Bi-invariant differential operators on the Euclidean motion group and applications to generalized Radon transforms

Fulton B. Gonzalez

Abstract

We determine the algebra of bi-invariant differential operators (i.e., the center of the universal enveloping algebra) of the group $M(n)$ of rigid motions of $\mathbf{R}^n$ by explicitly describing a set of $[\frac{1}{2}(n+1)]$ algebraically independent generators of orders $2, 4, 6, \ldots$. Passing to the complexification of the Lie algebra of $M(n)$ we then obtain a description of the algebra of bi-invariant differential operators on the connected Poincaré group $SO_0(p, q)\times\mathbf{R}^{p+q}$ (semidirect product). We also apply our main result to show how a certain generalization of the Radon transform, defined on the affine Grassmannian manifold of p-dimensional planes in $\mathbf{R}^n$, intertwines the $M(n)$-invariant differential operators on such manifolds.

1. Introduction

For a Lie group G let $\mathbf{D}(G)$ denote the algebra of left invariant differential operators on G and let $\mathbf{Z}(G)\subseteq\mathbf{D}(G)$ denote the algebra of left and right invariant differential operators on G. In this paper we determine the algebra $\mathbf{Z}(G)$ when G is the group $M(n)$ of rigid motions of the Euclidean space $\mathbf{R}^n$. We will show that $\mathbf{Z}(M(n))$ has $[\frac{1}{2}(n+1)]$ algebraically independent generators, having orders $2, 4, 6, \ldots$, and we will describe these generators explicitly.

Passing to the complexification of the Lie algebra of $M(n)$ we then obtain a description of the algebra $\mathbf{Z}(G)$, when G is the semidirect product $SO(n, \mathbf{C})\times\mathbf{C}^n$, and also when G is the general connected Poincaré group $SO_0(p, q)\times\mathbf{R}^{p+q}$.

Supported in part by NSF Grant DMS-8601965.

The problem of describing the algebra of bi-invariant differential operators on the above semidirect products was also considered by S. Takiff [14], but was only completely solved in the case $n \leqq 4$.

Next, let H be any closed subgroup of a Lie group G and let $\mathbf{D}(G/H)$ be the algebra of differential operators on the manifold G/H which are invariant under the action of G. If $\pi: G \to G/H$ is the natural projection, let $\mu: \mathbf{Z}(G) \to \mathbf{D}(G/H)$ be the homomorphism defined as in [7] by $(\mu(D)f) \circ \pi = D(f \circ \pi)$ for $D \in \mathbf{Z}(G)$ and $f \in C^\infty(G/H)$. Setting $G = M(n)$ and H the subgroup leaving a certain p-dimensional subspace of $\mathbf{R}^n$ invariant, the coset space G/H is then the manifold $G(p, n)$ of p-planes in $\mathbf{R}^n$. Using the description of $\mathbf{D}(G(p, n))$ in [4], we will show that the map $\mu: \mathbf{Z}(M(n)) \to \mathbf{D}(G(p, n))$ is surjective. Thus, in particular, $\mathbf{D}(G(p, n))$ is commutative.

As an application, we examine how certain generalizations of the Radon transform and its dual, considered by the author [3] and Strichartz [14], intertwine the invariant differential operators on the manifolds $G(p, n)$. Specifically, fix p and q between 0 and $n-1$ and choose an integer j with $\max(0, p+q-n) \leqq j \leqq \min(p, q)$. Define the transform $R(p, q, j)$ from functions on $G(p, n)$ to functions on $G(q, n)$ by

$$R(p, q, j)f(\eta) = \int f(\xi)\, d\xi, \quad \eta \in G(q, n)$$

when the integral is taken over all p-planes ξ which intersect a given q-plane η orthogonally in a j-dimensional plane. A result of Helgason on abstract Radon transforms [11] then enables us to show that for every $D \in \mathbf{Z}(M(n))$,

$$R(p, q, j) \circ \mu_p(D) = \mu_q(D) \circ R(p, q, j),$$

where μ_p and μ_q denote the projections of $\mathbf{Z}(M(n))$ onto $\mathbf{D}(G(p, n))$ and $\mathbf{D}(G(q, n))$, respectively. If $p+q=n-1$, $\mathbf{D}(G(p, n))$ and $\mathbf{D}(G(q, n))$ have the same number of algebraically independent generators [4] and in this special case one can find sets $\{E_i\}$ and $\{F_i\}$ of such generators of $\mathbf{D}(G(p, n))$ and $\mathbf{D}(G(q, n))$, respectively, such that

$$R(p, q, 0) \circ E_i = F_i \circ R(p, q, 0).$$

This generalizes a well-known formula for the Radon transform and its dual (Lemma 2.1 of [9]).

The author is indebted to Professor S. Helgason for introducing him to the subject and for offering valuable insights.

2. The algebra $\mathbf{Z}(M(n))$

The group $G=M(n)$ is isomorphic to the $(n+1)\times(n+1)$ matrix group

$$\left\{\begin{pmatrix} k & V \\ 0 & 1 \end{pmatrix} : k\in O(n),\ V\in\mathbf{R}^n\right\}, \tag{1}$$

and it acts on $\mathbf{R}^n$ by

$$\begin{pmatrix} k & V \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} Y \\ 1 \end{pmatrix} = k\cdot Y+V, \quad Y\in R^n.$$

Its Lie algebra $\mathfrak{g}$ is given by the set of matrices

$$S=\begin{pmatrix} T & Z \\ 0 & 0 \end{pmatrix}, \quad T\in so(n), \quad Z\in\mathbf{R}^n, \tag{2}$$

$so(n)$ being the Lie algebra of $O(n)$. The adjoint representation $\mathrm{Ad}=\mathrm{Ad}_G$ of the group G then satisfies

$$Ad\begin{pmatrix} k & V \\ 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} T & Z \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} kTk^{-1} & k\cdot Z-kTk^{-1}V \\ 0 & 0 \end{pmatrix}. \tag{3}$$

As usual, let E_{ij} denote the matrix $(\delta_{ri}\delta_{sj})_{1\leqq r,s\leqq n+1}$ and put

$$\begin{aligned} X_{ij} &= E_{ij}-E_{ji} \quad (1\leqq i<j\leqq n); \\ U_k &= E_{kn+1} \qquad (1\leqq k\leqq n). \end{aligned} \tag{4}$$

These vectors form a basis of $\mathfrak{g}$.

Let $S(\mathfrak{g})$ be the symmetric algebra over $\mathfrak{g}$ (consisting of polynomials in $\{X_{ij}, U_k\}$ with complex coefficients) and let $I(\mathfrak{g})$ be the algebra of $\mathrm{Ad}(G)$-invariant elements in $S(\mathfrak{g})$. As proved in [5], the symmetrization map

$$\lambda\colon S(\mathfrak{g})\to\mathbf{D}(G)$$

is a linear bijection. We recall that for any basis $\{Z_i\}$ of $\mathfrak{g}$ and any $f\in C^\infty(G)$,

$$\lambda(P)f(g)=\left\{P\left(\frac{\partial}{\partial t_1},\dots,\frac{\partial}{\partial t_s}\right)f(g\exp(\textstyle\sum_i t_iZ_i))\right\}_{(t_i)=0}, \quad P\in S(\mathfrak{g}),$$

where $g\in G$. Since λ commutes with the adjoint representation, its restriction to $I(\mathfrak{g})$ is a linear bijection onto $\mathbf{Z}(G)$. Although λ is not multiplicative, we have by Lemma 4.2 of [4] that if $P_1,\dots,P_m$ are algebraically independent generators of $I(\mathfrak{g})$, then $\lambda(P_1),\dots,\lambda(P_n)$ are algebraically independent generators of $\mathbf{Z}(G)$. Thus to characterize $\mathbf{Z}(G)$ it suffices to produce a set of algebraically independent generators of $I(\mathfrak{g})$.

To describe these generators of $I(\mathfrak{g})$ it is convenient to introduce some notation. Let $A=(a_{ij})$ be any $N\times N$ matrix, and for each $1\leqq k\leqq N$ let $1\leqq i_1<$

$<i_2<\ldots<i_k\leqq N$ be a choice of k indices in $\{1, \ldots, N\}$. For any such choice, let $D(i_1, i_2, \ldots, i_k)$ denote the $k\times k$ minor obtained from A by choosing entries a_{ij} when $i, j\in\{i_1, \ldots, i_k\}$. That is to say, $D(i_1, \ldots, i_k)=\det(a_{ij})_{k\times k}$ $(i, j\in\{i_1, \ldots, i_k\})$. Also, let

$$(5)\qquad P_k(A)=\sum_{i_1,\ldots,i_k} D(i_1, \ldots, i_k),\quad R_k(A)=\sum_{i_1,\ldots,i_{k-1}} D(i_1, \ldots, i_{k-1}, N)$$

where the sums extend over all choices of the given indices.

Theorem 2.1. *Consider the* $(n+1)\times(n+1)$ *skew-symmetric matrix with vector entries*

$$(6)\qquad A=\begin{pmatrix} 0 & X_{12}\ldots & X_{1n} & U_1 \\ -X_{12} & 0\ \ldots & X_{2n} & U_2 \\ \cdot & \cdot\ \ldots & \cdot & \cdot \\ \cdot & \cdot\ \ldots & \cdot & \cdot \\ \cdot & \cdot\ \ldots & \cdot & \cdot \\ -X_{1n} & -X_{2n}\ldots & 0 & U_n \\ -U_1 & -U_2\ \ldots & -U_n & 0 \end{pmatrix}$$

For $1\leqq j\leqq\left[\frac{1}{2}(n+1)\right]$ *let* $Q_j\in S(\mathfrak{g})$ *be the sum* $Q_j=R_{2j}(A)$. *(That is,* Q_j *is the sum of the* $2j\times 2j$ *skew-symmetric minors of* A *having vectors* U_k *in the last row and column.) Then the polynomials* Q_j *are algebraically independent generators of the algebra* $I(\mathfrak{g})$.

For the proof we view $S(\mathfrak{g})$ as the algebra of complex-valued polynomial functions on the dual space $\mathfrak{g}^*$. Then $I(\mathfrak{g})$ is identified with the algebra $I_0(\mathfrak{g}^*)$ of polynomial functions on $\mathfrak{g}^*$ invariant under the coadjoint representation Ad^* of G on $\mathfrak{g}^*$. Thus it suffices to obtain a set of generators for $I_0(\mathfrak{g}^*)$.

Consider now the linear bijection η of $so(n+1)$ onto $\mathfrak{g}^*$ given by

$$(7)\qquad \begin{pmatrix} X & U \\ -{}^tU & 0 \end{pmatrix}\to\eta_{X,U}\quad X\in so(n),\ U\in R^n$$

where, with S as in (2)

$$\eta_{X,U}(S)=\eta_{X,U}\begin{pmatrix} T & Z \\ 0 & 0 \end{pmatrix}=-\frac{1}{2}\operatorname{trace}\begin{pmatrix} X & U \\ -{}^tU & 0 \end{pmatrix}\begin{pmatrix} T & Z \\ -{}^tZ & 0 \end{pmatrix}=-\frac{1}{2}\operatorname{trace}(XT)+{}^tUZ.$$

Under this bijection, the coadjoint map $\mathrm{Ad}^*\begin{pmatrix} k & V \\ 0 & 1 \end{pmatrix}$ on $\mathfrak{g}^*$ corresponds to the transformation of $so(n+1)$ given by

$$(8)\qquad \begin{pmatrix} X & U \\ -{}^tU & 0 \end{pmatrix}\to\begin{pmatrix} k & V \\ 0 & 1 \end{pmatrix}\begin{pmatrix} X & U \\ -{}^tU & 0 \end{pmatrix}\begin{pmatrix} k^{-1} & 0 \\ {}^tV & 1 \end{pmatrix}.$$

$$=\begin{pmatrix} kXk^{-1}-V\,{}^tUk^{-1}+kU{}^tV & kU \\ -{}^tUk^{-1} & 0 \end{pmatrix}=\begin{pmatrix} X' & U' \\ -{}^tU' & 0 \end{pmatrix}.$$

Indeed,

$$\left\{\mathrm{Ad}^*\begin{pmatrix} k & V \\ 0 & 1\end{pmatrix}\cdot\eta_{X,U}\right\}\begin{pmatrix} T & Z \\ 0 & 0\end{pmatrix}=\eta_{X,U}\left(\mathrm{Ad}\begin{pmatrix} k & V \\ 0 & 1\end{pmatrix}^{-1}\cdot\begin{pmatrix} T & Z \\ 0 & 0\end{pmatrix}\right) \tag{9}$$

$$=-\frac{1}{2}\operatorname{trace}(Xk^{-1}Tk)+{}^tUk^{-1}TV+{}^tUk^{-1}Z.$$

On the other hand, by (8),

$$\eta_{X',U'}\begin{pmatrix} T & Z \\ 0 & 0\end{pmatrix}=-\frac{1}{2}\operatorname{trace}(kXk^{-1}T-V\,{}^tUk^{-1}T+kU\,{}^tVT)+{}^tUk^{-1}Z,$$

which is easily seen to agree with (9). Thus, under the bijection η, the algebra $I_0(\mathfrak{g}^*)$ consists by (8) of the polynomial functions on $so(n+1)$ invariant under the transformations

(i) $$\begin{pmatrix} X & U \\ -{}^tU & 0\end{pmatrix}\to\begin{pmatrix} k & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} X & U \\ -{}^tU & 0\end{pmatrix}\begin{pmatrix} k^{-1} & 0 \\ 0 & 1\end{pmatrix},\qquad k\in O(n)$$

(ii) $$\begin{pmatrix} X & U \\ -{}^tU & 0\end{pmatrix}\to\begin{pmatrix} I_n & V \\ 0 & 1\end{pmatrix}\begin{pmatrix} X & U \\ -{}^tU & 0\end{pmatrix}\begin{pmatrix} I_n & 0 \\ {}^tV & 1\end{pmatrix},\qquad V\in\mathbf{R}^n$$

I_n denoting the identity $n\times n$ matrix. Let x_{ij} $(1\leqq i,j\leqq n)$ and u_k $(1\leqq k\leqq n)$ denote the entry functions on the matrices $X\in so(n)$ and $U\in\mathbf{R}^n$, respectively. Then the bijection η identifies $\mathfrak{g}$ with the dual space $so(n+1)^*$ via

$$X_{ij}\to x_{ij},\quad U_k\to u_k \tag{10}$$

because $\eta_{X,U}(X_{ij})=x_{ij}$ and $\eta_{X,U}(U_k)=u_k$. Since the transformations (ii) consist of simultaneous elementary row and column operations involving the last row and column of the skew-symmetric matrix $\begin{pmatrix} X & U \\ -{}^tU & 0\end{pmatrix}$, it is clear from (10) and Lemma 2.2 at the end of this section that the polynomials Q_j do indeed belong to $I(\mathfrak{g})$.

Next let $(\mathfrak{g}^*)'$ be the subset of $\mathfrak{g}^*=so(n+1)$ consisting of the matrices $\begin{pmatrix} X & U \\ -{}^tU & 0\end{pmatrix}$ for which $|U|^2=u_1^2+\ldots+u_n^2\neq 0$. Then let $\mathfrak{g}_0^*\subset\mathfrak{g}^*$ be the subspace of matrices

$$\begin{pmatrix} 0 & 0 & u_1 \\ 0 & X' & 0 \\ -u_1 & 0 & 0\end{pmatrix}\qquad u_1\in\mathbf{R},\quad X'\in so(n-1). \tag{11}$$

Applying the transformations (i) and (ii) above, we see that the $\mathrm{Ad}^*(G)$-orbit of each point in $(\mathfrak{g}^*)'$ intersects $\mathfrak{g}_0^*$. Consider the subgroup $G_0\subset G$ of elements $g\in G$ in (1) with $V=0$ and k of the form

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & k_1\end{pmatrix}\qquad k_1\in O(n-1)$$

The action of $\mathrm{Ad}^*(G_0)$ on $\mathfrak{g}_0^*$ is given by

$$\begin{pmatrix} 0 & 0 & u_1 \\ 0 & X' & 0 \\ -u_1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & \pm u_1 \\ 0 & k_1 X' k_1^{-1} & 0 \\ \mp u_1 & 0 & 0 \end{pmatrix}.$$

Let $I_{G_0}(\mathfrak{g}_0^*)$ denote the corresponding algebra of $\mathrm{Ad}^*(G_0)$ — invariant polynomial functions on $\mathfrak{g}_0^*$. The restriction mapping $Q \to \bar{Q} = Q|\mathfrak{g}_0^*$ then maps $I_0(\mathfrak{g}^*)$ into $I_{G_0}(\mathfrak{g}_0^*)$. Since $\mathrm{Ad}^*(G)\cdot\mathfrak{g}_0^*$ contains $(\mathfrak{g}^*)'$, which is dense in $\mathfrak{g}^*$, the restriction map is injective. Now because of Lemma 2.2 below, $I_{G_0}(\mathfrak{g}_0^*)$ is generated by u_1^2 and the algebraically independent polynomials $P_{2k}(X')$ $\left(1 \leq k \leq l = \left[\frac{n-1}{2}\right]\right)$, where as in (5) $P_{2k}(X')$ is the sum of the $2k \times 2k$ skew-symmetric minors of X'. It follows that $u_1^2, u_1^2 P_2, \ldots, u_1^2 P_{2l}$ which coincide with $\bar{Q}_1, \bar{Q}_2, \ldots, \bar{Q}_{l+1}$ are algebraically independent so by the injectivity of the map $Q \to \bar{Q}$ the polynomials $Q_1, \ldots, Q_{l+1}$ are algebraically independent over $\mathbf{C}$.

It remains to prove that the algebra I generated by $Q_1, \ldots, Q_{l+1}$ equals $I_0(\mathfrak{g}^*)$. Suppose there exists $Q \in I_0(\mathfrak{g}^*)$ not in I. Then $\bar{Q}$ is a polynomial

$$\bar{Q} = S(u_1^2, P_2, \ldots, P_{2l}) = S(\bar{Q}_1, \bar{Q}_2/\bar{Q}_1, \ldots, \bar{Q}_{l+1}/\bar{Q}_1).$$

By the injectivity

$$Q = S(Q_1, Q_2/Q_1, \ldots, Q_{l+1}/Q_1) = \frac{S_1(Q_1, \ldots, Q_{l+1})}{Q_1^k}, \tag{12}$$

where S_1 is another polynomial. Since $Q \notin I$, we have $k \geq 1$. By the algebraic independence of the Q_i, we may assume that the variable t_1 does not divide $S_1(t_1, \ldots, t_{l+1})$. Write

$$S_1(t_1, \ldots, t_{l+1}) = S'(t_2, \ldots, t_{l+1}) + t_1 S''(t_1, \ldots, t_{l+1}).$$

Then $S'(t_2, \ldots, t_{l+1}) \neq 0$. We shall now show that there exists a complex matrix $\zeta_0 \in so(n+1, \mathbf{C})$ such that

$$Q_1(\zeta_0) = 0, \quad S'(Q_2(\zeta_0), \ldots, Q_{l+1}(\zeta_0)) \neq 0. \tag{13}$$

For this consider the complex skew-symmetric matrices of the form

$$\zeta = \begin{bmatrix} 0 & 0 & 0 & 0 \;\ldots\; 0 & 1 \\ 0 & 0 & z_{23} & z_{24} \;\ldots\; z_{2n} & i \\ 0 & -z_{23} & 0 & z_{34} \;\ldots\; z_{3n} & 0 \\ 0 & -z_{24} & -z_{34} & 0 \;\ldots\; z_{4n} & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & -z_{2n} & -z_{3n} & -z_{4n} \;\ldots\; 0 & 0 \\ -1 & -i & 0 & 0 \;\ldots\; 0 & 0 \end{bmatrix} \tag{14}$$

and put $Z=(z_{ij})_{2\leqq i,j\leqq n}$. Also, for each $k=1, \ldots, l$, let $Q_k'(Z)$ denote the sum of the $2k\times 2k$ skew-symmetric minors of Z with entries from $z_{23}, z_{24}, \ldots, z_{2n}$ in the first row and column. Then it is easy to see (because of a pairwise cancellation of minors) that

$$Q_1(\zeta)=0, \quad Q_k(\zeta)=Q_{k-1}'(Z) \quad (2\leqq k\leqq l+1).$$

Thus, $S'(Q_2(\zeta), \ldots, Q_{l+1}(\zeta))=S'(Q_1'(Z), \ldots, Q_l'(Z))$. However, the polynomial functions $Q_1', \ldots, Q_l'$ were already seen to be algebraically independent over $\mathbf{C}$ so there exists ζ_0 of the form (14) satisfying (13). This contradicts (12). Thus, $I=I_0(\mathfrak{g}^*)$.

To complete the proof of Theorem 2.1, we recall the following result [12, Ch. XII].

Lemma 2.2. *Let J be the algebra of polynomial functions on $so(n)$ invariant under the adjoint action $X\to kXk^{-1}$ of $O(n)$. Then J is generated by the polynomials P_{2k} $\left(1\leqq k\leqq\left[\frac{n}{2}\right]\right)$ where as in (5) $P_{2k}(X)$ is the sum of the $2k\times 2k$ skew-symmetric minors of X. Moreover the P_{2k} are algebraically independent over $\mathbf{C}$.*

Proof. Viewing each real $n\times n$ matrix A as a linear transformation of $\mathbf{R}^n$, we have $P_k(A)=\operatorname{trace}(\Lambda^k A\colon \Lambda^k\mathbf{R}^n\to\Lambda^k\mathbf{R}^n)$. Thus $P_k(A)$ is certainly invariant under any change of basis transformation $A\to\tau A\tau^{-1}$ $(\tau\in GL(n))$. (In fact, $\pm P_k(A)$ is the coefficient of λ^{n-k} in the characteristic polynomial $\det(\lambda I_n-A)$.) Now each $X\in so(n)$ is conjugate under $\operatorname{Ad}(O(n))$ to an element of the set D of matrices

$$\begin{pmatrix} 0 & s_1 & & & \\ -s_1 & 0 & & & \\ & & 0 & s_2 & \\ & & -s_2 & 0 & \\ & & & & \ddots \end{pmatrix}.$$

Let $Q\in J$ and $\bar{Q}$ the restriction $Q|D$. Since $\operatorname{Ad}(O(n))D=so(n)$, the map $Q\to\bar{Q}$ is injective. Also, $\bar{Q}$ is invariant under the transformation $s_i\to\varepsilon_i s_{\sigma(i)}$ where $\varepsilon_i=\pm 1$ and σ is any permutation (the Weyl group of $so(n)$). Thus, $\bar{Q}$ is a polynomial in the algebraically independent elementary symmetric polynomials of $s_1^2, \ldots, s_t^2$ $\left(t=\left[\frac{n}{2}\right]\right)$. However, these polynomials are just the restrictions to D of the polynomials P_{2k}. Thus, by the injectivity mentioned, the P_{2k} are algebraically independent and Q is a polynomial in them.

The proof of Theorem 2.1 is now complete.

3. Central operators on other semidirect products

Let G be any real Lie group with Lie algebra $\mathfrak{g}$. If $\mathfrak{A}(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$ (with complex coefficients), then we have the identities [6]

$$\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{g}^{\mathbf{C}}) = \mathbf{D}(G) = \mathbf{D}(G_0) \tag{15}$$

where G_0 is the component of G containing the identity and $\mathfrak{g}^{\mathbf{C}}$ is the complexification of $\mathfrak{g}$. Letting $\mathfrak{z}(\mathfrak{g})$ denote the center of $\mathfrak{A}(\mathfrak{g})$, we also have

$$\mathbf{Z}(G) \subseteqq \mathbf{Z}(G_0) = \mathfrak{z}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}^{\mathbf{C}}) \tag{16}$$

where $\mathbf{Z}(G_0)$ consists of the bi-invariant differential operators on G_0. Now extending each operator $\operatorname{ad} X$ $(X\in(\mathfrak{g}))$ to a derivation of the symmetric algebra $S(\mathfrak{g})$, we define the polynomial algebra $I_1(\mathfrak{g})$ to be the set $\{P\in S(\mathfrak{g})|\operatorname{ad}(X)P=0 \text{ for all } X\in\mathfrak{g}\}$. Then $I_1(\mathfrak{g})$ coincides with the $\operatorname{Ad}(G_0)$-invariants in $S(\mathfrak{g})$, $I(\mathfrak{g})\subseteqq I_1(\mathfrak{g})$ and the symmetrization map λ is a bijection of $I_1(\mathfrak{g})$ onto $\mathfrak{z}(\mathfrak{g})$.

Now take $G=M(n)$. Then G_0 is the semidirect product $SO(n)\times\mathbf{R}^n$. By the same proof as that of Theorem 2.1, with only the notation $k\in O(n)$ changed to $k\in SO(n)$, it can be shown that the algebra $I_1(\mathfrak{g})$ is also generated by the polynomials $Q_1, \dots, Q_{l+1}$. Thus $I_1(\mathfrak{g})=I(\mathfrak{g})$ and so by (16), $\mathfrak{z}(\mathfrak{g})=\mathbf{Z}(G)$. By passing to the complexification, we obtain generators for $I_1(\mathfrak{g}^{\mathbf{C}})$.

Theorem 3.1. *For the Lie group $SO(n,\mathbf{C})\times\mathbf{C}^n$ with Lie algebra $\mathfrak{g}^{\mathbf{C}}$ and basis vectors X_{ij}, U_k as in* (4), *the algebra $I_1(\mathfrak{g}^{\mathbf{C}})$ is generated by the algebraically independent polynomials $Q_1, \dots, Q_{l+1}$ in Theorem* 2.1.

Next let H be the connected general Poincaré group $SO_0(p, n-p)\times\mathbf{R}^n$, with Lie algebra $\mathfrak{H}=so(p,n-p)\times\mathbf{R}^n$. $\mathfrak{H}$ has basis vectors $X_{rs}=E_{rs}-E_{sr}$ $(1\leqq r<s<p$, $p+1\leqq r<s\leqq n)$, $Y_{rs}=E_{rs}+E_{sr}$ $(1\leqq r\leqq p,\ p+1\leqq s\leqq n)$, and $U_k=E_{k,n+1}$ $(1\leqq k\leqq n)$. The complex Lie algebra $\mathfrak{g}^{\mathbf{C}}$ is canonically isomorphic to the complexification $\mathfrak{H}^{\mathbf{C}}$ via the map

$$\varphi\colon \begin{pmatrix} X_1 & X_2 & U_1 \\ -{}^tX_2 & X_3 & U_2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} X_1 & iX_2 & U_1 \\ i{}^tX_2 & X_3 & -iU_2 \\ 0 & 0 & 0 \end{pmatrix}. \tag{17}$$

Here $X_1\in so(p,\mathbf{C})$, $X_3\in so(n-p,\mathbf{C})$, X_2 is any complex $p\times(n-p)$ matrix, $U_1\in\mathbf{C}^p$, and $U_2\in\mathbf{C}^{n-p}$. From Theorem 3.1 and (17) it follows that the polynomials $Q_j''\{\varphi(X_{rs})\}, \{\varphi(U_k)\}\})$ $(1\leqq j\leqq[\frac{1}{2}(n+1)])$ constitute a set of algebraically independent generators of $I(\mathfrak{H}^{\mathbf{C}})=I(\mathfrak{H})$. Now

$$\varphi(X_{rs})=\begin{cases}X_{rs} & 1\leqq r<s\leqq p \quad \text{or} \quad p+1\leqq r<s\leqq n;\\ iY_{rs} & 1\leqq r\leqq p,\ p+1\leqq s\leqq n;\end{cases}$$

$$\varphi(U_k)=\begin{cases}U_k & 1\leqq k\leqq p;\\ -iU_k & p+1\leqq k\leqq n.\end{cases}$$

From Theorem 2.1 we obtain the generators of the invariant algebra $I(\mathfrak{H})$

Theorem 3.2. *Consider the* $(n+1)\times(n+1)$ *matrix with vector entries*

(18)

$$B=\begin{bmatrix} 0 & X_{12} & \dots & X_{1p} & Y_{1,p+1} & Y_{1,p+2} & \dots\ Y_{1n} & U_1\\ -X_{12} & 0 & \dots & X_{2p} & Y_{2,p+1} & Y_{2,p+2} & \dots\ Y_{2n} & U_2\\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots\\ -X_{1p} & -X_{2p} & \dots & 0 & Y_{p,p+1} & Y_{p,p+2} & \dots\ Y_{pn} & U_p\\ Y_{1,p+1} & Y_{2,p+1} & \dots & Y_{p,p+1} & 0 & X_{p+1,p+2} & \dots\ X_{p+1,n} & U_{p+1}\\ Y_{1,p+2} & Y_{2,p+2} & \dots & Y_{p,p+2} & -X_{p+1,p+2} & 0 & \dots\ X_{p+2,n} & U_{p+2}\\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots\\ Y_{1n} & Y_{2n} & \dots & Y_{pn} & -X_{p+1,n} & -X_{p+2,n} & \dots\ 0 & U_n\\ -U_1 & -U_2 & \dots & -U_p & U_{p+1} & U_{p+2} & \dots\ U_n & 0 \end{bmatrix}.$$

Then the polynomials $R_{2j}(B)$ $\left(1\leqq j\leqq\left[\frac{n+1}{2}\right]\right)$ *are algebraically independent generators of* $I(\mathfrak{H})$.

In fact, $Q_j(\{\varphi(X_{kl})\}, \{\varphi(U_k)\})=R_{2j}(B)$.

As an example, let $n=4$ and $p=3$. Then computing by means of Theorem 3.2, the algebra of bi-invariant differential operators on the connected Poincaré group $SO_0(3, 1)\times R^4$ can be shown to have two algebraically independent generators, these being the images under the symmetrization λ of the second order polynomial $U_1^2+U_2^2+U_3^2-U_4^2$ and the fourth order polynomial $(X_{12}U_3-X_{13}U_2+X_{23}U_1)^2-(X_{12}U_4+X_{14}U_2-Y_{24}U_1)^2-(X_{13}U_4+Y_{14}U_3-Y_{34}U_1)^2-(X_{23}U_4+Y_{24}U_3-Y_{34}U_2)^2$. This result has been obtained previously by Varadarajan (see [15]).

4. Projections on Grassmannians and applications to Radon transforms

In this section G will denote the Euclidean motion group $M(n)$. For $0\leqq p\leqq n-1$, let E_p denote the subspace spanned by the first p basis elements of $\mathbf{R}^n$ ($E_p=0$ if $p=0$), and let H be the subgroup of G leaving E_p fixed. Then $H=M(p)\times O(n-p)$ and G/H is the affine Grassmann manifold $G(p, n)$ of p-planes in $\mathbf{R}^n$. Denote by

$\mathbf{D}(G/H)$ the algebra of differential operators on G/H which are invariant under the G-action. If $\pi\colon G\to G/H$ is the natural projection, we have a homomorphism μ of $\mathbf{Z}(G)$ into $\mathbf{D}(G/H)$ given by

$$(\mu(D)f)\circ\pi = D(f\circ\pi) \tag{19}$$

for $D\in\mathbf{Z}(G)$ and $f\in C^\infty(G/H)$ ([7]). We note that (19) also defines $\mu(D)\in\mathbf{D}(G/H)$ for any $\mathbf{D}\in\mathbf{D}(G)$ which is invariant under right translations by all $h\in H$.

Theorem 4.1. *μ maps $\mathbf{Z}(G)$ onto $\mathbf{D}(G/H)$.*

Remark. Since $\mathbf{Z}(G)$ is commutative, so is $\mathbf{D}(G/H)$ by Theorem 4.1. The commutativity of $\mathbf{D}(G/H)$ is also a consequence of the fact that the pair (G, H) is a symmetric pair ([1], [13]).

For the proof of Theorem 4.1, we decompose the Lie algebra $\mathfrak{g}$ of G into a direct sum of $\mathfrak{H}$, the Lie algebra of H, and an Ad(H)-invariant subspace $\mathfrak{M}$. Since H consists of the matrices

$$h=\begin{pmatrix} a & 0 & V\\ 0 & b & 0\\ 0 & 0 & 1\end{pmatrix}\quad \begin{matrix} a\in O(p),\ b\in O(n-p),\\ \\ V\in\mathbf{R}^p\end{matrix} \tag{20}$$

we define $\mathfrak{M}\subseteqq\mathfrak{g}$ as the subspace of matrices

$$T=\begin{pmatrix} 0 & Y & 0\\ -{}^tY & 0 & W\\ 0 & 0 & 0\end{pmatrix}\quad \begin{matrix} Y \text{ any real } p\times(n-p) \text{ matrix},\\ \\ W\in\mathbf{R}^{n-p}.\end{matrix}$$

Then $\mathfrak{g}=\mathfrak{H}\oplus\mathfrak{M}$ and since

$$\mathrm{Ad}(h)T=\begin{pmatrix} 0 & aYb^{-1} & 0\\ -b^tYa^{-1} & 0 & b\cdot W+b^tYa^{-1}V\\ 0 & 0 & 0\end{pmatrix}$$

$\mathfrak{M}$ is Ad(H)-invariant. Now for every $P\in S(\mathfrak{g})$, there exists a unique polynomial $\bar{P}\in S(\mathfrak{M})$ such that $P-\bar{P}\in S(\mathfrak{g})\mathfrak{H}$. Let $I(\mathfrak{M})$ be the algebra of Ad(H)-invariants in $S(\mathfrak{M})$. Then the map $P\to\bar{P}$ takes $I(\mathfrak{g})$ into $I(\mathfrak{M})$. Since the pair (G, H) is reductive [4], it suffices by [10, Chapter II, Proposition 5.32] to prove that the map $P\to\bar{P}$ takes $I(\mathfrak{g})$ onto $I(\mathfrak{M})$. Now $\mathfrak{M}$ has basis vectors X_{ij} $(1\leqq i\leqq p, p+1\leqq j\leqq n)$ and U_k $(p+1\leqq k\leqq n)$. We recall the characterization of $I(\mathfrak{M})$ in terms of these basis vectors [4].

Lemma 4.2. *Consider the $(p+1)\times(n-p)$ matrix with vector entries*

$$C=\begin{pmatrix} U_{p+1} & \cdots & U_n\\ X_{1,p+1} & \cdots & X_{1,n}\\ X_{p,p+1} & \cdots & X_{p,n}\end{pmatrix}.$$

For $1 \leqq k \leqq \min(p+1, n-p)$ *let* $T_k \in S(\mathfrak{M})$ *be the sum of the squares of the* $k \times k$ *minors of* C *having vectors* U_j *in the first row:*

$$T_k = \sum_{\substack{p+1 \leqq i_1 < \ldots < i_k \leqq n \\ 1 \leqq j_1 < \ldots < j_{k-1} \leqq p}} \det{}^2 \begin{pmatrix} U_{i_1} & \ldots & U_{i_k} \\ X_{j_1, i_1} & \ldots & X_{j_1, i_k} \\ & \ldots & \\ X_{j_{k-1}, i_1} & \ldots & X_{j_{k-1}, i_k} \end{pmatrix}.$$

Then the polynomials T_k *are algebraically independent generators of* $I(\mathfrak{M})$.

We will show that for the generators Q_k of $I(\mathfrak{g})$ in Theorem 2.1, $\bar{Q}_k = T_k$ when $1 \leqq k \leqq \min(p+1, n-p)$. Since the map $P \to \bar{P}$ is a homomorphism, this will show that it is surjective from $I(\mathfrak{g})$ to $I(\mathfrak{M})$.

For this purpose it is convenient to identify $S(\mathfrak{g})$ and $S(\mathfrak{M})$ with the algebras of polynomial functions on the dual spaces $\mathfrak{g}^*$ and $\mathfrak{M}^*$, respectively. Thus, as in the proof of Theorem 2.1, $I(\mathfrak{g}) = I_0(\mathfrak{g}^*)$. By the same token, $I(\mathfrak{M})$ is identified with the algebra $I_H(\mathfrak{M}^*)$ of polynomial functions Q on $\mathfrak{M}^*$ invariant under the co-isotropy representation $\mathrm{Ad}_G^*(H)$ on $\mathfrak{M}^*$

$$Q(\mathrm{Ad}^*(h)f) = Q(f) \quad h \in H, \ f \in \mathfrak{m}^*.$$

By letting each $f \in \mathfrak{M}^*$ be identically zero on $\mathfrak{H}$ we may assume $\mathfrak{M}^* \subset \mathfrak{g}^*$. If $P \in S(\mathfrak{g})$, $\bar{P}$ then coincides with the restriction $P|\mathfrak{M}^*$. Obviously if $P \in I(\mathfrak{g}^*)$ then $\bar{P} \in I_H(\mathfrak{M}^*)$. Under the bijection (7) of $so(n+1)$ onto $\mathfrak{g}^*$, the subspace $\mathfrak{M}^* \subset \mathfrak{g}^*$ corresponds to the subspace of skew-symmetric matrices of the form

$$(21) \qquad A = \begin{pmatrix} 0 & X & 0 \\ -{}^tX & 0 & U \\ 0 & -{}^tU & 0 \end{pmatrix} \quad \begin{array}{l} X \text{ any real } p \times (n-p) \text{ matrix,} \\ U \in \mathbf{R}^{n-p}. \end{array}$$

From this we also obtain a linear bijection of $\mathfrak{M}^*$ onto the space $M_{p+1, n-p}$ of real $(p+1) \times (n-p)$ matrices as follows:

$$(22) \qquad A \to \begin{pmatrix} {}^tU \\ X \end{pmatrix} = \begin{pmatrix} u_{p+1} & \ldots & u_n \\ x_{1, p+1} & \ldots & x_{1n} \\ x_{p, p+1} & \ldots & x_{pn} \end{pmatrix}.$$

By means of the transpose map, $\mathfrak{M}$ corresponds to the dual space $M^*_{p+1, n-p}$, and by (10) the basis vectors of $\mathfrak{M}$ correspond to the entry functions of $M_{p+1, n-p}$ via

$$(23) \qquad \begin{array}{ll} X_{ij} \to x_{ij} & 1 \leqq i \leqq p, \ p+1 \leqq j \leqq n; \\ U_k \to u_k & p+1 \leqq k \leqq n. \end{array}$$

Thus the polynomials T_k are polynomial functions on the space $M_{p+1, n-p}$, just as the Q_k are polynomial functions on $so(n+1)$. Moreover, it is easy to see that for any matrix A of the form (21),

$$Q_k(A) = T_k\begin{pmatrix} {}^tU \\ X \end{pmatrix} \quad k = 1, \ldots, \min(p+1, n-p).$$

Thus it follows that $\bar{Q}_k = T_k$, and this proves Theorem 4.1.

Remark. If $k > \min(p+1, n-p)$, then $Q_k|\mathfrak{M}^* = 0$.

Corollary 4.3. *The operators* $\mu(\lambda(Q_1)), \ldots, \mu(\lambda(Q_m))$ $(m = \min(p+1, n-p))$ *are algebraically independent generators of* $\mathbf{D}(G(p, n))$.

Proof. By [4, Lemma 4.2] the operators $\mu(\lambda(T_1)), \ldots, \mu(\lambda(T_m))$ are algebraically independent generators of $\mathbf{D}(G(p, n))$. Since $\bar{Q}_k = T_k$ $(1 \leqq k \leqq m)$, we have

$$\mu(\lambda(Q_k)) = \mu(\lambda(T_k)) + \text{lower order terms} \tag{24}$$

([7]). Now suppose $P = \sum a_{n_1, \ldots, n_m} x_1^{n_1} \ldots x_m^{n_m}$ is a nonzero polynomial such that the differential operator $D = P(\mu(\lambda(Q_1)), \ldots, \mu(\lambda(Q_m))) = 0$. Let $D' = P(\mu(\lambda(T_1)), \ldots, \mu(\lambda(T_m)))$. Then $D' \neq 0$ and by (24), order $(D'-D) <$ order (D'). This yields order $(D') <$ order (D'), a contradiction. Thus $\mu(\lambda(Q_1)), \ldots, \mu(\lambda(Q_m))$ are algebraically independent. Next let $D \in \mathbf{D}(G(p, n))$. Then we may write $D = P(\mu(\lambda(T_1)), \ldots, \mu(\lambda(T_m)))$ for some polynomial P. Setting

$$D_1 = P(\mu(Q_1), \ldots, \mu(\lambda(Q_m)))$$

we have $D = D_1 + D_2$, where order $(D_2) <$ order (D). The corollary follows by induction on the order of D_2.

Now fix a value of q between 0 and $n-1$, and fix j between $\max(0, p+q-n)$ and $\min(p, q)$. We consider a generalization of the Radon transform and its dual, due to Strichartz [14], from functions on $G(p, n)$ to functions on $G(q, n)$. For a fixed q-plane η, let $\hat{\eta}$ be the set of all p-planes ξ intersecting η orthogonally in a j-dimensional plane. Then $\hat{\eta}$ is a closed submanifold of $G(p, n)$ and there exists a canonical measure $d\mu(\xi)$ on $\hat{\eta}$ invariant under all Euclidean motions preserving η (cf. below). For any suitable function f on $G(p, n)$, the transform $R(p, q, j)f$ is a function on $G(q, n)$ defined by

$$R(p, q, j)f(\eta) = \int_{\hat{\eta}} f(\xi)\, d\mu(\xi), \quad \eta \in G(q, n). \tag{25}$$

For our purposes it is necessary to formulate this integral transform in terms of homogeneous spaces in duality. Let $e_1, \ldots, e_n$ be the usual basis of R^n, let E_p be as before the span of $e_1, \ldots, e_p$ and let E_q be the span of $e_{p-j+1}, \ldots, e_{p-j+q}$. Then E_p and E_q meet orthogonally in a j-dimensional plane. If H_p and H_q are the respective subgroups of G leaving E_p and E_q invariant, then H_p consists of the $(n+1)\times(n+1)$

matrices h in (20) while H_q consists of the $(n+1)\times(n+1)$ matrices

$$\begin{pmatrix} a' & 0 & 0 & 0 \\ 0 & b' & 0 & V' \\ 0 & 0 & c' & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{array}{l} a' \in O(p-j),\ b' \in O(q), \\ c' \in O(n+j-p-q),\ V' \in \mathbf{R}^q. \end{array}$$

Also, $G(p,n)=G/H_p$ and $G(q,n)=G/H_q$.

Proposition 4.4. *The manifolds G/H_p and G/H_q are homogeneous spaces in duality. That is to say, the groups G, H_p, H_q, and $H=H_p\cap H_q$ satisfy the following properties:*

(i) *They are all unimodular.*

(ii) *If $h_p \in H_p$ satisfies $h_p H_q \subset H_q H_p$, then $h_p \in H_q$.*

(iii) *$H_p H_q$ is a closed subset of G.*

The proof is straightforward and shall be omitted. (See [9] for the assumptions underlying homogeneous spaces in duality.)

The transform $R(p,q,j)$ is then the integral transform associated with the double fibration

$$\begin{array}{ccc} & G/H & \\ \swarrow & & \searrow \\ G/H_p & & G/H_q \end{array} \tag{26}$$

That is to say, if $\eta=g\cdot E_q$ $(g\in G)$, then $\hat{\eta}=\{gh_q\cdot E_p | h_q \in H_q\}$ and

$$R(p,q,j)f(\eta)=\int_{H_q/H} f(gh_q\cdot E_p)\,d(h_q)_H$$

where $d(h_q)_H$ is the H_q-invariant measure on H_q/H. (See [9].) A result of Helgason [11, Proposition 4.1] states that an integral transform associated with a double fibration such as (26) intertwines the G-invariant differential operators in G/H_p and G/H_q arising from operators in $\mathbf{Z}(G)$.

Proposition 4.5. *For any $D\in\mathbf{Z}(G)$, let $\mu_p(D)$ and $\mu_q(D)$ be the projections of D on G/H_p and G/H_q, respectively, as in* (19). *Then for any $f\in C_c^\infty(G(p,n))$,*

$$R(p,q,j)\big(\mu_p(D)f\big)=\mu_q(D)\big(R(p,q,j)f\big). \tag{27}$$

Note that by Theorem 4.1, (27) is a statement about how $R(p,q,j)$ intertwines *all* G-invariant differential operators on G/H_p and G/H_q.

Finally we consider the case $q=n-p-1$. By Corollary 4.3, the algebras $\mathbf{D}(G(p,n))$ and $\mathbf{D}(G(q,n))$ have the same number of algebraically independent generators, these being $E_i=\mu_p(\lambda(Q_i))$ and $F_i=\mu_q(\lambda(Q_i))$ $(1\leq i\leq \min(p+1,q+1))$ respectively. (27) shows how $R(p,q,j)$ intertwines these generators:

$$R(p,q,j)\circ E_i = F_i\circ R(p,q,j). \tag{28}$$

When $j=0$, the transform $R(p, q, 0)$ is injective (and in fact was inverted explicitly in [3]), and (28) generalizes the well known relations for the Radon transform $R(=R(0, n-1, 0))$ and its dual $R^t(=R(n-1, 0, 0))$ on $\mathbf{R}^n$:

$$R(Lf) = \square(Rf), \quad R^t(\square\varphi) = LR^t\varphi,$$

for all $f \in C_c^\infty(R^n)$, $\varphi \in C_c^\infty(G(n-1, n))$, where L is the Laplacian on $\mathbf{R}^n$ and $\square$ is the Laplacian on the fibers of the vector bundle $G(n-1, n)$ ([8], [9]).

References

1. Duflo, M., Opérateurs invariants sur un espace symétrique, *C.R. Acad. Sci. Paris Sér. A* **289** (1979), 135—137.
2. Gonzalez, F., *Ph. D. Thesis*, M.I.T. (1984).
3. Gonzalez, F., Radon transforms on Grassmann manifolds, *J. Funct. Anal.* (to appear).
4. Gonzalez, F. and Helgason, S., Invariant differential operators on Grassmann manifolds, *Adv. in Math.* **60** (1986), 81—91.
5. Harish-Chandra, On representations of Lie algebras, *Ann. Math.* **50** (1949), 900—915.
6. Harish-Chandra, The characters of semi-simple Lie groups, *Trans. Amer. Math. Soc.* **83** (1956), 98—163.
7. Helgason, S., Differential operators on homogeneous spaces, *Acta Math.* **102** (1959), 239—299.
8. Helgason, S., The Radon transform on Euclidean spaces, compact two-point homogeneous spaces, and Grassmann manifolds, *Acta Math.* **113** (1965), 153—180.
9. Helgason, S., *The Radon Transform,* Birkhäuser, Boston, 1980.
10. Helgason, S., *Groups and Geometric Analysis,* Academic Press, 1984.
11. Helgason, S., *Some results on Radon transforms, Huygens' principle, and X-ray transforms,* Contemporary Mathematics, Vol. 63 (Integral Geometry), Amer. Math. Soc. 1986.
12. Kobayashi, S. and Nomizu, K., *Foundations of Differential Geometry, II,* Wiley (Interscience), New York, 1969.
13. Lichnerowicz, A., Opérateurs différentiels sur un espace symétrique, *C.R. Acad. Sci. Paris* **257** (1963), 3548—3550.
14. Strichartz, R., Harmonic analysis on Grassmannian bundles, *Trans. Amer. Math. Soc.*, **296** (1986), 387—409.
15. Takiff, S., Invariant polynomials on Lie algebras of inhomogeneous unitary and special orthogonal groups, *Trans. Amer. Math. Soc.* **170** (1972), 221—230.

Received March 3, 1987

Department of Mathematics and Statistics
Case Western Reserve University
Cleveland, Ohio 44106
U.S.A.

Present Address: Department of Mathematics
Tufts University
Medford, Massachusetts 02155
U.S.A.