A note on existence of global classical solutions to nonlinear wave equations

P. Kumlin

1. Introduction

The aim of this paper is to obtain global classical solutions for classes of equations

$$(1) \qquad \begin{cases} (\partial_t^2 - \Delta_x) u = f(t, x, u), & x \in \mathbf{R}^3, \quad t > 0, \\ u|_{t=0} = \varphi, \\ \partial_t u|_{t=0} = \psi, \end{cases}$$

using the method with subsolutions and supersolutions. In particular, we want to study the case with superlinear growth of f in u.

Our main results are

Theorem 1.1. *Let* $g \in C_0^\infty(\mathbf{R}^3)$ *and assume that*

$$|g(x)| \leqq \sqrt{\frac{\sqrt{3\lambda}}{1+\lambda|x|^2}}, \quad x \in \mathbf{R}^3,$$

for some $\lambda > 0$. *Then there exists a unique global classical solution u to*

$$(2) \qquad \begin{cases} (\partial_t^2 - \Delta_x) u = t^{-4} u^5, & x \in \mathbf{R}^3, \quad t > 0, \\ u|_{t=0} = 0, \\ \partial_t u|_{t=0} = g. \end{cases}$$

and

Theorem 1.2. *Let* $g \in C_0^\infty(\mathbf{R}^3)$ *and assume that k is an odd integer greater than 5, or that k is an even integer greater than 5 and that* $g(x)$ *is nonnegative. Then there exists a unique global classical solution u to*

$$(3) \qquad \begin{cases} (\partial_t^2 - \Delta_x) u = t^{1-k} u^k, & x \in \mathbf{R}^3, \quad t > 0, \\ u|_{t=0} = 0, \\ \partial_t u|_{t=0} = g, \end{cases}$$

provided that

$$|g(x)| \leqq v(x), \quad x \in \mathbf{R}^3,$$

where $v(x)$ is a solution to the **Lane—Emden equation of index k.**

These results should be compared with the following known facts for problem (1) with compactly supported smooth data φ and ψ.

1) If $f(t, x, u) = -|u|^{\varrho-1}u$ and $1 < \varrho < 5$, then there exists a unique global classical solution (see Jörgens [9]).
2) If $f(t, x, u) = -|u|^{\varrho-1}u$ and $\varrho = 5$ and if (φ, ψ) is sufficiently small in $(H_2^1 \cap L^6) \times L^2$, then there exists a unique global classical solution (see Rauch [14]).
3) If $f(t, x, u) \geqq |u|^p$, where $1 < p < 1+\sqrt{2}$, then every nontrivial solution blows up in finite time (see John [6]).
4) If $f(t, x, u) \geqq |u|^p$, where $p > 1+\sqrt{2}$, then there exists a unique global classical solution for data small enough in $C^2 \times C^1$ (see John [6]).

We refer to Glassey [5], Kato [10], Kumlin [12] and von Wahl [15] for similar results. In particular, our result fits in between 1) and 2) on one side and 4) on the other side. The difference with 4) is that we don't have to impose smallness of data in a so strong norm, but in comparison with 1) and 2) we have a factor in the nonlinear term that decays in time. Moreover, we note that, according to [5], Theorem 1.1 and Theorem 1.2 will be false if the nonlinearities there are replaced by u^5 and u^k respectively.

In deriving result 1), where no restriction is put on the size of data, energy methods are used. This sets a sign constraint on the nonlinear term in the equation which our nonlinearity violates. We note that the only methods that have been used to obtain the global existence results above are energy methods and perturbation techniques.

The method with subsolutions and supersolutions is very old and suitable for nonlinear elliptic and parabolic equations, since in these cases there are appropriate maximum principles and apriori estimates. Our report is inspired by a paper by Korman [11], where he treats some noncoercive elliptic and hyperbolic problems.

Finally, we note that our result can't be generalized to more space dimensions than 3 since the Riemann function for the wave operator is no longer positive in those cases.

We organize the paper as follows. In Section 2 we give the solution of the linear wave equation and introduce the Euler—Poisson—Darboux equation that will be used in the succeeding Sections. In Section 3 we give a weak maximality principle and explicit subsolutions and supersolutions used in the proof of Theorem 1.1. Section 4 contains energy estimates and a local existence result for classical solutions and in Section 5 we conclude the proof of Theorem 1.2.

2. The Euler—Poisson—Darboux equation and the wave equation

For our problem the solution can be represented by

$$u(x,t) = \big(E_0^{(n)}(t)\varphi\big)(x) + \big(E_1^{(n)}(t)\psi\big)(x)$$
$$+\int_0^t \big(E_1^{(n)}(t-\tau)f(\tau,.,u(.,\tau))\big)(x)\,d\tau, \quad x\in\mathbf{R}^n, \quad t\geqq 0,$$

where $E_0^{(n)}(t)$ and $E_1^{(n)}(t)$ correspond to the Fourier multipliers $\cos(t|\xi|)$ and $|\xi|^{-1}\sin(t|\xi|)$ respectively. Moreover from Courant/Hilbert [2] we note that

$$\big(E_0^{(n)}(t)\varphi\big)(x) = \sum_{\nu=0}^{(n-3)/2}(\nu+1)a_\nu \frac{\partial^\nu}{\partial t^\nu}\frac{1}{|\Omega_n|}\int_{\Omega_n}\varphi(x+t\xi)\,d\xi$$
$$+t\sum_{\nu=0}^{(n-3)/2} a_\nu t^\nu \frac{\partial^{\nu+1}}{\partial t^{\nu+1}}\frac{1}{|\Omega_n|}\int_{\Omega_n}\varphi(x+t\xi)\,d\xi,$$

$$\big(E_1^{(n)}(t)\psi\big)(x) = t\sum_{\nu=0}^{(n-3)/2} a_\nu t^\nu \frac{\partial^\nu}{\partial t^\nu}\frac{1}{|\Omega_n|}\int_{\Omega_n}\psi(x+t\xi)\,d\xi$$

for n odd (greater than 1) and

$$\big(E_0^{(n)}(t)\varphi\big)(x) = \sum_{\nu=0}^{(n-2)/2}(\nu+1)b_\nu \frac{\partial^\nu}{\partial t^\nu}$$
$$\times\left(\frac{2\Gamma\left(\frac{n-1}{2}\right)}{\pi\Gamma\left(\frac{n}{2}\right)t^{n-1}}\int_0^t \frac{r^{n-1}}{|\Omega_n|(t^2-r^2)^{1/2}}\int_{\Omega_n}\varphi(x+r\xi)\,d\xi\,dr\right)$$
$$+t\sum_{\nu=0}^{(n-2)/2} b_\nu t^\nu \frac{\partial^{\nu+1}}{\partial t^{\nu+1}}\left(\frac{2\Gamma\left(\frac{n+1}{2}\right)}{\pi\Gamma\left(\frac{n}{2}\right)t^{n-1}}\int_0^t \frac{r^{n-1}}{|\Omega_n|(t^2-r^2)^{1/2}}\int_{\Omega_n}\varphi(x+r\xi)\,d\xi\,dr\right),$$

$$\big(E_1^{(n)}(t)\psi\big)(x) = t\sum_{\nu=0}^{(n-2)/2} b_\nu t^\nu \frac{\partial^\nu}{\partial t^\nu}$$
$$\times\left(\frac{2\Gamma\left(\frac{n+1}{2}\right)}{\pi\Gamma\left(\frac{n}{2}\right)t^{n-1}}\int_0^t \frac{r^{n-1}}{|\Omega_n|(t^2-r^2)^{1/2}}\int_{\Omega_n}\psi(x+r\xi)\,d\xi\,dr\right),$$

for n even. Hence we conclude (at least after recalling the formulas for $E_0^{(1)}(t)$ and $E_1^{(1)}(t)$) that $E_0^{(n)}(t)$ is a positive operator for $n=1$ and $E_1^{(n)}(t)$ is a positive operator for $n=1, 2$ and 3. Since this positivity is used for our weak maximality principle we are reduced to the cases $n \leqq 3$. In the course of the proof we consider the **Euler—Poisson—Darboux equation** below, which is related to the wave equation with a simple transformation.

Let $\Diamond$ denote the singular differential operator $\partial_t^2 + \frac{2}{t}\partial_t - \Delta_x$ and consider the equation

$$(4) \qquad \begin{cases} \Diamond u = f, \\ u|_{t=0} = u_0, \\ \partial_t u|_{t=0} = 0. \end{cases}$$

Comparing (4) with

$$(5) \qquad \begin{cases} \Box u = tf, \\ u|_{t=0} = 0, \\ \partial_t u|_{t=0} = u_0, \end{cases}$$

we observe that if v is a C^2-solution of (4), then $u = tv$ is a C^2-solution of (5), and if u is a C^2-solution of (5), then $v = t^{-1}u$ is a C^2-solution of (4).

3. A weak maximality principle, subsolutions and supersolutions

Define the operator $E_1(t)$ by

$$(E_1(t)\varphi)(x) = \frac{Ct}{|\Omega_3|}\int_{\Omega_3} \varphi(x + t\xi)\, d\xi, \quad x \in \mathbf{R}^3, \quad t \geqq 0,$$

where $\Omega_3 = \{x \in \mathbf{R}^3 \colon |x| = 1\}$. Hence the solution of (5) is given by

$$u(x, t) = (E_1(t)u_0)(x) + \int_0^t (E_1(t-\tau)\,\tau f(\tau, .))(x)\, d\tau$$

and we conclude from Section 2 that

Lemma 3.1. *If*

$$\begin{cases} \Diamond v \leqq 0, \quad x \in \mathbf{R}^3, \quad t \geqq 0, \\ v|_{t=0} \leqq 0, \\ \partial_t v|_{t=0} = 0, \end{cases}$$

then $v \leqq 0$ *in* $\mathbf{R}^3 \times [0, \infty)$.

Define a partial ordering on the real-valued functions defined on $\mathbf{R}^3 \times [0, \infty)$ by

$$u \preceq v,$$

if

$$u(x, t) \leqq v(x,t), \quad x \in \mathbf{R}^3, \quad t \geqq 0,$$

$$\partial_t u(x, 0) = \partial_t v(x, 0) = 0, \quad x \in \mathbf{R}^3.$$

Moreover, let $v=Tw$ denote the solution of

$$\text{(6)} \qquad \begin{cases} \Diamond\, v = w^5, \\ v|_{t=0} = g, \\ \partial_t v|_{t=0} = 0. \end{cases}$$

Using the definition we see that $u_1 \lesssim u_2$ yields $Tu_1 \lesssim Tu_2$. This follows easily from Lemma 3.1 since

$$\Diamond\,(Tu_1 - Tu_2) = u_1^5 - u_2^5 \leqq 0.$$

We say that V is a **supersolution** if

$$\text{(7)} \qquad \begin{cases} \Diamond\, V \geqq V^5, \\ V|_{t=0} \geqq g, \\ \partial_t V|_{t=0} = 0, \end{cases}$$

and v is a **subsolution** if v satisfies (7) with the inequalities reversed. In particular, we see that $v=-V$ is a subsolution. Moreover we recall from [3] that

$$V(x) = \sqrt{\frac{\sqrt{3\lambda}}{1+\lambda |x|^2}}, \quad x \in \mathbf{R}^3$$

satisfies

$$-\Delta V = V^5,$$

and thus provides a supersolution. From the definition of supersolution and Lemma 3.1 it follows that

$$TV \lesssim V$$

since

$$\Diamond\,(TV - V) \leqq V^5 - V^5 = 0.$$

Hence we get a nonincreasing sequence $(u_n)_{n=0}^{\infty}$, where

$$u_0 = V,$$

$$u_{n+1} = Tu_n, \quad n = 0, 1, 2, \dots .$$

Furthermore using $v \lesssim Tv$ we obtain

$$|u_n| \leqq V, \quad n = 0, 1, 2, \dots,$$

and hence the sequence $(u_n)_{n=0}^{\infty}$ converges pointwise to a function u.

Finally $(tu_n)_{n=0}^{\infty}$ converges pointwise to the function tu, which we claim has the desired smoothness and thus provides a solution to our original problem (2).

4. Energy estimates and local existence of classical solutions

In this section we show that the limit function tu is a global classical solution to (2), i.e. that tu has the desired smoothness and $Tu=u$. Let T_0 be an arbitrary large number and let $\varepsilon>0$ be a small number. Moreover for $m\in\mathbf{N}$, let H^m denote the m-th Sobolev space in (x, t)-space (with respect to L^2-norm) with norm $\|\cdot\|_m$.

Lemma 4.1. *Consider the equation*

$$(8)\qquad \begin{cases} \Box u = f(t, x)\in C^\infty, & x\in\mathbf{R}^3, \quad t\in[\varepsilon, T_0], \\ u|_{t=\varepsilon} = g\in C_0^\infty, \\ \partial_t u|_{t=\varepsilon} = h\in C_0^\infty. \end{cases}$$

Then there exists a constant $C=C(m, \varepsilon, T_0)$ *such that*

$$\|u\|_{m+1} \leqq C(\|f\|_m+\|g\|_{m+1}+\|h\|_m).$$

The proof of this is well-known so we just sketch the main idea. For $m=0$, standard energy estimates together with compactness of support of data and the finiteness of speed of propagation give the result. Higher order estimates are obtained by differentiating the equation $\Box u=f(t, x)$ and performing the same type of operations. The reason for $\varepsilon>0$ in Lemma 4.1 is the fact that the nonlinearity $t^{-4}u^5(t, x)$ is singular at $t=0$.

Let us assume that for some $\varepsilon>0$ tu is a classical solution of (2) for $t\in[0, \varepsilon]$. Then it follows easily that tu is a global classical solution from the argument below. We recall that H^m is a Banach algebra under pointwise multiplication, provided m is greater than half the dimension of the underlying space, i.e. $m\geqq 3$ in our case, and that we have the interpolation inequality

$$\|f\|_j \leqq C\|f\|_m^\theta\|f\|_0^{1-\theta}, \quad 0\leqq\theta\equiv\frac{j}{m}<1.$$

Now by applying Lemma 4.1 inductively we obtain that $(tu_n)_{n=0}^\infty$ is uniformly bounded for $m=6$ and by the interpolation inequality above that $(tu_n)_{n=0}^\infty$ is a Cauchy sequence in $H^{6-\tilde{\varepsilon}}$ for any $\tilde{\varepsilon}\in(0, 1)$ and thus in $C^2(\mathbf{R}^3\times(\varepsilon, T_0))\cap C^2(\mathbf{R}^3\times\{0\})$ by embedding and trace theorems for Sobolev spaces. Hence we can pass to the limit in

$$\Box(tu_{n+1}) = t^{-4}(tu_n)^5, \quad t\in[\varepsilon, T_0].$$

Here T_0 is arbitrary so we have the global result since we have uniqueness for classical solutions (see [7]).

Now it remains to prove what was assumed above: The sequence $(tu_n)_{n=0}^\infty$ has a limit function tu that is a weak solution of (2), i.e.

$$tu\in\mathcal{D}'(\mathbf{R}^3\times[0, \infty))\cap L^\infty([0, T_0], L^\infty(R^3))$$

for all $T_0>0$. Moreover we have

$$tu_{n+1}=E_1(t)g+\int_0^t E_1(t-\tau)\,\tau^{-4}(\tau u_n(\tau,.))^5\,d\tau$$

for $n=0, 1, \ldots$. By standard contraction methods we can show that (2) possesses a classical solution v for $t\in[0,\tilde{\varepsilon}]$ for some $\tilde{\varepsilon}>0$ and

$$v=E_1(t)g+\int_0^t E_1(t-\tau)\,\tau^{-4}(v(\tau,.))\,d\tau.$$

Hence we get

$$\left|\frac{v(t,.)}{t}-u_{n+1}(t,.)\right|\leqq\int_0^t\frac{\tau}{t}E_1(t-\tau)\left|\left(\frac{v(\tau,.)}{\tau}\right)^5-u_n(\tau,.)^5\right|d\tau,$$

where the positivity of the operator $E_1(t)$ has been used. Furthermore we note that

$$\sup\left\{\frac{1}{t}|v(x,t)|+|u_n(x,t)|:\ n\in N,\ x\in\mathbf{R}^3,\quad t\in(0,\tilde{\varepsilon}]\right\}\leqq C,$$

and it follows that

$$\left|\frac{v(t,.)}{t}-u_{n+1}(t,.)\right|\leqq C\int_0^t E_1(t-\tau)\left|\frac{v(\tau,.)}{\tau}-u_n(\tau,.)\right|d\tau.$$

Taking the L^4-norm with respect to x and applying the L^p estimates due to Miyachi (see [13]) we obtain

$$\|t^{-1}v(t,.)-u_{n+1}(t,.)\|\leqq C\int_0^t\|\tau^{-1}v(\tau,.)-u_n(\tau)\|\,d\tau$$

for $t\in[0,\tilde{\varepsilon}]$. Let n tend to infinity and apply Gronwall's lemma. This yields

$$v(t,x)=tu(t,x),\quad t\in[0,\tilde{\varepsilon}],$$

which proves our claim.

5. The case $k\geqq 6$

In Sections 2 to 4 we have assumed the nonlinearity to be u^5 in the Euler—Poisson—Darboux equation. The advantage in this case is that we can give explicit, analytic expressions for subsolutions and supersolutions. However, the same arguments work for the case u^k, $k\geqq 6$, with the obvious modifications for k being even, once we have the corresponding subsolutions and supersolutions. Moreover, since, given a positive supersolution V, $-V$ is a subsolution when k is an odd integer and the constant function 0 is a subsolution when k is an even integer, we confine the discussion to supersolutions.

It is natural to look for supersolutions that are independent of t or x. In particular, we restrict ourselves to t-independent supersolutions, i.e. solutions to the differential inequality

$$(9) \qquad \begin{cases} \Delta u + u^k \leqq 0, & x \in \mathbf{R}^3, \\ u > 0, \\ u \in C^2. \end{cases}$$

We first observe from Gidas/Spruck [4] that for $k=2$, 3 or 4 there are no such u so that (9) is satisfied with equality instead of inequality. We also observe that if u is a positive supersolution then the radially symmetric function we obtain by introducing polar coordinates around an arbitrary point x_0 and taking averages over spherical shells with respect to the point x_0 is a positive supersolution. This follows easily from Hölder's inequality. Here of course the supersolution need not be t-independent.

Inspired by the observation above we consider radially symmetric solutions to

$$\Delta u + u^k = 0, \quad x \in \mathbf{R}^3,$$

i.e. we study the equation

$$(10) \qquad \begin{cases} \dfrac{1}{r^2}\dfrac{d}{dr}\left(r^2 \dfrac{du}{dr}\right) + u^k = 0, & r > 0, \\ u'(0) = 0, \\ u > 0. \end{cases}$$

This equation is known as the **Lane—Emden equation of index** k and is studied in the context of stellar structure in astrophysics. The only known explicit solution apart from the linear cases $k=0$ and $k=1$, is the one we have given for $k=5$. However for $k>5$ it is known that (10) has a one-parameter family $\lambda^{2/(k-1)}u(\lambda x)$, $\lambda>0$, of solutions, which are positive, strictly decreasing with decay-rate $r^{-(2/(k-1))}$ as $r\to\infty$ and with $u'(0)=0$ and $u(0)>0$. This follows from a phase plane analysis of the equation (10) after performing a so called Emden transformation (see Chandrasekhar [1] and Joseph/Lundgren [8]). These solutions have the regularity needed to serve as supersolutions.

Finally we note that Theorem 1.1 and Theorem 1.2 cannot be improved using scaling arguments.

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P. Kumlin
Matematiska institutionen
Chalmers tekniska högskola
S-412 96 Göteborg
Sweden