Branched structures associated with Lamé's equation

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Branched projective and affine structures on compact Riemann surfaces of genus $g>1$ were first constructed by Mandelbaum [8]. More recently, elliptic functions have been used [9] to construct branched projective structures on an arbitrary compact Riemann surface M of genus $g=1$. In this case, it has been shown that each projective structure with unique arbitrary branch point $P \in M$ of fixed ramification order $\mu \in Z^+ - \{1\}$ is represented by a 2-connection of the form $\varphi_\lambda(z) = \frac{1-\mu^2}{2} \wp(z-\pi^{-1}(P)) + \lambda$, $\lambda \in \mathbf{C}$, where $\wp(z)$ is the Weierstrass function [10] for the universal covering group G of M and $\pi\colon \mathbf{C} \to M$ is the universal covering map. Furthermore, it is known [2] that the branched local coordinates for each structure are the (locally meromorphic) ratios of linearly independent solutions to the equation

$$y'' + \frac{\varphi_\lambda}{2} y = 0. \tag{1}$$

It should be noted that if

$$G = \{T_{(m,n)}(z) = z + m + n\tau,\ \forall (m,n) \in \mathbf{Z} \times \mathbf{Z},\ \text{fixed } \tau \ni \operatorname{Im}(\tau) > 0\}, \tag{2}$$

then Eq. (1) is a Hill's equation [7] with primitive periods 1 and τ and, in particular, is a Lamé's equation in Weierstrassian form.

In this paper, we use the spectral theory of Hill's equation in conjunction with results on projective and affine structures found in [5], [6], [8] and [9] to further classify the branched projective and affine structures on a compact torus M having modulus $\tau \in iR^+$. The reader is referred to [8] and [9] for detailed definitions of branched affine and projective structures and their associated divisors and connections and for preliminary results concerning these structures on surfaces of genus one.

Recall that Eq. (1) (and, in fact, any Hill's equation whose coefficient, excluding λ, is real-valued on some horizontal line in $\mathbf{C}$) has discriminant function $\Delta(\lambda)$ with corresponding characteristic roots λ satisfying $\Delta(\lambda)=\pm 2$. The double roots of $\Delta(\lambda)=\pm 2$ correspond to *coexisting* solutions y_i $(i=1, 2)$ of Eq. (1), i.e., solutions both of period 1 or 2; the simple roots are precisely all the endpoints of instability intervals for Equation (1).

We now proceed with a detailed description of our findings.

Lemma 1. *The general Lamé's equation*

$$y''(x)+[\lambda-m(m+1)k^2 \operatorname{sn}^2(x)]y(x)=0, \tag{3}$$

with $k^2\in(0, 1)$, $m\in\mathbf{R}-\{0, -1\}$ *and* $\lambda\in\mathbf{C}$, *can have coexisting solutions if and only if the coefficient set* $\{\varphi_\lambda|\lambda\in\mathbf{C}\}$ *(for fixed m) of its Weierstrassian form* $Y''(z)+(\varphi_\lambda(z)/2)Y(z)=0$ *(obtained by the substitution* $x=\sqrt{e_1-e_3}\,z$*) is the* 1-*(complex)-parameter family of* 2-*connections for all projective structures having unique branch point* $\pi(z_0)$ *of odd integer ramification order* $\mu=2m+1\in\{3, 5, \dots\}$, *if* $m>0$, *or* $\mu=2|m|-1\in\{3, 5, \dots\}$, *if* $m<0$, *on an arbitrary compact torus* $M=\mathbf{C}/G$ *with modulus* $\tau\in i\mathbf{R}^+$, *where G is given by* (2), $z_0=\tau/2$ *and* π *is defined as earlier.*

Proof. The change of variables $Y(z)=y(\sqrt{e_1-e_3}\,z)$, where $e_1=\wp(1/2)$, $e_3=\wp(\tau/2)$ and $\wp(z)$ is the Weierstrass function for G, transforms equation (3) into the equation

$$Y''(z)+\left[\frac{1-\mu^2}{4}\wp\left(z-\frac{\tau}{2}\right)+\lambda'\right]Y(z)=0, \tag{4}$$

where $\lambda'=\lambda(e_1-e_3)+m(m+1)e_3$ and $\mu=2m+1$ if $m\geqq-(1/2)$ or $\mu=2|m|-1$ if $m\leqq-(1/2)$. Lemmas 0 and 2 and Theorem 2 of [9] imply that the functions $\varphi_\lambda(z)=\frac{1-\mu^2}{2}\wp(z-z_0)+\lambda$, $\forall\lambda\in\mathbf{C}$ and fixed $\mu\in\{3, 5, \dots\}$, form a 1-(complex)-parameter family of 2-connections for the class of all projective structures on M having a single branch point $\pi(z_0)$ of ramification order μ. Also, the locally meromorphic ratios $f(z)=Y_1(z)/Y_2(z)$ of linearly independent solutions to the differential equations $Y''(z)+(\varphi_\lambda(z)/2)Y(z)=0$ are branched local coordinates associated with these structures. Hence, each equation (3) is naturally associated with a projective structure as specified in the statement of the Lemma iff either $2m+1\in\{3, 5, \dots\}$ or $2|m|-1\in\{3, 5, \dots\}$ or, equivalently, iff $m\in\mathbf{Z}-\{0, -1\}$. The condition $\tau\in i\mathbf{R}^+$ implies that $sn^2(x)$ (and $m(m+1)k^2sn^2(x))\in\mathbf{R}$ if $x\in\mathbf{R}$. In this case a theorem of Erdélyi [7] asserts that equation (3) can have coexisting solutions for some $\lambda\in C$ iff $m\in\mathbf{Z}-\{0, -1\}$, thus completing the proof of the Lemma. ∎

Remark. For arbitrary $P\in M$, a translation of C gives the equation $\pi(z_0)=P$. Hence, Lemma 1 implies that each equation (3) with $M\in\mathbf{Z}-\{0, 1\}$ is associated with a projective structure having an arbitrary fixed branch point on M.

Theorem 1. *Let M be any compact torus with modulus $\tau \in i\mathbf{R}^+$ and let $\mu \in \{3, 5, \ldots\}$ and $P \in M$ be fixed. Each projective structure on M with unique branch point P of ramification order μ must have at least one subordinate affine structure. Furthermore, for each given projective structure, the number of subordinate affine structures is one if the parameter λ in the associated Lamé's equation* (3) *is one of the* (μ) *endpoints of instability intervals for this equation, is two if $\lambda \in \mathbf{C} - \{\lambda | \Delta(\lambda) = \pm 2\}$ and is either one or two if λ is a value for which coexisting solutions of equation* (3) *exist.*

Proof. For each $\mu \in \{3, 5, \ldots\}$ and $\lambda \in \mathbf{C}$, the branched projective structure described in Lemma 1 has branched local coordinates with global analytic continuation $f(z)(=Y_1(z)/Y_2(z))$ satisfying $f \circ A(z) = \psi(A) \circ f(z)$ $\forall A \in G$ [2], where ψ: $G \to$ Möb is the monodromy homomorphism for Eq. (4). In particular, it is known [5] that ψ is conjugate in Möb to a homomorphism γ: $G \to$ Möb of exactly one of the following forms:

(A) $\gamma(A_i)(w) = a_i w$ with $a_i \in \mathbf{C}^*$, $i = 1, 2$,

(B) $\gamma(A_i)(w) = w + b_i$ with $b_i \in \mathbf{C}$, $i = 1, 2$,

(C) $\gamma(A_i)(w) = \dfrac{(-1)^{i-1}}{w}$, $i = 1, 2$,

where $A_1 = T_{(1,0)}$ and $A_2 = T_{(0,1)}$ are generators for G. Therefore, we may assume without loss of generality that Eq. (4) has monodromy homomorphism γ and monodromy group $\gamma(G)$. In (C), $\gamma(G) = D_2$ and $\gamma(G)$ is not conjugate in Möb to any affine group. In that case the ratio f of linearly independent solutions to Eq. (4) is doubly periodic with primitive periods 2 and 2τ. Furthermore, a classical result [1] implies that $Y_1(z) = (f'(z))^{-1/2}$ and $Y_2(z) = f(z)(f'(z))^{-1/2}$ are linearly independent doubly periodic solutions of Eq. (4) with primitive periods 4 and 4τ. However, Lamé's equation with $m \in Z - \{0, -1\}$ or, equivalently, with $\mu \in \{3, 5, \ldots\}$, cannot have two independent doubly periodic solutions [11]. Therefore, γ is of form (A) or (B), $\gamma(G)$ is affine and each projective structure considered in Theorem 1 has at least one subordinate affine structure. An argument in the proof of Theorem 4 of [6] establishes that any branched projective structure having a subordinate affine structure must have only one or two subordinate affine structures.

A theorem of Erdélyi [7] implies that Eqs. (3) and (4) have $m+1$ intervals of instability if $m \in Z^+$ or $|m|$ intervals of instability if $m \in Z^- - \{-1\}$. Lemma 1 implies that these intervals have μ endpoints, excluding the endpoint $\lambda = \lambda' = -\infty$ of the zeroth interval. The condition $\tau \in i\mathbf{R}^+$ implies that these endpoints and intervals (for Eqs. (3) and (4)) are on the real axis. Floquet's theorem implies that:

(1) If λ' (respectively, λ) is an endpoint of an instability interval for Eq. (4) (resp., Eq. (3)), then there exists a ratio f of linearly independent solutions which

is nontrivially additive for the period 1 and hence has monodromy homomorphism γ of form (B).

(2) If $\lambda' \in \mathbf{C} - \{\lambda' | \Delta(\lambda') = \pm 2\}$, then there exists such a ratio of solutions f which is nontrivially multiplicative for the period 1 and hence has monodromy homomorphism γ of form (A).

(3) If λ' is such that coexistence occurs, then there exists such a ratio f which is trivially multiplicative (multiplier $=1$) for the period 1; in this case f can be multiplicative or additive for the period τ and γ is of form (A) or (B).

If γ is a nontrivial homomorphism of form (B) and γ is conjugate in Möb to $\tilde{\gamma}$: $G \to GA(1, \mathbf{C})$ (the general affine group of dimension one), then ∞ is the only fixed point of all nontrivial elements in $\gamma(G)$ or $\tilde{\gamma}(G)$. Consequently, γ and $\tilde{\gamma}$ are conjugate in $GA(1, \mathbf{C})$. Hence, there is exactly one subordinate affine structure if λ' (respectively, λ) is an endpoint of an instability interval for Eq. (4) (resp., Eq. (3)) *or* is a coexistence value for which γ is a nontrivial homomorphism of form (B) (if any such λ' exists).

If γ is of form (A), then the homomorphism $\tilde{\gamma}$ defined by $\tilde{\gamma}(A) = D \circ \gamma(A) \circ D^{-1}$ $\forall A \in G$, where $D(w) = 1/w$, satisfies

$$\tilde{\gamma}(A_i)(w) = a_i^{-1} w, \quad i = 1, 2.$$

Furthermore, the divisor $(df)_F$ of zeroes and poles of df in a fixed fundamental set F for G is an invariant for each affine structure on M and satisfies $\deg (df)_F = 2g-2$, where g is the genus of M (see [8] and [9]). Also, the unique branch point $\pi(z_0) = P$ of each branched affine structure in Theorem 1 must be nonpolar, otherwise $\deg (df)_F < 0 = 2g-2$. Consequently, $(df)_F = (\mu-1) z_0 - 2 \sum_{i=1}^n z_i$, where z_i are the simple poles of f in F, $\deg (df)_F = \mu - 1 - 2n = 0$, $n = \frac{\mu-1}{2} \geqq 1$ and f has simple pole(s) in F. f and $\tilde{f} = D \circ f$ determine the same projective structure on M and $\tilde{f}$ has monodromy homomorphism $\tilde{\gamma}$. However, $(df)_F \neq (d\tilde{f})_F$ since $D(\infty) \neq \infty$ and f has pole(s) in F. Thus, f and $\tilde{f}$ determine inequivalent affine structures on M. Hence, there are two subordinate affine structures if λ' (respectively, λ) $\in \mathbf{C} - \{$the μ endpoints of instability intervals for Eq. (4) (resp., Eq. (3))$\}$ $-$ $\{$coexistence value(s) λ' (resp., λ) for which γ is a nontrivial homomorphism of form (B)$\}$. ∎

Corollary 1. *There is a natural* $1-1$ *correspondence between the set of affine structures on M having unique branch point P of fixed ramification order* $\mu \in \{3, 5, \ldots\}$ *and a singly punctured hyperelliptic Riemann surface of genus* $g \geqq \frac{\mu-1}{2}$ *(i.e.,* $g \geqq m$ *if* $m > 0$*).*

Proof. If $\lambda_1, \ldots, \lambda_\mu$ are the endpoints of the intervals of instability for Eq. (3) and $\tilde{\lambda}_k$ are the points (if any) for which coexistence occurs with nontrivial monodromy homomorphism (B), then the algebraic curve

$$w^2 = (t-\lambda_1) \ldots (t-\lambda_\mu)(t-\tilde{\lambda}_1) \ldots$$

determines a hyperelliptic surface $\tilde{M}$ of genus $g \geqq \frac{\mu-1}{2}$ and a degree two map $t\colon \tilde{M} \to \hat{C}$ having branch points $t^{-1}(\lambda_i)$, $i=1, \ldots, \mu$, and $t^{-1}(\tilde{\lambda}_k)$, $\forall k$, and branch point or accumulation point of branch points $t^{-1}(\infty)$. Lemma 1 and Theorem 1 together imply the existence of the commutative diagram

$$\begin{array}{ccc} \tilde{M} - \{t^{-1}(\infty)\} & \xrightarrow{\ t\ } & \mathbf{C}(=\{\lambda \text{ in Eq. (3)}\}) \\ h \big\uparrow \big\downarrow h^{-1} & & \big\updownarrow \\ \left\{\begin{array}{l} \text{affine structures} \\ \text{for Eq. (3), } \forall \lambda \in \mathbf{C}, \\ \text{fixed } m \in \mathbf{Z} - \{0, -1\} \end{array}\right\} & \to & \left\{\begin{array}{l} \text{projective structures} \\ \text{for Eq. (3), } \forall \lambda \in \mathbf{C}, \\ \text{fixed } m \in \mathbf{Z} - \{0, -1\} \end{array}\right\} \end{array}$$

containing the desired correspondence h (or h^{-1}). ▮

Corollary 2. *The monodromy group $\gamma(G)$ of any projective structure on M with unique branch point P of ramification order $\mu \in \{3, 5, \ldots\}$ is $\mathbf{C}_\infty \times \mathbf{C}_\infty$, $\mathbf{C}_\infty$ or $\mathbf{C}_\infty \times \mathbf{C}_n$ $(n \in \mathbf{Z}^+ - \{1\})$.*

Proof. As in the proof of Theorem 1, $\gamma(G)$ has generators either of form (A) or of form (B). The nonexistence of a basis of doubly periodic solutions to Lamé's equation for $m \in \mathbf{Z} - \{0, -1\}$ implies that the constants a_i, $i=1, 2$, in (A) are not both roots of unity and that the constants b_i, $i=1, 2$, in (B) are not both zero. The conclusion follows from group theory. ▮

In the next theorem P has even ramification order.

Theorem 2. *Let M be defined as in Theorem 1 and let $\mu \in \{2, 4, \ldots\}$ and $P \in M$ be fixed. Each projective structure on M with unique branch point P of ramification order μ is associated with a Lamé's equation (3) satisfying $m = \frac{\mu-1}{2} \in \frac{1}{2}\mathbf{Z}^+ - \mathbf{Z}^+$, if $m > 0$, or $m = -\frac{\mu+1}{2} \in \frac{1}{2}\mathbf{Z}^- - \mathbf{Z}^- - \left\{-\frac{1}{2}\right\}$, if $m < 0$, as well as $\Delta(\lambda) = 0$ and having a basis of linearly independent solutions with primitive periods $4\sqrt{e_1 - e_3}$ and $4\tau\sqrt{e_1 - e_3}$. Also,*

$$0 \leqq \operatorname{card} \{\lambda \in \mathbf{C} \mid \exists \text{ a projective structure on } M \text{ associated with Eq. (3)}\} \leqq \frac{\mu}{2}.$$

Proof. Lemma 2 of [9] implies that each projective structure having unique branch point $P \in M$ of ramification order $\mu \in \{2, 4, \ldots\}$ is represented by a 2-connection for G of the form $\varphi_{\lambda'}(z)=\frac{1-\mu^2}{2}\wp(z-z_0)+\lambda'$, where $\lambda'\in\mathbf{C}$ and $\pi(z_0)=P$. An argument analogous to part of the proof of Lemma 1 establishes that for each φ_λ, there is an associated Lamé's equation (3) with $m=\frac{\mu-1}{2}\in\frac{1}{2}\mathbf{Z}^+-\mathbf{Z}^+$ if $m>0$ or with $m=-\frac{\mu+1}{2}\in\frac{1}{2}\mathbf{Z}^--\mathbf{Z}^--\left\{-\frac{1}{2}\right\}$ if $m<0$. Furthermore, Theorem 3 of [9] implies that the monodromy homomorphism is of form (C). This observation and an argument in the proof of Theorem 1 together guarantee that all nontrivial solutions to the associated equation (3) (respectively, associated equation (4)) have primitive periods $4\sqrt{e_1-e_3}$ and $4\tau\sqrt{e_1-e_3}$ (resp., 4 and 4τ). Consequently, Floquet's theorem implies that $\Delta(\lambda)=\Delta(\lambda')=0$ for these equations. For each $\mu\in\{2,4,\ldots\}$, the proof of Theorem 2 of [9] implies that the set $I_\mu=\{\lambda'\in\mathbf{C}|\varphi_{\lambda'}$ represents a projective structure on $M\}$ is $\mathbf{C}$ or satisfies card $(I_\mu)\leqq\mu/2$. Therefore, the countability of the set $\{\lambda'\in\mathbf{C}|\Delta(\lambda')=0\}$ for each Eq. (4) implies that $I_\mu\neq\mathbf{C}$ and card $(I_\mu)\leqq\mu/2$. The observation that card $(I_\mu)=$card $\{\lambda\in\mathbf{C}|\exists$ a projective structure on M associated with Eq. (3)$\}$ completes the proof. ∎

Corollary 3. *If* $m\in\frac{1}{2}\mathbf{Z}-\mathbf{Z}-\{-\frac{1}{2}\}$ *and* $\Delta(\lambda)\neq 0$, *then the monodromy group of Eq.* (3) (*and of Eq.* (4)) *is an infinite, non-Abelian group.*

Proof. Eq. (3) is associated with a 2-connection $\varphi_{\lambda'}$ with $\mu\in\{2,4,\ldots\}$. However, Theorem 2 and the condition $\Delta(\lambda)\neq 0$ together imply that Eq. (3) (respectively, Eq. (4)) does not have an associated projective structure on M and therefore does have a locally multivalued ratio f of linearly independent solutions in a neighborhood of the singular point $\sqrt{e_1-e_3}\,z_0$ (resp., z_0). In fact, the roots of the indicial equation for Eq. (4) at z_0 are $\frac{1\pm\mu}{2}\in\frac{1}{2}\mathbf{Z}-\mathbf{Z}$ and the local multivaluedness of f is caused by a logarithmic term [3]. Thus, we obtain the identity $f(\Lambda z)=\gamma(\Lambda)\circ f(z)$, where $f(\Lambda z)$ represents the result of analytic continuation of f around a small, simple loop Λ enclosing z_0 and where $\gamma(\Lambda)$ is a parabolic element of Möb. Furthermore, Λ is homotopic on $M-\{\pi(z_0)\}$ to $\Lambda_1\Lambda_2\Lambda_1^{-1}\Lambda_2^{-1}$, where Λ_i, $i=1,2$, are the canonical cross cuts on M. Also, the monodromy group of Eq. (4) is generated by $\gamma(\Lambda_i)(=\gamma(A_i))$, $i=1,2$, with $\gamma(\Lambda)$ satisfying

$$\gamma(\Lambda)=\gamma(\Lambda_1\Lambda_2\Lambda_1^{-1}\Lambda_2^{-1})=\gamma(\Lambda_1)\circ\gamma(\Lambda_2)\circ(\gamma(\Lambda_1))^{-1}\circ(\gamma(\Lambda_2))^{-1}.$$

Therefore, $\gamma(\Lambda)$ is an infinite-order generator for the first commutator subgroup of the monodromy group and the monodromy group is an infinite, non-Abelian group. ∎

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