambridge Univ. Press, 1959.

Received May 10, 1982

Z. Nowak
Warsaw University of Technology
Nowowiejska 24
00-665 Warszawa
Poland