Some good unirational families of space curves

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The problem of classifying space curves, which goes back to Halphen and Noether, may be roughly divided into 3 parts:

(i) list the maximal families (i.e. Hilbert scheme components) of smooth curves in $\mathbf{P}^3$;
(ii) describe the properties of a general member Y of a given maximal family (e.g. number of moduli, vanishing of $H^1(N_Y)$, maximal rank etc.);
(iii) describe Y "explicitly" (e.g. by equations).

To date, little progress has been realized on part (i), while part (ii) has been answered in many cases. Part (iii), on the other hand has been answered, apart from curves which are (or almost are) complete intersections, only in a handful of cases. This is in part explained by the results of Harris and Mumford [6] showing that for g large, M_g, the moduli space of curves of genus g, is not unirational; hence a curve of genus g with general moduli cannot be described by equations depending on free parameters.

The purpose of this paper is to add some further cases to the list of those for which part (iii) above has a positive answer. Namely we prove the following result.

Theorem. *For* $d\leqq 15$, $5d-55\leqq g\leqq 2d-9$, *and* $(d,g)\neq(13,11)$, *the Hilbert scheme* $H_{d,g}$ *has a unirational component* H *such that the curve* Y *corresponding to a general point in* H *is linearly normal, of maximal rank, has* μ_0 *of maximal rank and* $H^1(N_Y)=0$.

The case $(d,g)=(13,11)$ is still open.

The cases $(d,g)=(12,10), (11,9)$ of the Theorem are proved using the method of monads, as in [3], [4]. The idea is to construct a curve Y with good properties,

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construct a reflexive sheaf from Y using the Serre construction, and then to describe this sheaf by means of a monad, where the family of monads involved is unirational. These arguments are carried out in § 1 (where the monad technique is presented) and § 2 (where the necessary curves Y are constructed).

The cases $(d, g)=(12, 10), (11, 9)$, by contrast, are proved using the method of liaison (cf. [8], [9]). This is curious because this was apparently the method envisaged by Severi for proving the unirationality of M_g in general. The failure of Severi's method was established by Lazarsfeld and Rao [7] who proved that in "most" cases, doing liaison on a general curve leads to curves of larger degree and genus. This also turns out to be the case for $(d, g)=(12, 10)$ and $(11, 9)$, namely liaison leads to curves of degree 13 (resp. 14), genus 13 (resp. 18). Fortunately, however, these curves have already been dealt with in [3] (resp. § 2), where the existence of a unirational component H of $H_{13,13}$ (resp. $H_{14,18}$) was established. Using this and other properties of the curves in H, we construct a unirational component of $H_{12,10}$ (resp. $H_{11,9}$).

Finally, we mention some notations and conventions to be used in this paper. A *curve* is, unless otherwise mentioned, a connected, reduced curve Y with nodes only. I_Y denotes the ideal sheaf, N_Y the normal bundle, and ω_Y the dualizing sheaf. Y has *maximal rank* if the restriction $H^0(\mathcal{O}_{\mathbf{P}^3}(k))\to H^0(\mathcal{O}_Y(k))$ is either injective or surjective for all k. $H_{d,g}$ denotes the Hilbert scheme of nonsingular curves of degree d, genus g in $\mathbf{P}^3$.

1. Monads for space curves

Our method for parametrizing space curves is to associate to the curves a reflexive sheaf using the Serre construction, and then to represent the sheaf as the cohomology of a monad (see [1]).

Theorem 1.1. *Let Y be a smooth curve in $\mathbf{P}^3$ with the following properties:*

(i) $H^1(I_Y(\leqq n))=0$;

(ii) μ: $H^0(\omega_Y(-n))\otimes H^0(\mathcal{O}_Y(1))\to H^0(\omega_Y(-n+1))$ *has maximal rank;*

(iii) $H^0(I_Y(\leqq n+1))=0$;

(iv) $H^1(\mathcal{O}_Y(n+1))=0$;

(v) *Y can be specialized to a curve Y' which has a k-secant line L, $3r\geqq k\geqq n+1$, disjoint from the base points of $\omega_{Y'}(-n)$ and from the singular points of Y', and $L\cap Y'$ imposes* min $\{k, r\}$ *conditions on $\omega_{Y'}(-n)$, also $r:=h^0(\omega_{Y'}(-n))=h^0(\omega_Y(-n))$;*

(vi) *in case μ is not injective, the multiplication map $v: H^1(I_Y(n+1))\otimes H^0(\mathcal{O}_Y(1))\to H^1(I_Y(n+2))$ has maximal rank.*

Then there is a reflexive sheaf E of rank $r+1=h^0(\omega_Y(-n))+1$ on $\mathbf{P}^3$ and an exact *sequence*

$$(1)\qquad 0 \to H^0(\omega_Y(-n))\hat{}\otimes\mathcal{O} \to E(2) \to I_Y(n+4) \to 0$$

Moreover, if μ *is surjective, then we have an exact sequence*

$$(2)\qquad 0 \to E \to (-\varrho)\mathcal{O}(-1)\oplus\sigma\mathcal{O} \to \tau\mathcal{O}(1) \to 0$$

if ν *is surjective or*

$$0 \to E \to (-\varrho)\mathcal{O}(-1) \to (-\sigma)\mathcal{O}\oplus\tau\mathcal{O}(1) \to 0$$

if ν *is injective.*

If μ *is injective, then* E *is the cohomology of a monad*

$$0 \to \varrho\mathcal{O}(-1) \to \sigma\mathcal{O} \to \sigma\mathcal{O}(1) \to 0.$$

Here

$$\varrho = h^0(\omega_Y(-n+1))-4h^0(\omega_Y(-n))$$
$$\sigma = 4h^1(I_Y(n+1))-h^1(I_Y(n+2))$$
$$\tau = h^1(I_Y(n+1)).$$

The proof is largely contained in [3] and [4], but for completeness we sketch it here. First, E is constructed using the standard Serre construction.

Lemma 1.2. *Let* E *and* Y *be as in sequence* (1). *Assume* Y *has the properties* $H^1(I_Y(\leqq n))=0$ *and* $H^1(\mathcal{O}_Y(n+1))=0$. *Then* $H^1(E(*))$ *is generated by* $H^1(E(-1))$ *and* $H^1(E)$.

Proof. Restricting (1) on a general line L, we conclude $H^1(E_L(\geqq 1))=0$ hence $H^1(E\otimes I_L(\geqq 1))\to H^1(E(\geqq 1))$ is surjective. Also $H^2(E(\geqq -1))=H^2(I_Y(\geqq n+1))=H^1(\mathcal{O}_Y(\geqq n+1))=0$ by hypothesis, so from the Koszul resolution of I_L we get $H^1(E(\geqq 0))\otimes H^0(I_L(1))\to H^0(E\otimes I_L(\geqq 1))$ surjective, hence $H^1(E(*))$ is generated by $H^1(E(\leqq 0))$. But (1) yields immediately $H^1(E(\leqq -2))=0$ hence the Lemma.

Lemma 1.3. *Let* E *and* Y *be as in Lemma* 1.2. *Assume for some* k, $3r\geqq k\geqq n+1$, Y *has a* k*-secant line* L *disjoint from the base points of* $\omega_Y(-n)$ *and from the singular points of* Y, $L\cap Y$ *imposes* $\min\{k, r\}$ *conditions on* $\omega_Y(-n)$, *then* $\operatorname{Ext}^1(E, \mathcal{O}(*))$ *is generated by* $\operatorname{Ext}^1(E, \mathcal{O}(-1))$.

Proof. By restricting (1) on L, it is easy to see that the image of $E_L(2)\to I_Y(n+4)\otimes\mathcal{O}_L$ is $\mathcal{O}_L(n+4-k)$. (Note that $n+4-k\leqq 3$) and the kernel is $\oplus\mathcal{O}_L(k_i)$ where $0\leqq k_i\leqq 3$. It follows that $H^1(E_L\check{}(\geqq 0))=H^0(E_L(\leqq -2))=0$. Now it is easy to see in general that $H^1(E_L\check{}(\geqq 0))\cong\operatorname{Ext}^1(E, \mathcal{O}_L(\geqq 0))$ (note that E_L is locally free); hence we get $\operatorname{Ext}^1(E, I_L(\geqq 0))\to\operatorname{Ext}^1(E, \mathcal{O}(\geqq 0))$ surjective. Now using the Koszul resolution of I_L and the vanishing of $\operatorname{Ext}^2(E, \mathcal{O}(\geqq -2))$ (Serre dual to $H^1(E(\leqq -2))=H^1(I_Y(\leqq n))$), we see that $\operatorname{Ext}^1(E, \mathcal{O}(-1))$ generates

$$\operatorname{Ext}^1(E, \mathcal{O}(\geqq -1)).$$

On the other hand, it follows easily from the construction and assumption that $\mathrm{Ext}^1(E, \mathcal{O}(k)) = H^2(E(-k-4))^{\check{}} = 0$ for $k \leqq -2$.

Remark 1.2.1. For $n+1$ small, e.g. $n+1 \leqq 3$, the restrictions on k and L are not essential, because $0 \leqq k_i \leqq 3$ is trivially true.

Proof of theorem. Let $0 \to A \to B \to C \to 0$ be the minimal monad for E (cf. [1]). First assume μ surjective. Then $\mathrm{Ext}^1(E, \mathcal{O}(-1)) = H^2(E(-3)) = 0$ implies that $A=0$, so we have $0 \to E \to B \to C \to 0$ exact. By Lemma 1.2 and hypothesis (vi), C is as stated in the theorem. By assumption (iii), we have $H^0(E(-1))=0$, hence $H^0(B(-1))=0$. Also, $H^0(E^{\check{}}(-2))=0$ by construction, hence $H^3(B(-2)) = H^3(E(-2))=0$. This implies $B = \oplus\mathcal{O}(-1) \oplus \oplus\mathcal{O}$. Now by comparing Chern classes and ranks we conclude B is as stated in the theorem.

Now assume μ injective. By Lemma 1.3 $A = \varrho\mathcal{O}(-1)$. By Lemma 1.2, we have $C = \oplus\mathcal{O} \oplus \tau\mathcal{O}(1)$. Let the display of the monad be

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & A & \to & K & \to & E & \to & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \to & A & \to & B & \to & P & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & C & = & C & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Applying $\mathrm{Hom}(\ \ , \mathcal{O}(-1))$ to the sequence $0 \to A \to K \to E \to 0$ and using the facts that $H^0(E^{\check{}}(-1))=0$ and $\mathrm{Ext}^1(K, \mathcal{O}(-1)) = H^1(K^{\check{}}(-1))=0$, we conclude that $H^0(K^{\check{}}(-1)) = \mathrm{Hom}(K, \mathcal{O}(-1)) = 0$. Hence $H^0(B^{\check{}}(-1))=0$ and $B = \oplus\mathcal{O}(n_i)$, $n_i \geqq 0$. But by minimality, $\min\{n_i\}$ is smaller than the smallest degree of a line bundle appearing in C. It follows that $C = \tau\mathcal{O}(1)$. Note from the display that $H^0(B(-2))=0$, hence $B = \oplus\mathcal{O} \oplus \oplus\mathcal{O}(1)$. Now finally a Chern class computation yields $B = \sigma\theta$.

Corollary 1.4. *Let notations be as in Theorem* 1.1. *If* $\varrho \geqq -3$, *then* ν *is injective.*

Proof. In the proof of Theorem 1.1, the map $B = (-\varrho)\mathcal{O}(-1) \oplus \oplus\mathcal{O} \to C = \oplus\mathcal{O} \oplus \tau\mathcal{O}(1)$ cannot be surjective unless $C = \oplus\mathcal{O}(1)$, if $-\varrho \leqq 3$.

We now show how under suitable hypotheses, monads as above can be used to produce unirational components of the Hilbert scheme.

Theorem 1.5. *Let* Y *be as in Theorem* 1.1. *Assume*

(i) $H^1(N_Y)=0$;
(ii) $H^1(I_Y(n+4))=0$;

and either

(iii) μ *is surjective; or*
(iii)′ $H^1(I_Y(n+3))=0$.

Then Y belongs to a unique component of the Hilbert scheme, which is unirational and smooth of dimension 4d near Y.

Proof. In case (iii), it is clear that the set V of bundles E' coming from monads as in Theorem 1.1 is irreducible and unirational. In case (iii)′, note that if $K=\ker(B\to C)$, then $H^i(K(1))=H^i(E(1))=0$, for $i>0$, so the set of bundles E' coming from monads can be parameterized by the choice of a map $\varphi: B\to C$ whose kernel K' satisfies $H^i(K'(1))=0$ $i>0$, plus ϱ general sections of $K'(1)$; so again the set of E' is irreducible and unirational. Finally, the set of curves Y' in a neighborhood of Y in the Hilbert scheme can be parameterized by a bundle E' in a neighborhood of E plus r general sections of $E'(2)$. As $H^i(E'(2))=H^i(I_{Y'}(n+4))=0$ for $i>0$, we have $h^0(E'(2))=h^0(E(2))$, hence the set of Y' is unirational.

2. Construction of good curves

In this section we shall prove the following theorem which, together with the result of § 1, will prove the main theorem stated in the Introduction for the case $(d, g)\neq(12, 10), (11, 9)$.

Theorem 2.1. *There exists a smoothable curve Y of degree d and genus g in* $\mathbf{P}^3$ *satisfying the hypotheses of Theorem* 1.5, *for* $n=1$, $d\leqq 15$, $5d-55\leqq g\leqq 2d-9$, *and* $(d, g)\neq(13, 11), (12, 10), (11, 9)$.

The curve Y we shall construct will be a reducible one. The main step in the construction of Y is to attach a 4- or 5-secant conic to a given curve, so we begin with some general properties of this operation.

Lemma 2.2. *Let Y be a curve in* $\mathbf{P}^3$, *C a* 4- *or* 5-*secant conic of Y on a general plane and* $Y'=Y\cup C$. *If Y has any of the properties* (i)~(vi) *below, so does* Y'.

(i) *Y is linearly normal;*
(ii) $\mathcal{O}_Y(2)$ *is nonspecial;*
(iii) $\mu_{0,Y}$ *is surjective;*
(iv) *Y is not in a quadric;*
(v) *Y is smoothable;*
(vi) $H^1(N_Y)=0$.

Moreover, if C is a 4-secant conic and $\mu_{0,Y}$ has maximal rank, so does $\mu_{0,Y'}$; if C is a 4-secant conic and ν_Y is surjective, then so is $\nu_{Y'}$; if C is a 5-secant conic and ν_Y, is injective, so is $\nu_{Y'}$.

Proof. The assertion about linear normality follows from general position, those about μ_0, smoothability, and $H^1(N)$ are contained in Sernesi [10] (cf. also [5]), and (iv) is trivial. To verify (ii), let $Z=Y\cap C$, and note the exact sequence

$$H^0(\mathcal{O}_{Y'}(2)) \to H^0(\mathcal{O}_Y(2)) \to H^1(\mathcal{O}_C(2-Z)) \to H^1(\mathcal{O}_{Y'}(2)) \to H^1(\mathcal{O}_Y(2)) \to 0.$$

If $\#Z\leqq 5$, then $H^1(\mathcal{O}_C(2-Z))=0$, hence $\mathcal{O}_{Y'}(2)$ is nonspecial if $\mathcal{O}_Y(2)$ is. Finally the assertions about ν follows from the sequence

$$0 \to I_{Y'}(k) \to I_Y(k)\oplus I_C(k) \to I_Z(k) \to 0$$

for $k=2, 3,$ plus the fact that I_Z is generated by quadrics (4-secant case) while $H^0(I_C(2)) \xrightarrow{\sim} H^0(I_Z(2))$ (5-secant case).

Next we show that under suitable conditions, a curve $Y\cup C$ as above has maximal rank if Y does.

Lemma 2.3. *Let Y, C, Y' be as in Lemma 2.2 and let $s+1=\#(Y\cap C)$. Suppose Y' has maximal rank and let $l=\min\{k\colon H^1(I_Y(k))=0\}$. Suppose moreover that*

(i) *if H is the plane of C and $Z'=(H\cap Y)\setminus C$, then Z' has maximal rank as a subscheme of H;*

(ii) $\binom{l+3}{3}-ld-1+g\leqq 2l-s$, *where d and g are the degree and genus of Y;*

(iii) $d-s-1\leqq\binom{l+1}{2}$.

Then Y has maximal rank and $l+1=\min\{k\colon H^1(I_{Y'}(k))=0\}$.

Proof. Note the exact sequences

$$0 \to I_{Y'}(l) \to I_Y(l) \to I_{Z,C}(l) \to 0$$

$$0 \to I_Y(l) \to I_{Y'}(l+1) \to I_{Z',H}(l-1) \to 0.$$

In view of hypothesis (i), hypothesis (ii) implies, by looking at H^0 terms, that $H^0(I_{Y'}(l))=0$, while hypothesis (iii) implies $H^1(I_{Y'}(l+1))=0$, so Y' has maximal rank.

Next, we consider the operation of obtaining a 6-secant twisted cubic to a given curve.

Lemma 2.4. *Let Y be a smoothable curve in $\mathbf{P}^3$ with $H^1(N_Y)=0$ and let Z be 6 distinct smooth points on Y which are linearly in general position (i.e. no 3 collinear, no 4 coplanar). Let D be a twisted cubic meeting Y precisely in Z and $Y'=Y\cup D$.*

Then Y' is smoothable and if ν_Y is surjective, then so is $\nu_{Y'}$; if $H^0(I_Y(3))=0$ and ν_Y is injective, so is $\nu_{Y'}$.

Proof. Smoothability follows from Sernesi (see [5], [11]) and the rest by observing the sequence

$$0 \to I_{Y\cup D}(k) \to I_Y(k) \oplus I_D(k) \to I_Z(k) \to 0$$

for $k=2, 3$ plus the fact that $H^1(I_D(2))=H^1(I_Z(2))=0$ and I_D and I_Z are generated by quadrics.

Given these general results, the proof of Theorem 2.1 can be divided into 3 steps. First, we construct, using mainly attachment of 4- and 5-secant conics, curves satisfying all of the conditions of Theorem 1.5 except perhaps for (vi). Second, we construct, using mainly attachment of 6-secant twisted cubic, curves satisfying the aforementioned condition (vi). Finally, we show that the above two types of curves possess good common specializations.

In Step 1 we successively add 4- or 5-secant (resp. 4-secant) conics to curves Y_0 with $(d_0, g_0)=(9, 9)$, $(9, 8)$ $(8, 7)$, (resp. $(d_0, g_0)=(8, 6)$, $(8, 5)$, $(8, 4)$, $(7, 4)$, $(9, 6)$, $(9, 5)$) to produce curves Y with all possible (d, g) such that $5d-55 \leqq g \leqq 2d-9$. It follows from Lemma 2.2 that $Y=Y_0\cup_i C_i$ has all the properties listed in the lemma, because Y_0 does (cf. [2]). In particular μ_0 is surjective (resp. injective). Y being of maximal rank follows from Lemma 2.3, except for the cases $(d, g)=(11, 10)$, $(12, 10)$, $(13, 11)$, $(13, 12)$, $(14, 16)$, $(15, 21)$ where some further degeneration is needed. To illustrate the argument, we consider the case $(15, 21)$, as the other cases are similar. We start with a curve Y_0 of degree 9 and genus 9, then the attach a 5-secant conic C_1, then a conic C_2 meeting C_1 in 2 points and Y_0 in 3 points, then a conic C_3 meeting C_1 in 2 points, C_2 in 2 points and Y_0 in 1 point. Maximal rank for $Y=Y_0\cup C_1\cup C_2\cup C_3$ means $h^0(I_Y(4))=0$ and $h^0(I_Y(5))=1$. We prove the second assertion, as the first is similar but simpler. First, it is easy to see that at least 6 of the points of $Y_0\cap H_1$ can be chosen generically, where H_1 is the plane of C_1 (e.g. by degenerating Y_0 to a twisted cubic plus three 4-secant conics). By Lemma 2.3 it follows that $h^0(I_{Y_0\cup C_1}(4))=3$. Now specialize C_2 and C_3 to conics on the same general plane H_2, thereby adding to Y two embedded points at two of the intersections of C_2 and C_3, say P_1, P_2. Now if F_5 is any quintic containing Y, then $F_5\cap H_2$ contains $C_2\cup C_3$ plus a line L containing $Y_0\cup C_1\cap(H_2\setminus(C_2\cup C_3))$ which is 3 noncollinear points. Hence $F_5=H_2\cup F_4$, where F_4 is a quartic containing $Y_0\cup C_1\cup P_1\cup P_2$. Since P_1 and P_2 are general w.r.t. $Y_0\cup C_1$, we have $h^0(I_{Y_0\cup C_1\cup P_1\cup P_2}(4))=1$, hence $h^0(I_Y(5))=1$, as desired.

Next, we establish property (vi) in each case. For $(d, g)=(13, 15)$, $(13, 16)$, $(14, 16)$, $(14, 17)$, $(14, 18)$, $(15, 20)$, we specialize Y to $\bar{Y}\cup D$, where D is a 6-secant twisted cubic. For $(d, g)=(11, 12)$, $(12, 14)$, $(14, 15)$, we specialize Y to $\bar{Y}\cup C$, where C is a 4-secant conic. In each case it follows by Lemmas 2.2 and 2.4 that the reducible curve is smoothable and has ν of maximal rank. The rest of cases have ν of maximal rank either for trivial reasons or by Corollary 1.4.

Finally, we show that the curves of each degree and genus, $(d, g)=(13, 15)$, $(13, 16)$, $(14, 16)$, $(14, 17)$, $(14, 18)$, $(15, 20)$, possess a good common specialization. To see this, note that by [2], the Hilbert scheme corresponding to Y_0 in Step 1 is irreducible, and from this it follows easily that Y_0 can be specialized to $Y_1 \cup D$, where D is a 6-secant twisted cubic of Y_1, and we may also assume that the conics attached to Y_0 specialize to conics meeting Y_1 but not D. Note that $\bar{Y}=Y_1 \cup_i C_i$ is smoothable, so $Y=D \cup \bar{Y}$ is as in Step 2.

3. Liaison

In this section we shall use techniques of liaison (cf. [8], [9]) to establish the following result.

Theorem 3.1. *For* $(d, g)=(12, 10)$, $(11, 9)$, *the Hilbert scheme* $H_{d,g}$ *of non-singular curves of degree* d, *genus* g *in* $\mathbf{P}^3$ *has a unirational component* H *whose general point corresponds to a curve* Y *which is linearly normal, of maximal rank and has* μ_0^c *injective.*

Proof. Our starting point is a unirational component H_1 of $H_{13,13}$ (respectively, $H_{14,18}$) whose general curve Y_1 is linearly normal, of maximal rank and has μ_0 of maximal rank; the existence of H_1 is established in [3] (resp. § 2). Y_1 is contained in precisely 3 independent quintics, necessarily irreducible. Let $E(2)$ be the rank $r+1=h^0(\omega_{Y_1}(-1))+1$ bundle on $\mathbf{P}^3$ corresponding to $\omega_{Y_1}(-1)$; i.e. we have an exact sequence

$$0 \to r\mathcal{O} \to E(2) \to I_{Y_1}(5) \to 0 \tag{3.1}$$

with dual

$$0 \to \mathcal{O}(-5) \to E^{\vee}(-2) \to r\mathcal{O} \to \omega_{Y_1}(-1) \to 0. \tag{3.2}$$

Now choose 2 general quintics F, F' containing Y_1 and let $X=F \cap F'$. Then we have an exact sequence

$$0 \to \mathcal{O}(-5) \to 2\mathcal{O} \to I_X(5) \to 0 \tag{3.3}$$

with dual

$$0 \to \mathcal{O}(-5) \to 2\mathcal{O} \to \mathcal{O}(5) \to \omega_X(-1) \to 0. \tag{3.4}$$

The natural maps $I_X(5) \to I_{Y_1}(5)$, $\omega_{Y_1}(-1) \to \omega_X(-1)$ extends to (dual) maps of complexes (3.1)→(3.3) and (3.2)→(3.4) and taking the mapping cone of each gives us dual sequences

(3.5) $$0 \to E^{\vee}(-2) \to (r+2)\mathcal{O} \to I_{Y_2}(5) \to 0$$

(3.6) $$0 \to \mathcal{O}(-5) \to (r+2)\mathcal{O} \to E(2) \to \omega_{Y_2}(-1) \to 0$$

where Y_2 is residual to Y_1 in X; thus Y_2 is Cohen—Macaulay.

We claim, in fact, that Y_2 is reduced. To see this note that we may assume the map $r\mathcal{O} \to E(2)$ in (3.1) corresponds to r general sections of $E(2)$, and since the map $(r+2)\mathcal{O} \to E(2)$ in (3.6) corresponds to the previous r sections plus 2 sections lifting F, F' respectively, it follows that this map is general, and since its dependency locus is, by (3.6), just Y_2, Y_2 will be reduced provided $E(2)$ is generated by its sections except on a finite set. By (3.1), the latter condition is equivalent to assertion that $I_{Y_1}(5)$ is generated by its sections except on a finite set. Now this last assertion is largely established in [3] (resp. § 2) (cf. Proposition 3.2, 3.3 (resp. Lemma 3.2 below)): namely it is shown there that Y_1 can be degenerated to $\tilde{Y} \cup C$ where $\tilde{Y}$ has degree 11, genus 10 (resp. degree 12, genus 14), C is a conic on a general plane H meeting $\tilde{Y}$ in 4 (resp. 5) points, and $Z = \tilde{Y} \cap H \setminus C$ is 7 generic points on H; moreover, it is shown that $h^0(I_{\tilde{Y} \cup C}(5)) = 3$ and that the restriction on H of $H^0(I_{\tilde{Y} \cup C}(5))$ is $[C]H^0(I_{Z,H}(3))$. But now it is well-known and classical that $I_{Z,H}(3)$,is generated by its sections. This implies that $H^0(I_{\tilde{Y} \cup C}(5))$ generates $I_{(\tilde{Y} \cup C) \cap H, H}(5)$; hence it generates $I_{\tilde{Y} \cup C}$ except on a finite set.

Now (3.6) plus the fact that $E(2)$ is generated by its sections except on a finite set show that $h^0(\omega_{Y_2}(-1)) = 1$ and the unique section of $\omega_{Y_2}(-1)$ generates it almost everywhere. Hence Y_2 is linearly normal and the μ_0-mapping $H^0(\mathcal{O}_{Y_2}(1)) \otimes H^0(\omega_{Y_2}(-1)) \to H^0(\omega_{Y_2})$ is injective. Also (3.5) plus $H^1(E(\leqq -2)) = 0$ (cf. [3] and § 2) show that Y_2 has maximal rank. By (3.5) again, Y_2 determines E. Now as Y_2 deforms to a general element Y_2' of the unique Hilbert scheme component H_2 to which it belongs, E deforms with it (by maximal rank) and since $H^i(E(2)) = 0$, $i > 0$, so does Y_1. It follows that Y_2 is already general in H_2, and that H_2 is unirational.

Now to complete the proof it suffices to prove Y_2 is smooth. Since Y_2 is linearly normal and has μ_0 injective and is general in H_2, it would suffice to prove that Y_2 is *abstractly* smoothable, a purely local question. This would follow if we can show that F, a general quintic containing Y_1, has at most ordinary nodes as singularities, because then any curve on F (e.g. Y_2) is locally the sum of a Cartier divisor plus a smooth curve, hence is abstractly smoothable.

Since F is irreducible, it would sufficeto prove that F can be degenerated to $H \cup F_4$, when H is a plane and F_4 is a quartic with nodes only and transverse to H. For this, it follows from [3], § 3 (resp. the lemma below) that Y_1 can be degenerated to $Y^1 \cup D$ where D is a cubic on a plane H meeting Y^1 in 4 (resp. 5) points and it is easy to see that $H^0(I_{Y^1 \cup D}(5)) = [H]H^0(I_{Y^1}(4))$ has Dimension 3, so F degenerates to $H \cup F_4$ where F_4 is a general quartic containing Y^1, and it suffices to prove F_4 has nodes only. Again by [3] (resp. Lemma 3.1 below), Y^1 can be degenerated to $Y^2 \cup D'$ where D' is a 4-secant (resp. 5-secant) cubic of Y^2 on a general plane H', and $H^0(I_{Y^2 \cup D'}(4)) = [H'] H^0(I_{Y^2}(3))$ has dimension 3, so it suffices finally, to prove that a general cubic surface F_3 containing Y^2 is smooth. Now the latter is easy and classical: indeed it is easy to see that the Hilbert scheme corresponding to Y^2 is irreducible and a general Y^2 is the residual intersection of 2 cubics F_3, F_3' containing a conic (resp. a line), and F_3 is smooth.

Lemma 3.1. *A general curve Y in the component of $H_{14,18}$ constructed in Theorem 2.1 can be degenerated to $Y^2 \cup D' \cup D$, where Y^2 is a curve of degree 8, genus 8, and D, D' are plane cubic such that $\#(D \cap (Y^2 \cup D')) = \#(D' \cap Y^2) = 5$.*

Proof. Recall that in § 2, Y was constructed as the specialization of $Y_0 \cup C_1 \cup C_2 \cup C_3$ where Y_0 is of degree 8, genus 7, C_i's are conics such that $\#(C_1 \cap Y_0) = 4$, and $\#(C_i \cap (Y_0 \cup \bigcup_{j=1}^{i-1} C_j)) = 5$ for $i = 2, 3$. Now we choose Y_0 to be $Q \cup D$, where Q is a curve of degree 5, genus 2 with the 5-secant cubic D. Then choose a 3-secant line L_1 of Q, a plane $H_1 \supset L_1$ and a line $L_2 \subset H_1$ such that $\#(L_2 \cap Q) = 1$, and let $C_1 = L_1 \cup L_2$. Then choose $C_2 \subset H_2$ such that $\#(C_2 \cap Q) = 5$, and finally $H_3 \supset L_2$ and $C_3 \subset H_3$ such that $\#(C_3 \cap C_2) = 2$ and $\#(C_3 \cap Q) = 1$. (Note that $\#(C_3 \cap L_2) = 2$). It is easy to see that $D' = L_2 \cup C_3$ is a 5-secant cubic of $Y^2 = Q \cup L_1 \cup C_2$.

Lemma 3.2. *A general Y in the component of $H_{14,18}$ constructed in Theorem 2.1 can be degenerated to $\hat{Y} \cup C$, where $\hat{Y}$ is of degree 12, genus 14 with $C \subset H$ a 5-secant conic, and $\hat{Y} \cap H \setminus C$ is 7 generic points on H.*

Proof. We degenerate the degree 8, genus 7 curve Y_0 to an elliptic quartic plus two 4-secant conics. It is easy to see that Y_0 plus the 4-secant conic C_1 has 7 generic points on H as a subset of a general hyperplane section. Now take $\tilde{Y} = Y_0 \cup C_1 \cup C_2$, where C_2 is a 5-secant conic of $Y_0 \cup C_1$, and let C be the last 5-secant conic, intersecting C_1 in 1 point, and Y_0, C_2 in 2 points each.

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