On Vervaat's sup vague topology

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1. Introduction and preliminaries

The topological space S is assumed to be locally compact, which means that whenever $s \in G$, where $G \subseteq S$ is open, there is a compact K and an open G' such that $s \in G' \subseteq K \subseteq G$. Write $\mathscr{G}$ and $\mathscr{K}$ for the collections of open and compact subsets of S, resp. Let I denote some compact interval on the extended real line $[-\infty, \infty]$, e.g. $I=[-\infty, 0]$. The topology on I is the usual one generated by the sets $I \cap [-\infty, x)$ and $I \cap (y, \infty]$ for $x, y \in I$.

We say a function $g: S \to I$ is *upper semicontinuous* provided $\{s \in S: g(s) < x\} \in \mathscr{G}$ for all $x \in I$. It is a nice exercise to show that this holds if, and only if, the *hypograph*

$$\text{hypo}(g) = \{(x, s) \in I \times S: \ x \leqq g(s)\}$$

is closed in the product topology of $I \times S$ (cf. Vervaat (1988)). Clearly two distinct functions cannot have the same hypograph. Write $\mathscr{F}(S, I)$ for the family of upper semicontinuous functions from S to I.

Vervaat's *sup vague* topology on $\mathscr{F}(S, I)$ is the coarsest topology containing the two families

$$\{\{g \in \mathscr{F}(S, I): \ g(s) < x \ \text{ for all } \ s \in K\}, \ \ K \in \mathscr{K}, \ \ x \in I\} \tag{1a}$$

and

$$\{\{g \in \mathscr{F}(S, I): \ g(s) > x \ \text{ for some } \ s \in G\}, \ \ G \in \mathscr{G}, \ \ x \in I\}. \tag{1b}$$

Endowed with the sup vague topology, $\mathscr{F}(S, I)$ is a compact Hausdorff space. Our aim with this short note is to give a nonstandard proof of this fact. Its main step is a characterization of the standard part map.

Standard proofs can be found in Vervaat (1988), Gerritse (1985) and, for Hausdorff S, Norberg (1986).

We continue with some remarks on topology. We write B° for the interior of $B \subseteq S$. Moreover, B is called *saturated* if B equals its *saturation*, sat(B), which by definition is the intersection of the open neighborhoods of B.

Clearly $K \subseteq S$ is compact if, and only if, $\mathrm{sat}(K)$ is so. Thus, if $s \in G \in \mathscr{G}$, then $s \in K^\circ \subseteq K \subseteq G$ for some saturated $K \in \mathscr{K}$. Similarly, if $g \in \mathscr{F}(S, I)$ satisfies $g(s) < x$ for all $s \in K \in \mathscr{K}$, then $g(s) < x$ for all $s \in \mathrm{sat}(K)$. Thus, in (1a) we may replace $\mathscr{K}$ by the collection $\mathscr{Q}$ of compact and saturated subsets of S. It is easily seen that $\mathscr{Q}$ and $\mathscr{K}$ coincide if S is Hausdorff.

We conclude this introduction with some remarks on our nonstandard setting. Let $\mathbf{N} = \{1, 2, \dots\}$. We work in a polysaturated enlargement of a superstructure containing $S \cup I \cup \mathbf{N}$ (see Lindstrøm's article p. 83 in Cutland (1988) or Stroyan & Bayod (1986), Section 0.4). The associated monomorphism satisfying the transfer principle is denoted *. The members of $S \cup I \cup \mathbf{N}$ are treated as individuals in the superstructure, so we write a instead of $^*a$ when $a \in S \cup I \cup \mathbf{N}$.

The article by Lindstrøm in Cutland (1988) is a short introduction to nonstandard analysis. Our main reference to nonstandard analysis is however Hurd & Loeb (1985), but see also Albeverio, Fenstad, Høegh-Krohn & Lindstrøm (1986) and Stroyan & Bayod (1986).

Assume, momentarily, that S is an arbitrary topological space. The set

$$\mathrm{m}(s) = \cap \{^*G : \; s \in G \in \mathscr{G}\} \subseteq {}^*S$$

is called the *monad* of $s \in S$ and we say that $t \in {}^*S$ is *near standard* if $t \in \mathrm{m}(s)$ for some $s \in S$.

Note that S is a Hausdorff space if, and only if, monads of distinct points in S are disjoint and that $K \subseteq S$ is compact if, and only if, every $t \in {}^*K$ is near standard. The latter result is Abraham Robinson's nonstandard characterization of compactness. (For proofs, see Hurd & Loeb (1985), Proposition III.1.12 and Theorem III.2.1.)

2. The compactness theorem

Let $h \in {}^*\mathscr{F}(S, I)$. Then, by the transfer principle, h is a mapping from $^*S$ into $^*I$. We let $\hat{h}$ be the unique member of $\mathscr{F}(S, I)$ satisfying the equivalence

$$x \leqq \hat{h}(s) \Leftrightarrow \exists y \in \mathrm{m}(x) \, \exists t \in \mathrm{m}(s) \colon \; y \leqq h(t) \tag{2}$$

for $x \in I$ and $s \in S$.

To see that $\hat{h}$ exists and is unique, write

$$H = \{(y, t) \in {}^*I \times {}^*S : \; y \leqq h(t)\}$$

and note that the set

$$\hat{H} = \{(x, s) \in I \times S : \; \mathrm{m}(x) \times \mathrm{m}(s) \cap H \neq \emptyset\}$$

is closed in the product topology on $I \times S$ (Hurd & Loeb (1985), Theorem III.1.22).

If $(x,s)\in\hat{H}$ and $y\leqq x$, then $(y,s)\in\hat{H}$ as the reader easily shows. Thus $\hat{H}$ is the hypograph of a unique upper semicontinuous function from S into I.

2.1. Example. Assume $h\in{}^*\mathscr{F}(\mathbf{R},I)$ is increasing ($\mathbf{R}$ denotes the real line $(-\infty,\infty)$). Then $\hat{h}$ is increasing and right continuous.

To see this, let $s<t$ and take $x\leqq\hat{h}(s)$. Then, for some $\tilde{x}\in\mathrm{m}(x)$ and $\tilde{s}\in\mathrm{m}(s)$, $\tilde{x}\leqq h(\tilde{s})$. If $u\in\mathrm{m}(t)$, then $u>\tilde{s}$ so we must have $\tilde{x}\leqq h(\tilde{s})\leqq h(u)$. But then $x\leqq\hat{h}(t)$. Thus $\hat{h}$ is increasing. Now right continuity follows because $\hat{h}$ is upper semicontinuous.

Fix $s\in\mathbf{R}$ and let

$$x=\mathrm{S}-\lim_{t\downarrow s}h(t).$$

Recall from Stroyan & Bayod (1986), p. 170, that, this means that $x\in\mathbf{R}$ and that, for some $u\in\mathrm{m}(s)$, we have $h(v)\in\mathrm{m}(x)$ whenever $v\in\mathrm{m}(s)$, $v\geqq u$. It is clear that $x\leqq\hat{h}(s)$ since $h(u)\in\mathrm{m}(x)$ and $u\in\mathrm{m}(s)$. If $x<y\leqq\hat{h}(s)$, then $\tilde{y}\leqq h(\tilde{s})$ for some $\tilde{y}\in\mathrm{m}(y)$ and $\tilde{s}\in\mathrm{m}(s)$. But then $h(\tilde{s})\notin\mathrm{m}(x)$, so we must have $\tilde{s}<u$. This implies $h(\tilde{s})\leqq h(u)$. Thus $\tilde{y}\leqq h(u)$ and we reach the contradiction $h(u)\notin\mathrm{m}(x)$. We conclude that

$$\hat{h}(s)=\mathrm{S}-\lim_{t\downarrow s}h(t),\quad s\in\mathbf{R}.\ \square$$

Our first result characterizes the monad of $g\in\mathscr{F}(S,I)$.

2.2. Theorem. *Let* $h\in{}^*\mathscr{F}(S,I)$ *and* $g\in\mathscr{F}(S,I)$. *Then* $h\in\mathrm{m}(g)$ *if, and only if,* $\hat{h}=g$.

Our proof of Theorem 2.2 uses the following lemma, whose proof is a routine exercise. Thus omitted.

2.3. Lemma. *Assume* $h\in\mathrm{m}(g)$. *Let* $K\in\mathscr{Q}$, $G\in\mathscr{G}$ *and* $x\in I$. *Then the following two implications hold true:*

(3a) $$\forall s\in K\colon\ g(s)<x\Rightarrow\forall s\in{}^*K\colon\ h(s)<x,$$

and

(3b) $$\exists s\in G\colon\ g(s)>x\Rightarrow\exists s\in{}^*G\colon\ h(s)>x.$$

Conversely, $h\in\mathrm{m}(g)$ *if these implications are true for all choices of* $K\in\mathscr{Q}$, $G\in\mathscr{G}$ *and* $x\in I$.

Proof of Theorem 2.2. Firstly, suppose $h\in\mathrm{m}(g)$. Fix $s\in S$. Take $x\in I$, $x<g(s)$, and let (G_i) be the filter of open neighborhoods of s. By (3b),

$$\{t\in{}^*G_i\colon\ x<h(t)\}\neq\emptyset$$

for all i. By polysaturation,

$$\bigcap_i\{t\in{}^*G_i\colon\ x<h(t)\}\neq\emptyset.$$

Thus $x<h(t)$ for some $t\in\bigcap_i {}^*G_i=\mathrm{m}(s)$. But then $x\leqq\hat{h}(s)$. Next, take $x\in I$, $x>g(s)$. Choose $K\in\mathcal{Q}$, $y\in I$ such that $s\in K^\circ$ and $x>y>g(t)$ for all $t\in K$. By (3a), $y>h(t)$ for all $t\in {}^*K$ and in particular for all $t\in\mathrm{m}(s)\subseteqq {}^*K^\circ\subseteqq {}^*K$. But $z>y$ for all $z\in\mathrm{m}(x)$. Hence $x>\hat{h}(s)$. This shows that $\hat{h}=g$.

Conversely, suppose $\hat{h}=g$. Take $x\in I$, $g(s)<x$ for all $s\in K\in\mathcal{Q}$. Fix $t\in {}^*K$. Then $t\in\mathrm{m}(s)$ for some $s\in K$. Now $\hat{h}(s)<x$ so $h(u)<y$ for all $u\in\mathrm{m}(s)$ and $y\in\mathrm{m}(x)$. In particular $h(t)<x$. Thus (3a) holds true. To see (3b), let $h(t)\leqq x$ for all $t\in {}^*G$, where $G\in\mathcal{G}$. Fix $s\in G$. If $x<y$, then $h(u)<z$ for all $u\in\mathrm{m}(s)\subseteqq {}^*G$ and $z\in\mathrm{m}(y)$. Hence $\hat{h}(s)<y$, and $\hat{h}(s)\leqq x$ follows. This shows (3b). By Lemma 3.2, $h\in\mathrm{m}(g)$. □

Now the main result of the paper is easy to prove.

2.4. Theorem. *The sup vague topology on $\mathcal{F}(S,I)$ is compact and Hausdorff.*

Proof. It follows from Theorem 2.2 that if $h\in {}^*\mathcal{F}(S,I)$ then $h\in\mathrm{m}(\hat{h})$, i.e., every member of ${}^*\mathcal{F}(S,I)$ is near standard. By Robinson's theorem, $\mathcal{F}(S,I)$ is compact. Theorem 2.2 also shows that if $h\in\mathrm{m}(g_1)\cap\mathrm{m}(g_2)$, where $g_1, g_2\in\mathcal{F}(S,I)$, then $g_1=\hat{h}=g_2$. Hence $\mathcal{F}(S,I)$ is a Hausdorff space. □

2.5. Remarks. Endow the collection $\mathcal{F}$ of closed subsets of S with Fell's topology (cf. Fell (1962)). This topology has the sets

$$\{F\in\mathcal{F}\colon\ F\cap K=\emptyset,\ \ F\cap G_1\neq\emptyset,\ \dots,\ F\cap G_n\neq\emptyset\},$$
$$K\in\mathcal{Q},\quad G_1,\dots,G_n\in\mathcal{G},$$

as open base. Let $H\in {}^*\mathcal{F}$. Then

$$\hat{H}=\{s\in S\colon\ \mathrm{m}(s)\cap H\neq\emptyset\}\in\mathcal{F}.$$

(Hurd & Loeb (1985), Theorem III.1.22). Let $F\in\mathcal{F}$. Then $H\in\mathrm{m}(F)$ if, and only if, $\hat{H}=F$. To see this, either proceed as in the proof of Theorem 2.2 or identify $F\in\mathcal{F}$ with its characteristic function $1_F\in\mathcal{F}(S,I)$ and use Theorem 2.2. We may conclude, as in Theorem 2.4, the well-known fact proved by Fell (1962) that $\mathcal{F}$ is a compact Hausdorff space. □

2.6. Remarks. Write $\mathcal{G}(S,I)$ for the collection of all lower semicontinuous functions from S into I. If $h\in {}^*\mathcal{G}(S,I)$, we write $\check{h}$ for the unique lower semicontinuous function from S into I satisfying

$$\check{h}(s)\leqq x\Leftrightarrow\exists y\in\mathrm{m}(x)\,\exists t\in\mathrm{m}(s)\colon\ h(t)\leqq y$$

for $x\in I$ and $s\in S$.

Endow $\mathcal{G}(S,I)$ with the topology generated by all sets of the form

$$\{g\in\mathcal{G}(S,I)\colon\ g(s)>x\ \text{ for all }\ s\in K\},$$

where $K \in \mathcal{Q}$ and $x \in I$, and all sets of the form

$$\{g \in \mathcal{G}(S, I) \colon \ g(s) < x \ \text{ for some } \ s \in G\},$$

where $G \in \mathcal{G}$ and $x \in I$. This is the analogue (or dual) of Vervaat's sup vague topology on $\mathcal{F}(S, I)$.

Assume $h \in {}^*\mathcal{G}(S, I)$ and let $g \in \mathcal{G}(S, I)$. Then $h \in \mathrm{m}(g)$, if, and only if, $\tilde{h} = g$. This follows by duality from Theorem 2.2. □

2.7. Remarks. Let $\mathcal{C}(S, I) = \mathcal{F}(S, I) \cap \mathcal{G}(S, I)$. We endow $\mathcal{C}(S, I)$ — the set of continuous functions from S into I — with the coarsest topology containing the relative topologies from both $\mathcal{F}(S, I)$ and $\mathcal{G}(S, I)$. Let $g \in \mathcal{C}(S, I)$. Write $\mathrm{m}_c(g)$, $\mathrm{m}_u(g)$ and $\mathrm{m}_l(g)$ for the monads of g relative to the topologies of $\mathcal{C}(S, I)$, $\mathcal{F}(S, I)$ and $\mathcal{G}(S, I)$, resp. Then $\mathrm{m}_c(g) = \mathrm{m}_u(g) \cap \mathrm{m}_l(g)$. It follows, e.g., that $h \in {}^*\mathcal{C}(S, I)$ is near standard if, and only if, $\hat{h} = \tilde{h}$.

In the case of a Hausdorff S, it is now easily seen that $h \in \mathrm{m}_c(g)$, if, and only if, $h(t) \in \mathrm{m}(g(s))$ whenever $t \in \mathrm{m}(s)$. By Keisler (1984), Proposition 1.17, $\mathrm{m}_c(g)$ is the monad of g taken with respect to the familiar compact-open topology generated by all sets of the form

$$\{g \in \mathcal{C}(S, I) \colon \ g(K) \subseteqq U\},$$

where $K \in \mathcal{K}$ and $U \subseteqq I$ is open. Two distinct topologies cannot have the same monads (in a polysaturated enlargement, see Cutland (1988), p. 86). So the topology we have equipped $\mathcal{C}(S, I)$ with is the compact-open topology. Also this result is known. Refer to Vervaat (1981). □

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