

# Divisible modules over integral domains

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## 1. Introduction

The aim of this paper is to describe an equivalence between the full subcategory of  $\text{Mod-}R$  whose objects are all the divisible modules over an integral domain  $R$  and a suitable full subcategory of modules over the endomorphism ring  $E$  of a fixed divisible module  $\partial$ . This equivalence corresponds to the similar equivalences for torsion divisible abelian groups due to Harrison [6] and for torsion  $h$ -divisible modules over an integral domain due to Matlis [7], [8] and [9].

Let  $R$  denote a commutative integral domain with 1 (not a field) and let  $\partial_R$  denote the divisible right  $R$ -module defined by L. Fuchs in [3] (see § 2 for the exact definition of  $\partial_R$ ). The module  $\partial_R$  has interesting properties that are shown in [3], in [4, § VI.3] and in §§ 2 and 3 of this paper. For instance, if  $E$  is the endomorphism ring of  $\partial_R$  and  $\partial$  is viewed as a left  $E$ -module  ${}_E\partial$ , then  $\text{End}({}_E\partial) \cong R$  and  ${}_E\partial \cong E/I$  for a suitable projective principal left ideal  $I$  of  $E$ . Moreover,  $\partial$  has flat and projective dimensions equal to one both as a right  $R$ -module and a left  $E$ -module, and this implies that the class  $\mathcal{F}$  of all right  $E$ -modules  $M$  such that  $\text{Tor}_1^E(M, \partial) = 0$  is the torsion-free class for a (non-hereditary) torsion theory  $(\mathcal{T}, \mathcal{F})$  in  $\text{Mod-}E$ . This torsion theory is generated by the cyclic right  $E$ -module  $\text{Ext}_R^1({}_E\partial_R, R)$ , and a right  $E$ -module  $M_E$  is a torsion-free module in this torsion theory (we say that  $M_E$  is *I-torsion-free*) if and only if the canonical homomorphism  $M \otimes_E I \rightarrow M \otimes_E E \cong M$  induced by the embedding  $I \rightarrow E$  is a monomorphism. Dually, we say that a module  $M_E$  is an *I-divisible* module if the canonical homomorphism  $M \otimes_E I \rightarrow M$  is an epimorphism, and that a right  $E$ -module  $N_E$  is *I-reduced* if it is cogenerated by the right  $E$ -module  $\partial^* = \text{Hom}_R(\partial, C)$ , where  $C$  is the minimal injective cogenerator in  $\text{Mod-}R$ . It is easy to show that a module  $M_E$  is *I-divisible* if and only if  $\text{Hom}(M, N) = 0$  for every *I-reduced*  $E$ -module  $N_E$ .

Now define a right  $E$ -module  $M$  to be an *I-cotorsion module* if it is *I-reduced* and  $\text{Ext}_E^1(N, M) = 0$  for every *I-divisible* *I-torsion-free* right  $E$ -module  $N$ . The

main result of this paper is the proof of the following theorem: the functors  $\text{Hom}_R(\partial, -): \text{Mod-}R \rightarrow \text{Mod-}E$  and  $-\otimes_E \partial: \text{Mod-}E \rightarrow \text{Mod-}R$  induce an equivalence between the full subcategory of  $\text{Mod-}R$  whose objects are the divisible  $R$ -modules and the full subcategory of  $\text{Mod-}E$  whose objects are the  $I$ -cotorsion  $E$ -modules. This generalizes the corresponding results of Harrison for torsion divisible abelian groups [6] and of Matlis for torsion  $h$ -divisible  $R$ -modules ([7] and [9]). In our equivalence the injective  $R$ -modules correspond to the  $I$ -reduced  $I$ -pure-injective  $E$ -modules. Here  $I$ -pure-injective means injective relatively to the  $I$ -pure exact sequences, that is, the sequences  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  of right  $E$ -modules for which the sequence  $0 \rightarrow M' \otimes_E \partial \rightarrow M \otimes_E \partial \rightarrow M'' \otimes_E \partial \rightarrow 0$  is exact. (This extends the corresponding result due to Warfield for Matlis' equivalence between torsion  $h$ -divisible modules and torsion-free cotorsion modules, see [4, Th. V.1.8].) Our  $I$ -purity is a purity in the sense of Warfield [14].

Finally, we prove that  $I$ -cotorsion  $E$ -modules are exactly the right  $E$ -modules of  $\partial^*$ -dominant dimension  $\cong 2$ , that is, the modules  $M_E$  for which there exists an exact sequence  $0 \rightarrow M \rightarrow \partial^{*X} \rightarrow \partial^{*Y}$  with  $\partial^{*X}$  and  $\partial^{*Y}$  suitable direct products of copies of  $\partial^*$ .

For technical reasons (proof of Lemma 2.2) the way we define the  $R$ -module  $\partial$  is a little different from the way Fuchs defines it in [3] and [4]. The difference is that our generators are the  $k$ -tuples  $(r_1, \dots, r_k)$  of non-zero elements  $r_i$  of  $R$ , and Fuchs' generators are the  $k$ -tuples  $(r_1, \dots, r_k)$  of non-zero and *non-invertible* elements  $r_i$  of  $R$ . Fuchs' results in [3] and [4] hold with this small modification as well.

## 2. The $R$ -module $\partial_R$ and its endomorphism ring $E$

In this paper  $R$  will be an integral domain and we will assume that it is not a field. We will denote the field of fractions of  $R$  by  $Q$ .

Let  $\partial$  be the right  $R$ -module generated by the set  $\mathcal{G}$  of all  $k$ -tuples  $(r_1, \dots, r_k)$  of non-zero elements  $r_i$  of  $R$ , for  $k \geq 0$ , with defining relations

$$(r_1, \dots, r_k)r_k = (r_1, \dots, r_{k-1}), \quad k \geq 1.$$

The right  $R$ -module  $\partial$  is obviously divisible, that is,  $\partial r = \partial$  for every  $r \in R$ ,  $r \neq 0$ . The length of  $(r_1, \dots, r_k)$  is defined to be  $k$ , and the unique generator  $w = \emptyset$  in  $\mathcal{G}$  of length 0 generates a submodule  $wR$  of  $\partial$  isomorphic to  $R$  [4, § VI.3]. Note that for every  $x \in \partial$  there exists  $r \in R$ ,  $r \neq 0$ , such that  $xr \in wR$  (possibly  $xr = 0$ ).

The fundamental property of  $\partial$  is the following one:

**Proposition 2.1** [4, Lemma VI.3.2]. *Let  $D$  be a divisible right  $R$ -module and  $a \in D$ . Then there exists a homomorphism  $f: \partial \rightarrow D$  with  $f(w) = a$ .*

Let  $\partial_n$  be the submodule of  $\partial$  generated by the elements of  $\mathcal{G}$  of length  $\leq n$ , so that in particular  $\partial_0 = wR$ .

**Lemma 2.2.** *Fix a nonnegative integer  $n$  and an element  $a$  of  $R$ ,  $a \neq 0$  and  $a \neq 1$ . Then the correspondence  $\mathcal{G} \rightarrow \partial$  defined by*

$$(r_1, \dots, r_k) \in \mathcal{G} \mapsto \begin{cases} 0 & \text{if } k \leq n \\ (r_1, \dots, r_n, 1, r_{n+1}, \dots, r_k) - (r_1, \dots, r_n, a, r_{n+1}, \dots, r_k)a & \text{if } k > n \end{cases}$$

*extends to an endomorphism of  $\partial$  whose kernel is  $\partial_n$  and whose image is a direct summand of  $\partial$ .*

*Proof.* It is easy to show that the defining relations of  $\partial$  are preserved by the correspondence; for instance, when  $k = n + 1$ , the relation  $(r_1, \dots, r_k)r_k = (r_1, \dots, r_{k-1})$  is preserved because  $[(r_1, \dots, r_n, 1, r_{n+1}) - (r_1, \dots, r_n, a, r_{n+1})a]r_k = (r_1, \dots, r_n, 1) - (r_1, \dots, r_n, a)a = (r_1, \dots, r_n) - (r_1, \dots, r_n) = 0$ . Therefore the correspondence extends to an endomorphism  $\varphi$  of  $\partial$ . Note that  $\partial_n \subset \ker \varphi$  because  $\varphi(r_1, \dots, r_n) = 0$  for every  $(r_1, \dots, r_n)$ . In particular  $\varphi = \varphi' \circ \pi$  where  $\pi: \partial \rightarrow \partial/\partial_n$  is the canonical projection and  $\varphi': \partial/\partial_n \rightarrow \partial$  is a homomorphism.

Now consider the correspondence  $\mathcal{G} \rightarrow \partial/\partial_n$  defined by

$$(r_1, \dots, r_k) \in \mathcal{G} \mapsto \begin{cases} \partial_n & \text{if } k \leq n+1 \\ \partial_n & \text{if } k > n+1 \text{ and } r_{n+1} \neq 1 \\ ((r_1, \dots, r_n, \widehat{r_{n+1}}, r_{n+2}, \dots, r_k) + \partial_n) & \text{if } k > n+1 \text{ and } r_{n+1} = 1, \end{cases}$$

where  $(r_1, \dots, r_n, \widehat{r_{n+1}}, r_{n+2}, \dots, r_k)$  denotes the  $(k-1)$ -tuple in which  $r_{n+1}$  has been deleted. The defining relations of  $\partial$  are preserved by this correspondence as well; for instance, when  $k = n + 2$  and  $r_{n+1} = 1$ , the relation  $(r_1, \dots, r_k)r_k = (r_1, \dots, r_{k-1})$  is preserved because  $[(r_1, \dots, r_n, \widehat{r_{n+1}}, r_k) + \partial_n]r_k = (r_1, \dots, r_n) + \partial_n = \partial_n$ . Therefore this correspondence also extends to a homomorphism  $\psi: \partial \rightarrow \partial/\partial_n$ .

The composed homomorphism  $\psi\varphi: \partial \rightarrow \partial/\partial_n$  is defined by  $\psi\varphi(r_1, \dots, r_k) = \partial_n$  if  $k \leq n$  and  $\psi\varphi(r_1, \dots, r_k) = \psi[(r_1, \dots, r_n, 1, r_{n+1}, \dots, r_k) - (r_1, \dots, r_n, a, r_{n+1}, \dots, r_k)a] = (r_1, \dots, r_n, r_{n+1}, \dots, r_k) + \partial_n$  if  $k > n$ , i.e.,  $\psi\varphi: \partial \rightarrow \partial/\partial_n$  is the canonical projection  $\pi$ . Therefore  $\pi = \psi\varphi = \psi\varphi'\pi$ , hence  $\psi\varphi'$  is the identity of  $\partial/\partial_n$ , so that  $\varphi'$  is injective and  $\partial = \varphi'(\partial/\partial_n) \oplus \ker \psi$ . Since  $\varphi'$  is injective,  $\ker \varphi = \ker(\varphi'\pi) = \ker \pi = \partial_n$ . Moreover  $\varphi(\partial) = \varphi'(\partial/\partial_n)$  is a direct summand of  $\partial$ .

Fix the following notations:

- $E$  is the endomorphism ring  $\text{End}(\partial_R)$  of the  $R$ -module  $\partial_R$ ;
- $\varphi$  is a fixed  $R$ -endomorphism of  $\partial$  (i.e.,  $\varphi \in E$ ) with  $\ker \varphi = wR$  and  $\varphi(\partial)$  a direct summand of  $\partial$  (it exists by Lemma 2.2);

—  $\varepsilon$  is a fixed idempotent  $R$ -endomorphism of  $\partial$  (i.e.,  $\varepsilon \in E$  and  $\varepsilon^2 = \varepsilon$ ) with  $\varepsilon(\partial) = \varphi(\partial)$ ;

—  $I$  is the left ideal  $\{f \in E \mid f(w) = 0\}$  of  $E$ ;

—  $J$  is the two sided ideal  $\{f \in E \mid f(\partial) \subset t(\partial)\}$  of  $E$ , where  $t(\partial)$  denotes the torsion submodule of  $\partial$ .

Since  $R$  is a commutative ring and  $\partial_R$  is a faithful module, the ring  $R$  is a subring of the center  $Z(E)$  of  $E$ . In the next theorem we prove that  $R$  is equal to  $Z(E)$ .

**Theorem 2.3.** *The integral domain  $R$  is the center of  $E = \text{End}(\partial_R)$ .*

*Proof.* It is sufficient to show that if  $f$  belongs to the center of  $E$  then there exists  $r \in R$  such that  $f(x) = xr$  for every  $x \in \partial$ . If  $f$  is in the center of  $E$  and  $\varphi$  denotes the endomorphism defined before the statement of this theorem, then  $\varphi f(w) = f\varphi(w) = f(0) = 0$ , so that  $f(w) \in \ker \varphi = wR$ ; hence there exists  $r \in R$  with  $f(w) = wr$ . If  $x \in \partial$ , then there is a homomorphism  $g: \partial \rightarrow \partial$  with  $g(w) = x$  by Proposition 2.1, and  $f(x) = f(g(w)) = g(f(w)) = g(wr) = g(w)r = xr$ . This concludes the proof of the theorem.

If  $\alpha: \partial \rightarrow Q$  is the  $R$ -module homomorphism defined by  $\alpha(r_1, \dots, r_k) = (r_1 \dots r_k)^{-1}$  for  $k \geq 1$  and  $\alpha(w) = 1$ , then  $\ker \alpha$  is the torsion submodule  $t(\partial)$  of  $\partial$ . This is easily seen, because  $t(\partial) \subset \ker \alpha$  since  $Q$  is torsion-free, and if  $x \in \ker \alpha$  and  $r \in R$ ,  $r \neq 0$ , is such that  $xr \in wR$ ,  $xr = ws$  say, then  $0 = \alpha(xr) = \alpha(ws) = \alpha(w)s = s$ ; therefore  $xr = 0$  and  $x \in t(\partial)$ . In particular  $\partial/t(\partial) \cong Q$ .

If we apply the functor  $\text{Hom}_R(\partial, -)$  to the exact sequence  $0 \rightarrow t(\partial) \rightarrow \partial \xrightarrow{\alpha} Q \rightarrow 0$ , we obtain the exact sequence  $0 \rightarrow J \rightarrow E \rightarrow \text{Hom}_R(\partial, Q) \rightarrow \text{Ext}_R^1(\partial, t(\partial))$ . But  $\text{Hom}_R(\partial, Q) \cong \text{Hom}_R(\partial/t(\partial), Q) \cong \text{Hom}_R(Q, Q) \cong Q$  and  $\text{Ext}_R^1(\partial, t(\partial)) = 0$  because  $t(\partial)$  is a divisible  $R$ -module [4, Prop. VI.3.4]. Hence  $E/J \cong Q$  and  $J$  is an ideal of  $E$  maximal among the two sided ideals of  $E$ .

Note that the left annihilator of  $\varphi$ ,  $l(\varphi) = \{g \in E \mid g\varphi = 0\}$ , is  $E(1 - \varepsilon)$ . In fact,  $(1 - \varepsilon)\varphi = 0$  because  $\varepsilon(\partial) = \varphi(\partial)$ , so that  $E(1 - \varepsilon) \subset l(\varphi)$ . And if  $g \in l(\varphi)$ , then  $g\varphi = 0$ , i.e.,  $\ker g \supset \varphi(\partial) = \varepsilon(\partial)$ ; it follows that  $g\varepsilon = 0$  and  $g = g - g\varepsilon = g(1 - \varepsilon) \in E(1 - \varepsilon)$ . The right annihilator of  $\varphi$ ,  $r(\varphi) = \{g \in E \mid \varphi g = 0\}$ , is 0, because if  $\varphi g = 0$ , then  $g(\partial) \subset \ker \varphi = wR$ . Since  $g(\partial)$  is a divisible module, it must be the zero submodule of  $wR$ , i.e.,  $g = 0$ .

**Theorem 2.4.** *If  $B_R$  is any right  $R$ -module and  $f: \partial \rightarrow B$  is a homomorphism such that  $f(w) = 0$ , then there exists  $g: \partial \rightarrow B$  such that  $f = g\varphi$ . In particular,  $I = \{f \in E \mid f(w) = 0\}$  is the left principal ideal  $E\varphi$  generated by  $\varphi$  and is a projective ideal of  $E$  isomorphic to  $E\varepsilon$ .*

*Proof.* Since  $\ker \varphi = wR$  and  $\varphi(\partial)$  is a direct summand of  $\partial$ , there exists  $\psi: \partial \rightarrow \partial/wR$  such that  $\psi\varphi$  is the canonical projection  $\pi: \partial \rightarrow \partial/wR$  (this had been

also shown in the proof of Lemma 2.2). Since  $f: \partial \rightarrow B$  annihilates  $w$ ,  $f$  can be written as  $f=f'\pi$  for a suitable  $f': \partial/wR \rightarrow B$  induced by  $f$ . If  $g=f'\psi: \partial \rightarrow B$ , then  $f=f'\pi=f'\psi\varphi=g\varphi$ . This proves the first assertion.

In particular,  $I=\{f \in E \mid f(w)=0\} \subset \{g\varphi \mid g \in \text{Hom}_R(\partial, \partial)\}=E\varphi$ , so that  $I=E\varphi$ , the other inclusion being trivial.

Finally, since  $l(\varphi)=E(1-\varepsilon)=l(\varepsilon)$ , the ideal  $I=E\varphi \cong E\varepsilon$  is projective.

### 3. The $E$ -modules ${}_E\partial$ and $\partial_E^\circ$

Since  $E=\text{End}(\partial_R)$ , the module  $\partial$  can be viewed as a left  $E$ -module, and  $R=\text{End}({}_E\partial)$  by Theorem 2.3. In this section we shall study the  $E$ -module  ${}_E\partial$ .

**Lemma 3.1.** *The left  $E$ -module  ${}_E\partial$  is isomorphic to  $E/I$ .*

*Proof.* Consider the mapping  $E \rightarrow \partial$  defined by  $f \mapsto f(w)$  for every  $f \in E$ . Obviously it is a left  $E$ -module homomorphism. It is surjective by proposition 2.1 and its kernel is  $I$ .

Fuchs [4, Lemma VI.3.1] has proved that the projective dimension of  $\partial_R$ ,  $\text{proj. dim } \partial_R$ , is equal to one (this can also be shown by proving that the relations  $(r_1, \dots, r_k)r_k - (r_1, \dots, r_{k-1})$  generate a free submodule  $H$  of the module  $F$  freely generated by  $\mathcal{G}$ ); since  $\partial_R$  is not flat (every flat  $R$ -module is torsion-free, and  $\partial_R$  is not torsion-free) and  $\text{proj. dim } \partial_R \cong \text{flat. dim } \partial_R$ , where  $\text{flat. dim } \partial_R$  is the flat dimension of  $\partial_R$ , it follows that  $\text{flat. dim } \partial_R = \text{proj. dim } \partial_R = 1$ . This holds for the module  ${}_E\partial$  too.

**Corollary 3.2.**  $\text{flat. dim } {}_E\partial = \text{proj. dim } {}_E\partial = 1$ .

*Proof.* By Lemma 3.1 and Theorem 2.4  $\text{proj. dim } {}_E\partial \leq 1$ . If  $\text{proj. dim } {}_E\partial < 1$ , then  ${}_E\partial$  is projective, so that  $I=E\varphi$  is a direct summand of  $E$ , i.e.,  $E\varphi=E\beta$  for an idempotent  $\beta \in E$ . Then  $wR = \cap \{\ker f \mid f \in E\varphi\} = \cap \{\ker f \mid f \in E\beta\} = \ker \beta$  is a direct summand of the divisible module  $\partial_R$ , contradiction, because  $wR$  is not divisible. This proves that  $\text{proj. dim } {}_E\partial = 1$ . Moreover  $\text{flat. dim } {}_E\partial \leq \text{proj. dim } {}_E\partial = 1$ , and  ${}_E\partial$  is not flat, because  ${}_E\partial$  is finitely presented by Lemma 3.1 and Theorem 2.4 and every finitely presented flat module is projective [13, Cor. I.11.5]. Therefore  $\text{flat. dim } {}_E\partial = 1$ .

By Corollary 3.2  $\text{Tor}_n^E(-, {}_E\partial) = \text{Ext}_E^n({}_E\partial, -) = 0$  for  $n \geq 2$ . In the sequel we need the exact formulas for the functors  $\text{Tor}_1^E(-, {}_E\partial)$  and  $\text{Ext}_E^1({}_E\partial, -)$  that are calculated in the next corollary.

**Corollary 3.3.** *If  $M_E$  is any right  $E$ -module, then  $\text{Tor}_1^E(M, \partial) \cong (0:{}_M\varphi)\varepsilon$ , where  $(0:{}_M\varphi) = \{x \in M \mid x\varphi = 0\}$ .*

If  ${}_E N$  is any left  $E$ -module, then  $\text{Ext}_E^1(\partial, N) \cong \varepsilon N / \varphi N$ .

*Proof.* Consider the exact sequence  $0 \rightarrow I \rightarrow E \rightarrow \partial \rightarrow 0$ . By applying the functor  $M \otimes_E -$ , we obtain that the sequence  $0 \rightarrow \text{Tor}_1^E(M, \partial) \rightarrow M \otimes I \rightarrow M \otimes E$  is exact. Since  $I = E\varphi \cong E\varepsilon$  and  $M \otimes_E E \cong M$ , it follows that  $\text{Tor}_1^E(M, \partial)$  is isomorphic to the kernel of the abelian group homomorphism  $M\varepsilon \rightarrow M$  defined by  $x\varepsilon \rightarrow x\varphi$  for every  $x \in M$ . It follows that  $\text{Tor}_1^E(M, \partial) \cong (0 :_{M\varphi})\varepsilon$ . Similarly for  $\text{Ext}_E^1(\partial, N)$ .

Note that since  $\text{proj. dim } \partial_R = 1$ , the torsion submodule  $t(\partial_R)$  of  $\partial_R$  is isomorphic to a submodule of  $K^{(X)}$ , where  $K = Q/R$  and  $K^{(X)}$  is a direct sum of copies of  $K$ . Namely, if  $M_R$  is any module with  $\text{proj. dim } M_R = 1$ , fix a free resolution  $0 \rightarrow R^{(X)} \rightarrow R^{(Y)} \rightarrow M \rightarrow 0$  of  $M$  (this is possible by [10, page 90, Ex. 3]) and apply the functor  $- \otimes_R K$  to this sequence. Then the sequence  $\text{Tor}_R^1(R^{(Y)}, K) \rightarrow \text{Tor}_R^1(M, K) \rightarrow R^{(X)} \otimes K \rightarrow R^{(Y)} \otimes K$  can be rewritten as  $0 \rightarrow t(M) \rightarrow K^{(X)} \rightarrow K^{(Y)}$  by [8, page 10].

Since  $\text{proj. dim } \partial_R = 1$ , it follows that  $\text{Ext}_R^n(\partial, -) = 0$  for  $n \geq 2$ . Consider  $\text{Ext}_R^1(\partial, R)$ . Since  $\text{Ext}_R^1(-, R)$  is a contravariant functor, every  $R$ -homomorphism  $f: \partial \rightarrow \partial$  induces an  $R$ -homomorphism  $\text{Ext}_R^1(f, R): \text{Ext}_R^1(\partial, R) \rightarrow \text{Ext}_R^1(\partial, R)$ , so that  $\text{Ext}_R^1(\partial, R)$  is a right  $E$ -module.

**Theorem 3.4.** *The right  $E$ -module  $\text{Ext}_R^1(\partial, R)$  is isomorphic to  $\varepsilon E / \varphi E$ .*

*Proof.* Let  $C$  be the image of the endomorphism  $1 - \varepsilon$  of  $\partial$ , so that  $\partial = \varepsilon(\partial) \oplus (1 - \varepsilon)(\partial) = \varphi(\partial) \oplus C$ . Consider the exact sequence of  $R$ -modules

$$S: 0 \rightarrow R \xrightarrow{\alpha} \partial \oplus C \xrightarrow{\beta} \partial \rightarrow 0,$$

where  $\alpha(r) = (wr, 0)$  for every  $r \in R$  and  $\beta(x, y) = \varphi(x) + y$  for every  $(x, y) \in \partial \oplus C$ . Let  $\bar{S}$  be the image of the extension  $S$  into  $\text{Ext}_R^1(\partial, R)$ . In order to prove the theorem it is sufficient to show that  $\Phi: \varepsilon E \rightarrow \text{Ext}_R^1(\partial, R)$  defined by  $\Phi(\varepsilon f) = \bar{S}f$  for every  $f \in E$  is a well defined surjective  $E$ -homomorphism with kernel  $\varphi E$ .

If  $f \in E$  and  $\varepsilon f = 0$ , then  $f(\partial) \subset \ker \varepsilon = C$ , so that it is possible to define a homomorphism  $g: R \oplus \partial \rightarrow \partial \oplus C$  by setting  $g(r, x) = (wr, f(x))$  for every  $(r, x) \in R \oplus \partial$ . If  $Z$  denotes the trivial extension, the diagram

$$\begin{array}{ccccccc} Z: & 0 & \rightarrow & R & \rightarrow & R \oplus \partial & \rightarrow & \partial & \rightarrow & 0 \\ & & & \parallel & & \downarrow g & & \downarrow f & & \\ S: & 0 & \rightarrow & R & \xrightarrow{\alpha} & \partial \oplus C & \xrightarrow{\beta} & \partial & \rightarrow & 0 \end{array}$$

commutes. This shows that  $\bar{S}f$  is zero in  $\text{Ext}_R^1(\partial, R)$  and proves that  $\Phi$  is a well defined homomorphism of right  $E$ -modules.

Now we shall show that  $\Phi$  is surjective. Let

$$T: 0 \rightarrow R \xrightarrow{\gamma} A \xrightarrow{\delta} \partial \rightarrow 0$$

be any extension and  $\bar{T}$  its image into  $\text{Ext}_R^1(\partial, R)$ . Since  $\text{Ext}_R^1(\partial, \partial) = 0$  [4, Prop. VI.3.4], the  $R$ -homomorphism  $\gamma^*: \text{Hom}_R(A, \partial) \rightarrow \text{Hom}_R(R, \partial)$  is surjective. Hence

there exists  $\chi \in \text{Hom}_R(A, \partial)$  such that  $(\gamma^*(\chi))(1) = w$ , that is,  $\chi(\gamma(1)) = w$ . Define  $h: A \rightarrow \partial \oplus C$  by  $h(a) = (\chi(a), 0)$  for every  $a \in A$ . Then  $h(\gamma(1)) = (\chi(\gamma(1)), 0) = (w, 0) = \alpha(1)$ , so that  $h\gamma = \alpha$  and the left-hand square in the diagram

$$\begin{array}{ccccccc} T: & 0 & \rightarrow & R & \xrightarrow{\gamma} & A & \xrightarrow{\delta} & \partial & \rightarrow & 0 \\ & & & \parallel & & \downarrow h & & \downarrow f & & \\ S: & 0 & \rightarrow & R & \xrightarrow{\alpha} & \partial \oplus C & \xrightarrow{\beta} & \partial & \rightarrow & 0 \end{array}$$

commutes; it follows that there exists an  $f \in E$  making the right-hand square commute. Then  $\bar{S}f = \bar{T}$ , and  $\Phi$  is surjective.

In order to prove that  $\ker \Phi = \varphi E$ , fix an  $f \in E$ , so that  $\varepsilon f \in \varepsilon E$ . Then  $\varepsilon f \in \ker \Phi$  if and only if  $\bar{S}f = \bar{Z}$ , i.e., if and only if there exists a homomorphism  $g: R \oplus \partial \rightarrow \partial \oplus C$  making the diagram

$$\begin{array}{ccccccc} Z: & 0 & \rightarrow & R & \rightarrow & R \oplus \partial & \rightarrow & \partial & \rightarrow & 0 \\ & & & \parallel & & \downarrow g & & \downarrow f & & \\ S: & 0 & \rightarrow & R & \xrightarrow{\alpha} & \partial \oplus C & \xrightarrow{\beta} & \partial & \rightarrow & 0 \end{array}$$

commute. This means that  $g(r, 0) = (wr, 0)$  and  $\beta g(r, x) = f(x)$  for every  $r \in R$  and  $x \in \partial$ . Since the homomorphisms  $g: R \oplus \partial \rightarrow \partial \oplus C$  such that  $g(r, 0) = (wr, 0)$  for every  $r \in R$  are exactly of the form  $g(r, x) = (wr + h(x), l(x))$  for suitable  $h: \partial \rightarrow \partial$  and  $l: \partial \rightarrow C$ , it follows that  $\varepsilon f \in \ker \Phi$  if and only if there exists  $h: \partial \rightarrow \partial$  and  $l: \partial \rightarrow C$  such that  $f(x) = \beta g(r, x) = \beta(wr + h(x), l(x)) = \varphi(wr + h(x)) + l(x) = (\varphi h + l)(x)$ , i.e.,  $f - \varphi h = l$ . But  $C = (1 - \varepsilon)(\partial) = \ker \varepsilon$ , so that  $\varepsilon f \in \ker \Phi$  if and only if  $\varepsilon(f - \varphi h) = 0$  for some  $h: \partial \rightarrow \partial$ , i.e.,  $\varepsilon f = \varepsilon \varphi h = \varphi h \in \varphi E$ . This proves that  $\ker \Phi = \varphi E$ .

We shall often need the right  $E$ -module  $\text{Ext}_R^1(\partial, R)$  in the sequel, and we shall denote it by  $\partial^\circ$ . Hence  $\partial^\circ = \text{Ext}_R^1(\partial, R) \cong \varepsilon E / \varphi E$  as a right  $E$ -module. There are other "presentations" of the module  $\partial^\circ$ . For instance the right  $E$ -modules  $\partial^\circ$  and  $\text{Ext}_E^1(\partial, E)$  are isomorphic right  $E$ -modules by Corollary 3.3. Moreover the functor  $\text{Hom}_R({}_E\partial_R, -)$  applied to the exact sequence of  $R$ -modules  $0 \rightarrow wR \rightarrow \partial \rightarrow \partial/wR \rightarrow 0$  gives the exact sequence of right  $E$ -modules  $0 \rightarrow E \rightarrow \text{Hom}_R(\partial, \partial/wR) \rightarrow \text{Ext}_R^1(\partial, wR) \rightarrow \text{Ext}_R^1(\partial, \partial)$ . The last module is zero by [4, Prop. VI.3.4], so that the right  $E$ -modules  $\partial^\circ \cong \text{Ext}_R^1(\partial, wR)$  and  $\text{Hom}_R(\partial, \partial/wR)/E$  are isomorphic.

Furthermore, the functor  $\text{Hom}_R({}_E\partial_R, -)$  applied to the exact sequence of  $R$ -modules  $0 \rightarrow R \rightarrow Q \rightarrow K \rightarrow 0$  gives the exact sequence of right  $E$ -modules  $0 \rightarrow \text{Hom}_R(\partial, Q) \rightarrow \text{Hom}_R(\partial, K) \rightarrow \text{Ext}_R^1(\partial, R) \rightarrow \text{Ext}_R^1(\partial, Q)$ . The last module is zero by [4, Prop. VI.3.4], and the first module is  $\text{Hom}_R(\partial, Q) \cong \text{Hom}_R(\partial/t(\partial), Q) \cong Q$  by the remarks after proposition 2.3. Therefore  $\partial^\circ \cong \text{Hom}_R(\partial, K)/Q$  as  $E$ -modules.

If we are only interested in the structure of  $\partial^\circ$  as an  $R$ -module, there is one more "presentation" of  $\partial^\circ$ : the functor  $\text{Hom}_R(-, R)$  applied to the exact sequence  $0 \rightarrow H \rightarrow F \rightarrow \partial \rightarrow 0$  (where  $F$  is the  $R$ -module freely generated by  $\mathcal{G}$  and  $H$  is the free submodule of  $F$  generated by the relations) gives  $0 \rightarrow \text{Hom}_R(F, R) \rightarrow \text{Hom}_R(H, R) \rightarrow$

$\text{Ext}_R^1(\partial, R) \rightarrow 0$ , which is a presentation of  $\partial^\circ$  as a quotient of two  $R$ -modules isomorphic to direct products of copies of  $R$ .

**Corollary 3.5.**  $\text{flat. dim } \partial_E^\circ = \text{proj. dim } \partial_E^\circ = 1$ .

*Proof.* Since  $r(\varphi) = 0$ , it follows that  $\varphi E \cong E$  is projective, so that  $\partial^\circ \cong \varepsilon E / \varphi E$  has projective dimension  $\cong 1$ . Hence  $1 \cong \text{proj. dim } \partial^\circ \cong \text{flat. dim } \partial^\circ$ . It remains to prove that  $\varepsilon E / \varphi E$  is not flat. But  $\varepsilon + \varphi E \in \varepsilon E / \varphi E$  is annihilated by  $\varphi$  (because  $\varepsilon\varphi = \varphi$ ) so that it belongs to  $(0 : \varphi)\varepsilon \cong \text{Tor}_1^E(\partial^\circ, \partial)$  (Corollary 3.3). Thus  $\text{Tor}_1^E(\partial^\circ, \partial) \neq 0$  and  $\partial^\circ$  is not flat.

**Theorem 3.6.**  $\text{End}(\partial_E^\circ) \cong R$ .

*Proof.* First of all observe that  $\partial^\circ$  is a torsion-free  $R$ -module, because if  $r \in R$  and  $r \neq 0$ , the functor  $\text{Hom}_R(\partial, -)$  applied to the exact sequence  $0 \rightarrow R \xrightarrow{r} R \rightarrow R/rR \rightarrow 0$  gives the exact sequence  $\text{Hom}_R(\partial, R/rR) \rightarrow \partial^\circ \xrightarrow{r} \partial^\circ$ . The first module is zero because  $\partial$  is divisible and  $R/rR$  is torsion of bounded order. Hence the multiplication by  $r$  is an injective endomorphism of  $\partial^\circ$ , and  $\partial^\circ$  is a torsion-free  $R$ -module.

Since  $\partial^\circ \cong \varepsilon E / \varphi E \cong E / (\varphi E + (1 - \varepsilon)E)$  is a cyclic  $E$ -module, it follows that  $\text{End}_E(\partial^\circ) \cong U / (\varphi E + (1 - \varepsilon)E)$ , where  $U$  is the subring  $\{f \in E \mid f(\varphi E + (1 - \varepsilon)E) \subset \varphi E + (1 - \varepsilon)E\}$  of  $E$  (for instance see [10, page 24]). Similarly, since  $\partial \cong E / E\varphi$ , the ring  $\text{End}_E(\partial)$  is isomorphic to  $V / E\varphi$ , where  $V = \{g \in E \mid E\varphi g \subset E\varphi\}$ . But  $\text{End}_E(\partial)$  is canonically isomorphic to  $R$  (Theorem 2.3), and thus  $V = R + E\varphi$ .

Now we prove that  $U = R + \varphi E + (1 - \varepsilon)E$ . The inclusion  $U \supset R + \varphi E + (1 - \varepsilon)E$  is trivial. Conversely, if  $f \in U$ , that is,  $f \in E$  and  $f(\varphi E + (1 - \varepsilon)E) \subset \varphi E + (1 - \varepsilon)E$ , then  $\varepsilon f \varphi \in \varepsilon(\varphi E + (1 - \varepsilon)E) = \varphi E$ . Therefore  $\varepsilon f \varphi = \varphi g$  for some  $g \in E$ . In particular  $E\varphi g = E\varepsilon f \varphi \subset E\varphi$ , that is,  $g \in V = R + E\varphi$ . Hence  $g = r + h\varphi$  for some  $r \in R$  and  $h \in E$ , and  $\varepsilon f \varphi = \varphi g = \varphi(r + h\varphi) = (r + \varphi h)\varphi$ . Then  $(\varepsilon f - r - \varphi h)\varphi = 0$ , and since  $l(\varphi) = E(1 - \varepsilon) = l(\varepsilon)$  (§ 2), we have  $(\varepsilon f - r - \varphi h)\varepsilon = 0$ , so that  $\varepsilon f \varepsilon = r\varepsilon - \varphi h \varepsilon = r - (1 - \varepsilon)r - \varphi h \varepsilon \in R + (1 - \varepsilon)E + \varphi E$ . Moreover  $f \in U$  implies  $f(1 - \varepsilon) \in \varphi E + (1 - \varepsilon)E$ , so that  $f = f(1 - \varepsilon) + (1 - \varepsilon)f \varepsilon + \varepsilon f \varepsilon \in (\varphi E + (1 - \varepsilon)E) + (1 - \varepsilon)E + (R + (1 - \varepsilon)E + \varphi E) = R + \varphi E + (1 - \varepsilon)E$ . This proves that  $U = R + \varphi E + (1 - \varepsilon)E$ .

It follows that

$$\begin{aligned} \text{End}_E(\partial^\circ) &\cong U / (\varphi E + (1 - \varepsilon)E) = (R + \varphi E + (1 - \varepsilon)E) / (\varphi E + (1 - \varepsilon)E) \\ &\cong R / (R \cap (\varphi E + (1 - \varepsilon)E)), \end{aligned}$$

i.e., every element of  $\text{End}_E(\partial^\circ)$  is induced by the multiplication by an element of  $R$ . But  $\partial^\circ$  is a torsion-free  $R$ -module, so that  $\text{End}_E(\partial^\circ) \cong R$ .

4. The functors  $\text{Hom}_R(\partial, -)$  and  $- \otimes_E \partial$

Consider the two functors  $\text{Hom}_R({}_E\partial_R, -): \text{Mod-}R \rightarrow \text{Mod-}E$  and  $- \otimes_E \partial_R: \text{Mod-}E \rightarrow \text{Mod-}R$ . Then  $\text{Hom}_R({}_E\partial_R, -)$  is the right adjoint of  $\otimes_E \partial_R$ , for each  $M \in \text{Mod-}E$  there is a canonical  $E$ -module homomorphism

$$\eta_M: M \rightarrow \text{Hom}_R(\partial, M \otimes_E \partial)$$

defined by  $\eta_M(m)(x) = m \otimes x$  for every  $m \in M$  and  $x \in \partial$  (the unit of the adjunction), and for each  $A \in \text{Mod-}R$  there is a canonical  $R$ -module homomorphism  $\varepsilon_A: \text{Hom}_R(\partial, A) \otimes_E \partial \rightarrow A$  defined by  $\varepsilon_A(f \otimes x) = f(x)$  for every  $f \in \text{Hom}_R(\partial, A)$  and  $x \in \partial$  (the counit of the adjunction).

Note that if  $M_E$  is any  $E$ -module, the  $R$ -module  $M \otimes_E \partial$  is divisible (because  $\partial_R$  is divisible and  $- \otimes_E \partial_R$  is right exact). Hence  $- \otimes_E \partial$  is a functor of  $\text{Mod-}E$  into the full subcategory  $\mathcal{D}_R$  of  $\text{Mod-}R$  whose objects are the divisible  $R$ -modules.

**Theorem 4.1.** *Let  $A_R$  be a right  $R$ -module. Then  $\varepsilon_A: \text{Hom}_R(\partial, A) \otimes_E \partial \rightarrow A$  is an isomorphism if and only if  $A$  is a divisible  $R$ -module.*

*Proof.* If  $\varepsilon_A$  is an isomorphism and  $F_E \rightarrow \text{Hom}_R(\partial, A)$  is a surjective  $E$ -homomorphism of a free  $E$ -module  $F_E$  onto  $\text{Hom}_R(\partial, A)$ , then  $F \otimes_E \partial \rightarrow \text{Hom}_R(\partial, A) \otimes_E \partial$  is a surjective  $R$ -homomorphism of the  $R$ -module  $F \otimes_E \partial$  onto  $\text{Hom}_R(\partial, A) \otimes_E \partial \cong A$ . Hence  $A$ , homomorphic image of the divisible  $R$ -module  $F \otimes_E \partial$ , is divisible.

Conversely, suppose  $A_R$  divisible and apply the functor  $\text{Hom}_R(\partial, A) \otimes_E -$  to the exact sequence  $0 \rightarrow E\varphi \rightarrow E \rightarrow \partial \rightarrow 0$ , where the first homomorphism is the inclusion and the second is defined by  $1 \mapsto w$  (Theorem 2.4 and Lemma 3.1). The first homomorphism in the obtained sequence

$$\text{Hom}_R(\partial, A) \otimes_E E\varphi \rightarrow \text{Hom}_R(\partial, A) \rightarrow \text{Hom}_R(\partial, A) \otimes_E \partial \rightarrow 0$$

is induced by the multiplication, so that its image is  $\{g\varphi | g \in \text{Hom}_R(\partial, A)\}$ , which is equal to  $B = \{f | f \in \text{Hom}_R(\partial, A), f(w) = 0\}$  by Theorem 2.4.

The homomorphism  $\chi: \text{Hom}_R(\partial, A) \rightarrow A$  defined by  $\chi(f) = f(w)$  for every  $f \in \text{Hom}_R(\partial, A)$  is surjective by proposition 2.1 because  $A$  is divisible, and has  $B$  as its kernel. Moreover the diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & B & \rightarrow & \text{Hom}_R(\partial, A) & \rightarrow & \text{Hom}_R(\partial, A) \otimes_E \partial \rightarrow 0 \\ & & \parallel & & \parallel & & \downarrow \varepsilon_A \\ 0 & \rightarrow & B & \rightarrow & \text{Hom}_R(\partial, A) & \xrightarrow{\chi} & A \rightarrow 0 \end{array}$$

commutes, because  $\chi(f) = f(w) = \varepsilon_A(f \otimes w)$  for every  $f \in \text{Hom}_R(\partial, A)$ . It follows that  $\varepsilon_A$  is an isomorphism.

If  $\mathcal{D}_R$  denotes the full subcategory of  $\text{Mod-}R$  whose objects are the divisible modules, the functor  $\text{Hom}_R(\partial, -): \mathcal{D}_R \rightarrow \text{Mod-}E$  is full and faithful by Theorem 4.1

[11, prop. 5.2], so that  $\mathcal{D}_R$  is equivalent to the full subcategory  $\mathcal{S}_E$  of  $\text{Mod-}E$  whose objects are the  $E$ -modules isomorphic to  $\text{Hom}_R(\partial, A)$  for some  $A \in \text{Mod-}R$ .

In the next sections we shall study and characterize the right  $E$ -modules isomorphic to  $\text{Hom}_R(\partial, A)$  for some  $A \in \text{Mod-}R$ . In order to do this we shall often need the following result.

**Proposition 4.2.** *For every  $R$ -module  $A_R$ ,  $\text{Tor}_1^E(\text{Hom}_R(\partial, A), {}_E\partial) = 0$ .*

*Proof.* By Corollary 3.3 we must show that  $(0: \varphi)\varepsilon = 0$ , where  $(0: \varphi) = \{f \in \text{Hom}_R(\partial_R, A) \mid f\varphi = 0\}$ . Now  $f\varphi = 0$  if and only if  $\varphi(\partial) \subset \ker f$ . But  $\varphi(\partial) = \varepsilon(\partial)$ . Hence if  $f \in (0: \varphi)$ , then  $\varepsilon(\partial) \subset \ker f$ , so that  $f\varepsilon = 0$ . This concludes the proof of the proposition.

**Theorem 4.3.** *Let  $\mathcal{S}$  be the class of all right  $E$ -modules isomorphic to  $\text{Hom}_R(\partial, A)$  for some right  $R$ -module  $A$ . Let  $0 \rightarrow L_E \rightarrow M_E \rightarrow N_E \rightarrow 0$  be a short exact sequence of right  $E$ -modules.*

- (i) *If  $L, N \in \mathcal{S}$ , then  $M \in \mathcal{S}$ .*
- (ii) *If  $M, N \in \mathcal{S}$ , then  $L \in \mathcal{S}$ .*
- (iii) *If  $L, M \in \mathcal{S}$  and  $\text{Tor}_1^E(N, \partial) = 0$ , then  $N \in \mathcal{S}$ .*

*Proof.* In all of the three cases  $\text{Tor}_1^E(N, \partial) = 0$  by proposition 4.2. Hence the functor  $-\otimes_E \partial$  applied to the sequence of the statement of the theorem gives the exact sequence  $0 \rightarrow L \otimes \partial \rightarrow M \otimes \partial \rightarrow N \otimes \partial \rightarrow 0$ . The functor  $\text{Hom}_E(\partial, -)$  applied to this sequence and the naturality of the transformation  $\eta$  give the commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N \longrightarrow 0 \\
 & & \downarrow \eta_L & & \downarrow \eta_M & & \downarrow \eta_N \\
 0 & \rightarrow & \text{Hom}_R(\partial, L \otimes_E \partial) & \rightarrow & \text{Hom}_R(\partial, M \otimes_E \partial) & \rightarrow & \text{Hom}_R(\partial, N \otimes_E \partial) \rightarrow 0.
 \end{array}$$

The second row in this diagram is exact because  $\text{Ext}_R^1(\partial, L \otimes_E \partial) = 0$  by [4, Prop. VI.3.4]. Hence if two of the mappings  $\eta_L, \eta_M, \eta_N$  are isomorphisms, so is the third. It remains to prove that for a module  $P_E$  the mapping  $\eta_P: P \rightarrow \text{Hom}_R(\partial, P \otimes \partial)$  is an isomorphism if and only if  $P \in \mathcal{S}$ . But if  $P \in \mathcal{S}$ , then the functors  $-\otimes_E \partial$  and  $\text{Hom}_R(\partial, -)$  give an equivalence  $\mathcal{D} \rightarrow \mathcal{S}$ , so that  $\eta_P$  is an isomorphism. And if  $P \cong \text{Hom}_R(\partial, P \otimes \partial)$ , then  $P \cong \text{Hom}_R(\partial, A) \in \mathcal{S}$  with  $A = P \otimes \partial$ .

The hypothesis  $\text{Tor}_1^E(N, \partial) = 0$  in part (iii) of Theorem 4.3 cannot be eliminated as the following example shows: set  $L = M = E$  and let  $r$  be any non-zero and non-invertible element of  $R$ . Since  $E = \text{Hom}_R(\partial, \partial)$  is a torsion-free  $R$ -module (because  $\partial$  is divisible), the multiplication by  $r$  gives an exact sequence  $0 \rightarrow E \rightarrow E \rightarrow E/Er \rightarrow 0$  of  $E$ -modules. In this sequence the first two modules are in  $\mathcal{S}$  and the third  $E$ -module  $E/Er$  is torsion of bounded order as an  $R$ -module. But  $E \neq Er$ ,

otherwise  $r$  would be invertible in  $E$ , that is,  $1=fr$  for some  $f \in E$ , contradiction, because the multiplication by  $r$  is not an injective mapping  $\partial \rightarrow \partial$ . Hence  $E/Er \neq 0$  is not a torsion-free  $R$ -module, and in particular  $E/Er \notin \mathcal{T}$  (every module in  $\mathcal{T}$  is torsion-free as an  $R$ -module).

### 5. The torsion theory $(\mathcal{T}, \mathcal{F})$ and its cotorsion theory

In this section  $S$  is an arbitrary associative ring with identity and  $I=S\varphi$  is a projective principal left ideal of  $S$ .

If  $M_S$  is any right  $S$ -module, the inclusion  $I \rightarrow S$  induces a homomorphism  $M \otimes_S I \rightarrow M$ , and we say that  $M$  is  $I$ -torsion-free if this mapping  $M \otimes_S I \rightarrow M$  is injective, and say that  $M$  is  $I$ -divisible if it is surjective. Note that the definition of  $I$ -divisible module is obtained by dualizing the definition of  $I$ -torsion-free module. Moreover  $M$   $I$ -divisible simply means  $M\varphi=M$ .

Denote the class of all  $I$ -torsion-free right  $S$ -modules by  $\mathcal{F}$ .

**Lemma 5.1.** *If  $S$  is an algebra over a commutative ring  $R$ ,  $C$  is an injective cogenerator in  $\text{Mod-}R$ ,  $(S/I)^*$  is the right  $S$ -module  $\text{Hom}_R(S/I, C)$ , and  $M$  is a right  $S$ -module, then*

(i)  $M$  is  $I$ -torsion-free if and only if  $\text{Tor}_1^S(M, S/I)=0$ , if and only if

$$\text{Ext}_S^1(M, (S/I)^*) = 0;$$

(ii)  $M$  is  $I$ -divisible if and only if  $M \otimes_S (S/I)=0$ .

*Proof.* From the exact sequence  $0 \rightarrow I \rightarrow S \rightarrow S/I \rightarrow 0$  we obtain the exact sequence  $0 \rightarrow \text{Tor}_1^S(M, S/I) \rightarrow M \otimes_S I \rightarrow M \rightarrow M \otimes_S (S/I) \rightarrow 0$ . Hence  $M$  is  $I$ -torsion-free if and only if  $\text{Tor}_1^S(M, S/I)=0$ , and  $M$  is  $I$ -divisible if and only if  $M \otimes_S (S/I)=0$ . Moreover  $\text{Hom}_R(\text{Tor}_1^S(M, S/I), C) \cong \text{Ext}_S^1(M, (S/I)^*)$ , so that  $\text{Tor}_1^S(M, S/I)=0$  if and only if  $\text{Ext}_S^1(M, (S/I)^*)=0$ .

**Proposition 5.2.** *The class  $\mathcal{F}$  is the torsion-free class for a torsion theory  $(\mathcal{T}, \mathcal{F})$ .*

*Proof.* We must show that  $\mathcal{F}$  is closed under submodules, products and extensions [13, Prop. VI.2.2]. Since  $I$  is projective, the flat dimension of  $S/I$  is  $\leq 1$ , so that  $\text{Tor}_2^S(-, S/I)=0$ . In particular the functor  $\text{Tor}_1^S(-, S/I)$  is left exact. Hence if  $\text{Tor}_1^S(M, S/I)=0$ , then  $\text{Tor}_1^S(N, S/I)=0$  for every submodule  $N$  of  $M$ . Therefore  $\mathcal{F}$  is closed under submodules. Moreover if  $N \leq M$ ,  $\text{Tor}_1^S(N, S/I)=0$  and  $\text{Tor}_1^S(M/N, S/I)=0$ , then  $\text{Tor}_1^S(M, S/I)=0$ , that is,  $\mathcal{F}$  is closed under extensions. Finally, since  $I$  is a projective principal ideal,  $I$  is a finitely presented module, so that if  $\{M_\lambda | \lambda \in A\} \subset \mathcal{F}$  is a family of  $S$ -modules,  $\prod_\lambda (M_\lambda \otimes I)$  and  $(\prod_\lambda M_\lambda) \otimes I$  are canonically isomorphic [13, Lemma I.13.2]. Then the mapping  $(\prod_\lambda M_\lambda) \otimes I \cong \prod_\lambda (M_\lambda \otimes I) \rightarrow \prod_\lambda M_\lambda$  is injective, and  $\mathcal{F}$  is closed under products.

In the statement of Proposition 5.2 the torsion class  $\mathcal{T}$  consists of all right  $S$ -modules  $T$  with  $\text{Hom}_S(T, M) = 0$  for all  $M \in \mathcal{F}$ . Note that  $S_S$  is an  $I$ -torsion-free module. Moreover the torsion theory  $(\mathcal{T}, \mathcal{F})$  is not hereditary in general. Our torsion theory  $(\mathcal{T}, \mathcal{F})$  generalizes the  $p$ -torsion theory of abelian groups, where  $p$  is a prime. In fact, it is easy to see that for  $S = \mathbb{Z}$  and  $I = p\mathbb{Z}$  the  $I$ -torsion-free,  $I$ -divisible and  $I$ -torsion modules are exactly the  $p$ -torsion-free,  $p$ -divisible and  $p$ -torsion abelian groups respectively.

**Proposition 5.3.** *Let  $\varphi$  be a generator of the projective principal left ideal  $I$  of  $S$ , so that the left annihilator  $l(\varphi)$  of  $\varphi$  is equal to  $S(1 - \varepsilon)$  for an idempotent  $\varepsilon \in S$ . Then the torsion theory  $(\mathcal{T}, \mathcal{F})$  is generated by the right  $S$ -module  $\varepsilon S / \varphi S$ .*

*Proof.* In order to prove that the torsion theory  $(\mathcal{T}, \mathcal{F})$  is generated by  $\varepsilon S / \varphi S$ , we must prove that a right  $S$ -module  $F$  belongs to  $\mathcal{F}$  if and only if  $\text{Hom}_S(\varepsilon S / \varphi S, F) = 0$ .

Suppose  $F \in \mathcal{F}$  and fix an  $f \in \text{Hom}_S(\varepsilon S / \varphi S, F)$ . Set  $x = f(\varepsilon + \varphi S) \in F$ . Then  $x\varepsilon = f(\varepsilon + \varphi S)\varepsilon = f(\varepsilon + \varphi S) = x$  and  $x\varphi = f(\varepsilon + \varphi S)\varphi = f(\varepsilon\varphi + \varphi S) = f(\varphi + \varphi S) = 0$ . Consider the element  $x \otimes \varphi \in F \otimes I$ . Since  $x\varphi = 0$  and the mapping  $F \otimes I \rightarrow F$  is injective because  $F \in \mathcal{F}$ , it follows that  $x \otimes \varphi = 0$ . Apply the functor  $F \otimes -$  to the exact sequence  $0 \rightarrow S(1 - \varepsilon) \rightarrow S \rightarrow I \rightarrow 0$ , where the first homomorphism is the inclusion and the second homomorphism is defined by  $1 \rightarrow \varphi$ . Then the sequence  $0 \rightarrow F \otimes_S S(1 - \varepsilon) \rightarrow F \otimes_S S \rightarrow F \otimes_S I \rightarrow 0$  is exact because  $I$  is projective, hence flat. The last sequence can be rewritten as  $0 \rightarrow F(1 - \varepsilon) \rightarrow F \rightarrow F \otimes_S I \rightarrow 0$  where the first homomorphism is the inclusion and the second homomorphism maps  $x$  into  $x \otimes \varphi$ . Since  $x \otimes \varphi = 0$ , it follows that  $x \in F(1 - \varepsilon)$ , so that  $x\varepsilon = 0$ . In particular  $f(\varepsilon + \varphi S) = x = x\varepsilon = 0$  and  $f: \varepsilon S / \varphi S \rightarrow F$  is the zero homomorphism. This proves that  $\text{Hom}_S(\varepsilon S / \varphi S, F) = 0$ .

Conversely, suppose that  $\text{Hom}_S(\varepsilon S / \varphi S, F) = 0$ . We must prove that  $F \otimes I \rightarrow F$  is injective. Since  $I = S\varphi$ , every element in  $F \otimes I$  can be written as  $x \otimes \varphi$ ,  $x \in F$ . Suppose  $x \otimes \varphi$  is in the kernel of  $F \otimes I \rightarrow F$ , i.e.,  $x\varphi = 0$ . The mapping  $f: \varepsilon S / \varphi S \rightarrow F$  defined by  $f(\varepsilon s + \varphi S) = x\varepsilon s$  is a well defined homomorphism, because if  $\varepsilon s \in \varphi S$ , then  $x\varepsilon s \in x\varphi S = \{0\}$ . It follows that  $f$  must be zero, hence  $x\varepsilon = 0$ . Then  $x \otimes \varphi = x \otimes \varepsilon \varphi = x\varepsilon \otimes \varphi = 0$ . This proves that  $F \in \mathcal{F}$ .

Our concept of  $I$ -divisibility differs from the concept of divisibility in [13, § VI.9], because our  $I$ -torsion-free modules and  $I$ -divisible modules are both right  $S$ -modules.

Define a right  $S$ -module  $M$  to be  $I$ -reduced if it is cogenerated by  $(S/I)^*$ , that is, if it is isomorphic to a submodule of a direct product of copies of  $(S/I)^*$ . Here  $(S/I)^* = \text{Hom}_R(S/I, C)$ , where  $R$  is a commutative ring such that  $S$  is an  $R$ -algebra and  $C$  is an injective cogenerator of  $\text{Mod-}R$ . Therefore  $M_S$  is  $I$ -reduced if and only

if for every  $x \in M, x \neq 0$ , there exists  $\vartheta_x: M \rightarrow (S/I)^*$  such that  $\vartheta_x(x) \neq 0$ . Since  $\text{Hom}_S(M, (S/I)^*) \cong \text{Hom}_R(M \otimes_S (S/I), C) \cong \text{Hom}_R(M/MI, C)$ , this happens if and only if for every  $x \in M, x \neq 0$ ,  $xS$  is not contained in  $MI$ . Therefore a right  $S$ -module  $M$  is  $I$ -reduced if and only if  $MI$  does not contain nonzero right  $S$ -submodules of  $M$ .

Note that a module  $N_S$  is  $I$ -divisible if and only if  $\text{Hom}_S(N, M) = 0$  for every  $I$ -reduced  $S$ -module  $M_S$ . In fact,  $\text{Hom}_S(N, M) = 0$  for every  $I$ -reduced  $S$ -module  $M_S$  if and only if  $\text{Hom}_S(N, (S/I)^*) = 0$ . This happens if and only if  $N \otimes (S/I) = 0$ , that is, if and only if  $N$  is  $I$ -divisible (Lemma 5.1(ii)).

We conclude this section with a last definition. We say that a right  $S$ -module  $M$  is an  $I$ -cotorsion module if it is  $I$ -reduced and  $\text{Ext}_S^1(N, M) = 0$  for every  $I$ -divisible  $I$ -torsion-free right  $S$ -module  $N$ .  $I$ -cotorsion modules will be studied in § 7.

### 6. Purity

In this section  $S$  is an arbitrary (associative) ring with identity and  $I = S\varphi$  is a fixed projective principal left ideal of  $S$ . We say that a short exact sequence  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  of right  $S$ -modules is  $I$ -pure if one of the equivalent conditions of next lemma holds.

**Lemma 6.1.** *The following properties of a short exact sequence  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  of right  $S$ -modules are equivalent:*

- (a) *The short exact sequence  $0 \rightarrow \text{Hom}_S(S/\varphi S, M') \rightarrow \text{Hom}_S(S/\varphi S, M) \rightarrow \text{Hom}_S(S/\varphi S, M'') \rightarrow 0$  is exact.*
- (b) *The short exact sequence  $0 \rightarrow M' \otimes S/S\varphi \rightarrow M \otimes S/S\varphi \rightarrow M'' \otimes S/S\varphi \rightarrow 0$  is exact.*
- (c)  *$M'\varphi = M' \cap M\varphi$ .*

Under these equivalent conditions we shall also say that  $M'$  is an  $I$ -pure submodule of  $M$ . The proof of this lemma is analogous to the proof of [14, Prop. 2 and 3]. Our purity is a particular case of Warfield's  $\mathcal{S}$ -purity [14] with  $\mathcal{S} = \{S/\varphi S, S\}$ . (See also [12].) It would also be possible to apply Gruson's and Jensen's idea developed in [5] to the study of  $I$ -purity: if  $\mathcal{O} = \{S, S/S\varphi\}$  is viewed as a full subcategory of  $S\text{-Mod}$  and  $D(S)$  is the category of additive functors of  $\mathcal{O}$  into the category of abelian groups  $\mathcal{A}\mathcal{B}$ , then the functor  $M \mapsto M \otimes_S -$  of  $\text{Mod-}S$  into  $D(S)$  is the left adjoint to the functor  $F \mapsto F(S)$  of  $D(S)$  into  $\text{Mod-}S$  and is an equivalence of  $\text{Mod-}S$  onto a full subcategory of  $D(S)$ ; in this equivalence short exact sequences of  $D(S)$  correspond to  $I$ -pure short exact sequences of  $\text{Mod-}S$ , and the injective

objects in  $D(S)$  correspond to the  $I$ -pure-injective  $S$ -modules. See also [2]. We shall not need this remark in the sequel.

Note that if  $M$  is an  $I$ -torsion-free  $S$ -module, that is,  $M \in \mathcal{F}$ , then a submodule  $M'$  of  $M$  is  $I$ -pure in  $M$  if and only if  $M/M'$  is  $I$ -torsion-free. This can be seen from the exact sequence  $\text{Tor}_1^S(M, S/S\varphi) \rightarrow \text{Tor}_1^S(M/M', S/S\varphi) \rightarrow M' \otimes S/S\varphi \rightarrow M \otimes S/S\varphi$ , where  $\text{Tor}_1^S(M, S/S\varphi) = 0$  because  $M \in \mathcal{F}$  (Lemma 5.1), so that  $M' \otimes S/S\varphi \rightarrow M \otimes S/S\varphi$  is injective if and only if  $\text{Tor}_1^S(M/M', S/S\varphi) = 0$ .

The theory developed in [12] applies to our notion of  $I$ -purity. If  $\mathcal{E}$  is the class of  $I$ -pure short exact sequences of  $S$ -modules, then  $\mathcal{E}$  is a *flatly generated, proper* class [12, § 3], closed under direct limits and *projectively closed* [12, Prop. 3.1 and 2.2]. For every right  $S$ -module  $M''$  there is an  $I$ -pure exact sequence  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  with  $M$   $I$ -pure-projective (i.e.,  $M$   $\mathcal{E}$ -projective). Moreover a module  $M$  is  $I$ -pure-projective if and only if it is isomorphic to a direct summand of a direct sum of copies of  $S_S$  and  $S/\varphi S$ . These statements follow immediately from [12, Prop. 2.3].  $I$ -pure-injective modules (that is,  $\mathcal{E}$ -injectives) are characterized as the direct summands of direct products of copies of  $\text{Hom}_R(S, C)$  and  $\text{Hom}_R(S/\varphi S, C)$ ; here  $R$  is any commutative ring such that  $S$  is an  $R$ -algebra, and  $C$  is an injective cogenerator in  $\text{Mod-}R$  [12, Prop. 3.3]. Finally, every module has a suitably defined  $I$ -pure-injective envelope [12, Prop. 4.5], and  $I$ -pure-injective modules are direct summands of every module which contains them as  $I$ -pure submodules.

## 7. The equivalences

Now we apply the theory developed in §§ 5 and 6 to the study of the functors  $\text{Hom}_R({}_E\partial_R, -): \text{Mod-}R \rightarrow \text{Mod-}E$  and  $- \otimes_E \partial_R: \text{Mod-}E \rightarrow \text{Mod-}R$  introduced in § 4.

As in the first four sections  $R$  is an integral domain,  $\partial_R$  is the  $R$ -module of § 2,  $E$  is its endomorphism ring  $\text{End}(\partial_R)$ ,  $\varphi$  is an endomorphism of  $\partial_R$  whose kernel is  $wR$  and image is a direct summand of  $\partial_R$ . The left ideal  $I = E\varphi$  of  $E$  is a projective principal ideal by Theorem 2.4, so that the theory developed in § 5 can be applied. Let  $C$  be the minimal injective cogenerator in  $\text{Mod-}R$  and  $\partial^* = \text{Hom}_R(\partial, C)$ . There is a torsion theory  $(\mathcal{T}, \mathcal{F})$  for  $\text{Mod-}E$  where the  $I$ -torsion-free class  $\mathcal{F}$  consists of the right  $E$ -modules  $M$  with  $\text{Tor}_1^E(M, \partial) = 0$ , or, equivalently, with  $\text{Ext}_E^1(M, \partial^*) = 0$  (Lemmas 3.1 and 5.1). The class of  $I$ -divisible  $E$ -modules consists of the right  $E$ -modules  $M$  with  $M \otimes_E \partial = 0$ . The torsion theory  $(\mathcal{T}, \mathcal{F})$  is generated by the right  $E$ -module  $\partial^\circ = \text{Ext}_R^1(\partial, R)$  (Proposition 5.3 and Theorem 3.4) and  $E_E$  is a torsion-free  $E$ -module in the torsion theory  $(\mathcal{T}, \mathcal{F})$ .

The  $I$ -reduced  $E$ -modules are the right  $E$ -modules cogenerated by  $\partial^*$ ; and a module  $M_E$  is  $I$ -reduced if and only if  $MI$  does not contain nonzero right  $E$ -submodules of  $M$ .

**Theorem 7.1.** *Let  $R$  be an integral domain and  $A$  a right  $R$ -module. Then  $\text{Hom}_R(\partial, A)$  is an  $I$ -cotorsion  $E$ -module.*

*Proof.* Since  $C$  is an injective cogenerator in  $\text{Mod-}R$ ,  $A \cong C^X$  for some set  $X$ , so that  $\text{Hom}_R(\partial, A) \cong \text{Hom}_R(\partial, C^X) \cong (\partial^*)^X$ ; hence  $\text{Hom}_R(\partial, A)$  is cogenerated by  $\partial^*$ , that is, it is  $I$ -reduced.

Now let  $N_E$  be an  $I$ -divisible  $I$ -torsion-free  $E$ -module and let  $D$  be an injective  $R$ -module containing  $A$ . Then the functor  $\text{Hom}_R(\partial, -)$  applied to the exact sequence  $0 \rightarrow A \rightarrow D \rightarrow D/A \rightarrow 0$  gives an exact sequence  $0 \rightarrow \text{Hom}_R(\partial, A) \rightarrow \text{Hom}_R(\partial, D) \rightarrow P \rightarrow 0$  for a suitable  $E$ -submodule  $P$  of  $\text{Hom}_R(\partial, D/A)$ . Apply the functor  $\text{Hom}_E(N, -)$  to this sequence and obtain the exact sequence  $\text{Hom}_E(N, P) \rightarrow \text{Ext}_E^1(N, \text{Hom}_R(\partial, A)) \rightarrow \text{Ext}_E^1(N, \text{Hom}_R(\partial, D))$ . But

$$\text{Hom}_E(N, P) \cong \text{Hom}_E(N, \text{Hom}_R(\partial, D/A)) \cong \text{Hom}_R(N \otimes_E \partial, D/A) = 0$$

because  $N \otimes_E \partial = 0$  since  $N$  is  $I$ -divisible. Moreover  $\text{Tor}_1^E(N, \partial) = 0$  (because  $N$  is  $I$ -torsion-free) and  $D$  is injective, and thus

$$\text{Ext}_E^1(N, \text{Hom}_R(\partial, D)) \cong \text{Hom}_R(\text{Tor}_1^E(N, \partial), D) = 0.$$

Therefore  $\text{Ext}_E^1(N, \text{Hom}_R(\partial, A)) = 0$  and  $\text{Hom}_R(\partial, A)$  is  $I$ -cotorsion.

Note that  $E/\varphi E \cong ((1-\varepsilon)E \oplus \varepsilon E)/\varphi E \cong (1-\varepsilon)E \oplus (\varepsilon E/\varphi E) \cong (1-\varepsilon)E \oplus \partial^\circ$  (Theorem 3.4), so that  $E/\varphi E$  is projective relatively to an exact sequence of right  $E$ -modules if and only if  $\partial^\circ$  is projective relatively to that exact sequence. It follows that an exact sequence  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  of right  $E$ -modules is  $I$ -pure, that is,  $M'I = M' \cap MI$ , if and only if  $0 \rightarrow M' \otimes_E \partial \rightarrow M \otimes_E \partial \rightarrow M'' \otimes_E \partial \rightarrow 0$  is exact, if and only if  $0 \rightarrow \text{Hom}_E(\partial^\circ, M') \rightarrow \text{Hom}_E(\partial^\circ, M) \rightarrow \text{Hom}_E(\partial^\circ, M'') \rightarrow 0$  is exact. Moreover, if  $C$  is the minimal injective cogenerator in  $\text{Mod-}R$  and  $\partial^*$  is the right  $E$ -module  $\text{Hom}_R(\partial, C)$  then  $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$  is  $I$ -pure if and only if  $0 \rightarrow \text{Hom}_E(M'', \partial^*) \rightarrow \text{Hom}_E(M, \partial^*) \rightarrow \text{Hom}_E(M', \partial^*) \rightarrow 0$  is exact.

By the general theory developed in § 6, the  $I$ -pure-projective  $E$ -modules are exactly the direct summands of direct sums of copies of  $E_E$  and  $\partial^\circ$ , and the  $I$ -pure-injective  $E$ -modules are exactly the direct summands of direct products of copies of  $\text{Hom}_R(E, C)$  and  $\text{Hom}_R(\partial, C) = \partial^*$ .

**Theorem 7.2.** *Let  $M$  be a right  $E$ -module and let  $\eta_M: M \rightarrow \text{Hom}_R(\partial, M \otimes_E \partial)$  be the canonical homomorphism. Then:*

- (a)  $\ker \eta_M$  is the largest  $E$ -submodule of  $M$  contained in  $MI$ .
- (b) The image of  $\eta_M$  is an  $I$ -pure submodule of  $\text{Hom}_R(\partial, M \otimes_E \partial)$ .
- (c)  $\text{coker } \eta_M$  is an  $I$ -torsion-free  $I$ -divisible  $E$ -module.

*Proof.* (a) Since  $\partial \cong E/I$ , the  $R$ -module  $M \otimes_E \partial$  is isomorphic to  $M/MI$ , so that  $x \in M$  is in the kernel of  $\eta_M$  if and only if  $xe \in MI$  for every  $e \in E$ , that is, if and only if  $xE \subset MI$ . In particular  $\ker \eta_M$  is an  $E$ -submodule of  $M$  contained in  $MI$ . And if  $N$  is any  $E$ -submodule of  $M$  contained in  $MI$ , then  $xE \subset MI$  for every  $x \in N$ , that is,  $x \in \ker \eta_M$  for every  $x \in N$ . This proves that  $N \subset \ker \eta_M$ .

(b) By Theorem 2.4  $\text{Hom}_R(\partial, M \otimes_E \partial)I = \{f \in \text{Hom}_R(\partial, M \otimes_E \partial) \mid f(w) = 0\}$ . Therefore  $\eta_M(M) \cap \text{Hom}_R(\partial, M \otimes_E \partial)I = \{\eta_M(x) \mid x \in M, \eta_M(x)(w) = 0\} = \{\eta_M(x) \mid x \in M, x \otimes w = 0\}$ . Since the homomorphism  $\partial \rightarrow E/I, w \mapsto 1+I$  is an isomorphism of  $E$ -modules (Lemma 3.1), it follows that  $M \otimes \partial \cong M \otimes E/I \cong M/MI$ , and  $x \otimes w = 0$  if and only if  $x \in MI$ . Hence  $\eta_M(M) \cap \text{Hom}_R(\partial, M \otimes_E \partial)I = \{\eta_M(x) \mid x \in MI\} = \eta_M(MI) = \eta_M(M)I$ .

(c) Suppose that  $\eta_M$  is injective (by Part (a) this happens if and only if  $M$  is  $I$ -reduced). Under this hypothesis consider the exact sequence

$$0 \rightarrow M \rightarrow \text{Hom}_R(\partial, M \otimes_E \partial) \rightarrow \text{coker } \eta_M \rightarrow 0.$$

This sequence is  $I$ -pure by Part (b) and  $\text{Hom}_R(\partial, M \otimes_E \partial)$  is  $I$ -torsion-free by Proposition 4.2. Therefore  $\text{coker } \eta_M$  is  $I$ -torsion-free.

Now apply the functor  $-\otimes_E \partial$  to the above  $I$ -pure exact sequence and obtain the exact sequence  $0 \rightarrow M \otimes \partial \rightarrow \text{Hom}_R(\partial, M \otimes_E \partial) \otimes_E \partial \rightarrow \text{coker } \eta_M \otimes_E \partial \rightarrow 0$ . The homomorphism  $\eta_M \otimes \partial: M \otimes \partial \rightarrow \text{Hom}_R(\partial, M \otimes_E \partial) \otimes_E \partial$  is equal to  $\varepsilon_{M \otimes \partial}^{-1}$  (where  $\varepsilon$  is the counit of the adjunction and  $\varepsilon_{M \otimes \partial}$  is an isomorphism by Theorem 4.1) because if  $x \in M$  and  $y \in \partial$  then  $\eta_M \otimes \partial(x \otimes y) = f_x \otimes y$ , where  $f_x \in \text{Hom}_R(\partial, M \otimes_E \partial)$  and  $f_x(z) = x \otimes z$  for every  $z \in \partial$ . Therefore  $\varepsilon_{M \otimes \partial}(\eta_M \otimes \partial(x \otimes y)) = \varepsilon_{M \otimes \partial}(f_x \otimes y) = f_x(y) = x \otimes y$ , i.e.,  $\eta_M \otimes \partial(x \otimes y) = \varepsilon_{M \otimes \partial}^{-1}(x \otimes y)$  and  $\eta_M \otimes \partial = \varepsilon_{M \otimes \partial}^{-1}$ . Hence  $\eta_M \otimes \partial$  is an isomorphism, and the exactness of the above sequence gives  $(\text{coker } \eta_M) \otimes_E \partial = 0$ , i.e.,  $\text{coker } \eta_M$  is  $I$ -divisible.

This proves Part (c) under the additional hypothesis that  $\eta_M$  is injective. In the general case the naturality of  $\eta$  applied to the canonical projection  $\pi: M \rightarrow M/\ker \eta_M$  gives the equality  $\eta_{M/\ker \eta} \cdot \pi = \text{Hom}(\partial, \pi \otimes \partial) \cdot \eta_M$ . But  $\pi \otimes \partial: M \otimes \partial \rightarrow (M/\ker \eta_M) \otimes \partial$  is an isomorphism because

$$\begin{aligned} (M/\ker \eta_M) \otimes \partial &\cong (M/\ker \eta_M) \otimes (E/I) \cong (M/\ker \eta_M)/(M/\ker \eta_M)I \\ &\cong M/(\ker \eta_M + MI) \cong M/MI \cong M \otimes (E/I) \cong M \otimes \partial. \end{aligned}$$

Therefore  $\text{Hom}(\partial, \pi \otimes \partial)$  is an isomorphism and

$$\text{coker } \eta_M \cong \text{coker}(\text{Hom}(\partial, \pi \otimes \partial) \cdot \eta_M) = \text{coker}(\eta_{M/\ker \eta} \cdot \pi) = \text{coker } \eta_{M/\ker \eta}.$$

Now  $M/\ker \eta$  is  $I$ -reduced by Part (a), so that  $\text{coker } \eta_M \cong \text{coker } \eta_{M/\ker \eta}$  is  $I$ -torsion-free and  $I$ -divisible by the previous case.

As a corollary to Theorem 7.2 it must be noted that every  $I$ -reduced  $E$ -module is  $I$ -torsion-free. This holds because if  $M_E$  is  $I$ -reduced, then  $\eta_M$  is injective (Theorem 7.2(a)) and  $\text{Hom}_R(\partial, M \otimes \partial)$  is  $I$ -torsion-free (Proposition 4.2), so that  $M$  is  $I$ -torsion-free too. Nevertheless this fact does not hold for an arbitrary ring  $S$  (take  $S = \mathbb{Z}$ ,  $I = 2\mathbb{Z}$  and  $M$  any abelian group with  $2M = 0$ , so that  $M$  is  $I$ -reduced and is not  $I$ -torsion-free).

**Theorem 7.3.** *Let  $M$  be a right  $E$ -module. Then  $\eta_M: M \rightarrow \text{Hom}_R(\partial, M \otimes_E \partial)$  is an isomorphism if and only if  $M$  is  $I$ -cotorsion.*

*Proof.* If  $M \cong \text{Hom}_R(\partial, M \otimes \partial)$ ,  $M$  is  $I$ -cotorsion by Theorem 7.1. Conversely, if  $M$  is  $I$ -cotorsion, the homomorphism  $\eta_M$  is injective by Theorem 7.2(a) and the exact sequence  $0 \rightarrow M \rightarrow \text{Hom}_R(\partial, M \otimes_E \partial) \rightarrow \text{coker } \eta_M \rightarrow 0$  splits because

$$\text{Ext}_E^1(\text{coker } \eta_M, M) = 0$$

( $\text{coker } \eta_M$  is  $I$ -torsion-free and  $I$ -divisible by Theorem 7.2(c)). Hence  $\text{coker } \eta_M$  is isomorphic to a submodule of  $\text{Hom}_R(\partial, M \otimes_E \partial)$ . But  $\text{coker } \eta_M$  is  $I$ -divisible, and  $\text{Hom}_R(\partial, M \otimes_E \partial)$  is  $I$ -reduced. Therefore  $\text{coker } \eta_M = 0$  and  $\eta_M$  is an isomorphism.

Theorem 7.3 has the following corollary: if  $M$  is any right  $E$ -module, every  $E$ -homomorphism from  $M$  into an  $I$ -cotorsion module  $N_E$  can be uniquely factored over  $\eta_M: M \rightarrow \text{Hom}_R(\partial, M \otimes_E \partial)$ . Hence  $\text{Hom}_R(\partial, M \otimes_E \partial)$  is a sort of “ $I$ -cotorsion completion” of  $M$ . The factorization of  $f: M \rightarrow N$  is  $f = (\eta_N^{-1} \cdot \text{Hom}_R(\partial, f \otimes \partial)) \cdot \eta_M$  (this equality is given by the naturality of the transformation  $\eta$ ). The uniqueness of the factorization is proved as follows: if  $f = f_1 \cdot \eta_M = f_2 \cdot \eta_M$ , then  $(f_1 - f_2) \cdot \eta_M = 0$ , so that  $f_1 - f_2: \text{Hom}_R(\partial, M \otimes_E \partial) \rightarrow N$  induces a mapping  $\text{coker } \eta_M \rightarrow N$ . But  $\text{coker } \eta_M$  is  $I$ -divisible (Theorem 7.2(c)) and  $N$  is  $I$ -reduced, so that this mapping is zero. Hence  $f_1 - f_2 = 0$ . This proves the corollary.

It must be remarked that our “ $I$ -cotorsion completion”  $\text{Hom}_R(\partial, - \otimes_E \partial)$  is substantially different from the cotorsion hull in a hereditary torsion theory developed in [1], since our torsion theory  $(\mathcal{T}, \mathcal{F})$  is not hereditary.

**Theorem 7.4.** *If  $R$  is an integral domain and  $E = \text{End}(\partial_R)$ , the functors  $\text{Hom}_R(\partial, -): \mathcal{D}_R \rightarrow \mathcal{C}_E$  and  $- \otimes_E \partial: \mathcal{C}_E \rightarrow \mathcal{D}_R$  give an equivalence between the full subcategory  $\mathcal{D}_R$  of divisible  $R$ -modules and the full subcategory  $\mathcal{C}_E$  of  $\text{Mod-}E$  whose objects are the  $I$ -cotorsion  $E$ -modules. In this equivalence injective  $R$ -modules correspond to  $I$ -reduced  $I$ -pure-injective  $E$ -modules.*

*Proof.* By Theorems 4.1 and 7.3  $\text{Hom}_R(\partial, -)$  and  $- \otimes_E \partial$  give an equivalence between the categories  $\mathcal{D}_R$  and  $\mathcal{C}_E$ . Let us prove that if  $B_R$  is an injective right  $R$ -module then  $\text{Hom}_R(\partial, B)$  is an  $I$ -pure-injective  $E$ -module. If  $B_R$  is injective, then  $B$  is isomorphic to a direct summand of  $C^X$ , where  $C$  is a minimal injective cogenerator in  $\text{Mod-}R$ . Then  $\text{Hom}_R(\partial, B)$  is isomorphic to a direct summand in  $\text{Hom}_R(\partial, C^X) \cong$

$\text{Hom}_R(\partial, C)^X = \partial^{*X}$ . By the remark immediately above Theorem 7.2,  $\text{Hom}_R(\partial, B)$  is an  $I$ -pure-injective  $E$ -module.

Conversely, if  $M_E$  is an  $I$ -reduced,  $I$ -pure-injective  $E$ -module, then  $\eta_M: M \rightarrow \text{Hom}_R(\partial, M \otimes_E \partial)$  is an  $I$ -pure monomorphism (Theorem 7.2). Let  $D$  be an injective  $R$ -module containing  $M \otimes \partial$ , so that  $\text{Hom}_R(\partial, M \otimes \partial) \cong \text{Hom}_R(\partial, D)$ . The submodule  $\text{Hom}_R(\partial, M \otimes \partial)$  is  $I$ -pure in  $\text{Hom}_R(\partial, D)$ , because  $\text{Hom}_R(\partial, D)I = \{f \in \text{Hom}_R(\partial, D) \mid f(w) = 0\}$  by Theorem 2.4, so that  $\text{Hom}_R(\partial, D)I \cap \text{Hom}_R(\partial, M \otimes \partial) = \{f \in \text{Hom}_R(\partial, M \otimes \partial) \mid f(w) = 0\} = \text{Hom}_R(\partial, M \otimes \partial)I$  by Theorem 2.4 again. Therefore  $M$  is isomorphic to an  $I$ -pure submodule of  $\text{Hom}_R(\partial, D)$ . Since  $M$  is  $I$ -pure-injective,  $M$  is isomorphic to a direct summand of  $\text{Hom}_R(\partial, D)$ . Then  $M \otimes \partial$  is isomorphic to a direct summand of  $\text{Hom}_R(\partial, D) \otimes \partial \cong D$ . This proves that  $M \otimes \partial$  is an injective  $R$ -module.

Thus we have seen that the class we had denoted by  $\mathcal{I}$  in Theorem 4.3, i.e., the image of the functor  $\text{Hom}_R(\partial, -): \text{Mod-}R \rightarrow \text{Mod-}E$ , is exactly the class  $\mathcal{C}_E$  of  $I$ -cotorsion  $E$ -modules. There is a further characterization of these modules: they are exactly the right  $E$ -modules of  $\partial^*$ -dominant dimension  $\cong 2$ , that is, the right  $E$ -modules  $M$  for which there exists an exact sequence  $0 \rightarrow M \rightarrow \partial^{*X} \rightarrow \partial^{*Y}$  for suitable direct powers  $\partial^{*X}$  and  $\partial^{*Y}$  of the  $E$ -module  $\partial^*$ . In order to see this, note that if  $M$  is an  $I$ -cotorsion  $E$ -module, then there is an exact sequence of  $R$ -modules  $0 \rightarrow M \otimes_E \partial \rightarrow C^X \rightarrow C^Y$  because  $C$  is an injective cogenerator in  $\text{Mod-}R$ , so that by applying the left exact functor  $\text{Hom}_R(\partial, -)$  to this sequence one obtains an exact sequence  $0 \rightarrow M \cong \text{Hom}_R(\partial, M \otimes \partial) \rightarrow \partial^{*X} \rightarrow \partial^{*Y}$ . Conversely, if  $M$  has  $\partial^*$ -dominant dimension  $\cong 2$ , from the exact sequence  $0 \rightarrow M \rightarrow \partial^{*X} \rightarrow \partial^{*Y}$  we obtain that  $M$  is cogenerated by  $\partial^*$  (i.e., it is  $I$ -reduced) and that there is an exact sequence  $0 \rightarrow M \rightarrow \partial^{*X} \rightarrow N \rightarrow 0$  with  $N \cong \partial^{*Y}$ . If  $F$  is any  $I$ -divisible  $I$ -torsion-free  $E$ -module then the sequence  $\text{Hom}_E(F, N) \rightarrow \text{Ext}_E^1(F, M) \rightarrow \text{Ext}_E^1(F, \partial^{*X})$  is exact,  $\text{Hom}_E(F, N) = 0$  (because  $F$  is  $I$ -divisible and  $N$  is  $I$ -reduced), and  $\text{Ext}_E^1(F, \partial^{*X}) = 0$  (because  $\partial^{*X} \cong \text{Hom}_R(\partial, C^X)$  is in  $\mathcal{I}$ , i.e., it is  $I$ -cotorsion). Therefore  $\text{Ext}_E^1(F, M) = 0$  and  $M$  is  $I$ -cotorsion.

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