Complementary spaces and multipliers of double Fourier series

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In his paper [9], Goes shows, how by means of known theorems on summability factors for the Cesàro summability method C^α of nonnegative order α, it is possible to obtain effective sufficient conditions for the multipliers of Fourier series, imposed directly on them. The foundation for this is the theory of C^α-complementary spaces of Fourier coefficients, developed by him [6, 7]. The theory was extended to Toepliz methods T by Tynnov [24]. The notion of T-complementary space was generalized in [2] for double Fourier series, and with the help of known theorems for summability factors for double series, we found effective sufficient conditions for various classes of multipliers for double Fourier series.

In the present paper, we find A-complementary spaces of Fourier coefficients to Orlicz spaces L_Φ and L_Ψ of functions of two variables, and to the space C of continuous functions of two variables, for an arbitrary matrix method A of summability of double series, with bounded double sequence of Lebesgue constants. We first prove criterions for a double trigonometric series to belong to these spaces, expressed in terms of A-means of the double Fourier series. Theorems are obtained on connections between summability factors of double series, A-complementary spaces, and multipliers of classes of double Fourier series. It is shown that the A-complementary space of coefficients of a double Fourier series is a BK-space. We get also theorems on identities of classes of multipliers for various BK-spaces of double Fourier series, under the hypothesis that the preimage spaces are norm-determining manifolds for a certain BK-space.

1. Notations and introductory remarks

In what follows if limits of summation are not indicated, they are 0 and ∞ on each index, and the free indices assume all values $0, 1, \dots$. If $x_{mn} \to a$ as $m, n \to \infty$ and $x_{mn} = O(1)$ we write $b\text{-}\lim_{m,n} x_{mn} = a$.

Throughout this paper $f, g, h, \dots$ will denote real-valued functions of two variables, defined almost everywhere on the plane, 2π-periodic in each variable and Lebesgue integrable on the square $Q = [-\pi, \pi]^2$. As in [2, 3] let f^0 be the double Fourier series of the function f, i.e.

$$f^0(s, t) = \sum_{k,l} A_{kl}(s, t),$$

denoting

$$A_{kl}(s, t) = 2^{-\varkappa}(a_{kl} \cos ks \cos lt + b_{kl} \sin ks \cos lt + c_{kl} \cos ks \sin lt + d_{kl} \sin ks \sin lt),$$

where $\varkappa = 2$ if $k = l = 0$, $\varkappa = 1$ if $k + l = 1$, and $\varkappa = 0$ otherwise.

We shall denote by the same symbol both a set of functions f and the set of double Fourier series f^0 of these functions f. Moreover if X is a normed space and $f \in X$, then we define $\|f^0\|_X = \|f\|_X$, and consequently the set of all $f^0 \in X$ is also a normed space. This convention will be applied to the Lebesgue spaces $X = L^p := L^p(Q)$ for $1 \leqq p < \infty$ with the norm

$$\|f\|_p = \left(\iint_Q |f(s, t)|^p \, ds \, dt \right)^{1/p};$$

$X = M := L^\infty(Q)$ with the norm

$$\|f\|_M = \operatorname*{ess\,sup}_Q |f(s, t)|,$$

and to the space $X = C := C(Q)$ of all continuous functions on Q with the maximum norm.

We shall assume that Φ and Ψ are two absolutely continuous functions of one nonnegative variable forming a pair of complementary Young functions (see [16], p. 134, or [27], p. 77). In this case their derivatives $\Phi' = \varphi$ and $\Psi' = \psi$ are mutually inverse in a generalized sense ([16], p. 135). We recall that Φ is called a Young function if

$$\Phi(u) = \int_0^u \varphi(s) \, ds,$$

and φ is a real nondecreasing function defined on $[0, +\infty)$ such that $\varphi(0) = 0$ and φ is left continuous for $s \geqq 0$ (see e.g. [16], p. 195, or [27], p. 76).

We require throughout that the Young function Φ satisfies the so-called Δ_2-*condition*

$$\Phi(2u) = O(1)\Phi(u) \quad \forall u \geqq u_0$$

(for some $u_0 > 0$). We denote by L_Φ the Orlicz space (see [27], p. 79, or [16], p. 145) of all measurable functions f, for which the norm

$$\|f\|_{\Phi} = \sup\left\{\iint_Q |f(s,t)\,g(s,t)|\,ds\,dt\colon\ g\in M_{\Psi}\right\}$$

is finite, where

$$M_{\Psi} = \left\{g\colon \iint_Q \Psi(|g(s,t)|)\,ds\,dt \leqq 1\right\}.$$

Since Φ satisfies the Δ_2-condition, then (see corollaries of [26], p. 154, and [27], p. 81)

$$L_{\Phi} = \left\{f\colon \iint_Q \Phi(|f(s,t)|)\,ds\,dt < \infty\right\}.$$

By L_{Ψ} we denote the Orlicz space of all measurable functions f for which the norm

$$\|f\|_{\Psi} = \sup\left\{\iint_Q |f(s,t)\,g(s,t)|\,ds\,dt\colon\ g\in M_{\Phi}\right\}$$

is finite, where

$$M_{\Phi} = \left\{g\colon \iint_Q \Phi(|g(s,t)|)\,ds\,dt \leqq 1\right\}.$$

We do not require that Ψ satisfies the Δ_2-condition.

The Orlicz spaces L_{Φ} and L_{Ψ} are Banach spaces (see [27], p. 101, or [16], p. 156, or [15], p. 71).

If $\Phi(u)=cu^p$, where $0<c=\text{const}$, then $L_{\Phi}=L^p$. For $p>1$ we have that $L_{\Psi}=L^q$, where $p^{-1}+q^{-1}=1$ and for $p=1$ we have $L_{\Psi}=M$ (see [26], p. 154, [27], p. 82, [16], p. 195—196).

If $\Phi(u)=u\ln^+ u$, where $\ln^+ u=\max(0, \ln u)$, then Φ is a Young function, satisfying the Δ_2-condition (see [16], pp. 133 and 138) and $L_{\Phi}=L\ln^+ L$, where (cf. [30], p. 16)

$$L\ln^+ L = \left\{f\colon \iint_Q |f(s,t)|\ln^+|f(s,t)|\,ds\,dt < \infty\right\},$$

but its complementary function Ψ does not satisfy the Δ_2-condition (see [16], p. 136 and 138).

2. Conditions for a double trigonometric series to be a double Fourier series

Let $A=(\alpha_{mnkl})$ be a triangular matrix summability method, given by means of transformation matrices of a double series into a double sequence. Let us denote the *A-means* of the double series f^0 by $\sigma_{mn}f$, i.e.

$$(\sigma_{mn}f)(s,t) = \sum\nolimits_{k,l\leqq m,n} \alpha_{mnkl} A_{kl}(s,t).$$

Denote by K_{mn} the *kernel of the summability method A*, i.e.

$$K_{mn}(s,t) = \sum\nolimits_{k,l\leqq m,n} 2^{-\varkappa}\alpha_{mnkl}\cos ks\cos lt,$$

and by L_{mn} the *Lebesgue constants of A,* i.e.

$$L_{mn} = \pi^{-2} \iint_Q |K_{mn}(s, t)| \, ds \, dt.$$

We define the functions e, e_1 and e_2 by the formulas

$$e(s, t) \equiv 1, \quad e_1(s, t) = \cos ks \cdot \cos lt, \quad e_2(s, t) = \sin ks \cdot \cos lt.$$

We shall give criterions for double series to belong to Orlicz spaces first for L_Ψ and then for L_Φ.

Theorem 2.1. *If the method A satisfies the conditions*

$$b\text{-}\lim_{m,n} \alpha_{mnkl} = 1 \tag{2.1}$$

and

$$L_{mn} = O(1), \tag{2.2}$$

then in order that $f^0 \in L_\Psi$, *it is necessary and sufficient that*

$$\|\sigma_{mn} f\|_\Psi = O(1). \tag{2.3}$$

Proof. Necessity. Let $f^0 \in L_\Psi$. Then there exist (see [27], p. 80, theorem 2) positive constants β and N such that

$$\iint_Q \Psi(\beta |f(s, t)|) \, ds \, dt \leqq N.$$

Relying on (2.2), we denote

$$L = \sup_{m,n} L_{mn}, \quad \delta_{mn} = \pi^2 L_{mn}.$$

For the A-means of the double series f^0 we have the formula (cf. [30], p. 303)

$$(\sigma_{mn} f)(u, v) = \pi^{-2} \iint_Q K_{mn}(s, t) f(u+s, v+t) \, ds \, dt.$$

Since $\delta_{mn} > 0$ and Ψ is nondecreasing and convex (see [15], p. 7, or [16], p. 129), it follows by means of Jensen's inequality (cf. [16], p. 133, or [3], p. 159) that

$$\Psi(L^{-1}\beta |(\sigma_{mn} f)(u, v)|) \leqq \Psi\Big(\delta_{mn}^{-1} \beta \iint_Q |f(u+s, v+t)| \, |K_{mn}(s, t)| \, ds \, dt\Big)$$

$$\leqq \delta_{mn}^{-1} \iint_Q \Psi(\beta |f(u+s, v+t)|) \, |K_{mn}(s, t)| \, ds \, dt.$$

This yields by periodicity in both variables

$$\iint_Q \Psi(L^{-1}\beta |(\sigma_{mn} f)(u, v)|) \, du \, dv$$

$$\leqq \delta_{mn}^{-1} \iint_Q du \, dv \iint_Q \Psi(\beta |f(s, t)|) \, |K_{mn}(s-u, t-v)| \, ds \, dt$$

$$= \iint_Q \Psi(\beta |f(s, t)|) \, ds \, dt \leqq N.$$

Furthermore, since (cf. theorem 1 of [26], p. 152, or [27], p. 79)

$$\|L^{-1}\beta\sigma_{mn}f\|_{\Psi} \leqq \iint_Q \Psi(L^{-1}\beta\,|(\sigma_{mn}f)(u, v)|)\,du\,dv+1,$$

then by what was proved above, we obtain

$$\|L^{-1}\beta\sigma_{mn}f\|_{\Psi} \leqq N+1,$$

whence (2.3) follows, because $\beta\neq 0$.

Sufficiency. Suppose that (2.3) is true. Then (cf. [5], p. 10, theorem 2) there exists a convergent subsequence $(\|\sigma_{m_\nu n_\nu}f\|_{\Psi})$. According to Theorem 9 of [27], p. 159, there exists a subsequence $(\mu)\subseteqq(\nu)$ and a function $h\in L_{\Psi}$ with the property that for every $g\in L_{\Phi}$, the equality

$$\lim_{\mu\to\infty}\iint_Q g(s, t)(\sigma_{m_\mu n_\mu}f)(s, t)\,ds\,dt = \iint_Q g(s, t)\,h(s, t)\,ds\,dt \tag{2.4}$$

holds. Putting $g=\pi^{-2}e_1$ and observing that $g\in L_{\Phi}$ (see for example [27], p. 128, ex. 8), we conclude from (2.4) that

$$\iint_Q h(s, t)\,e_1(s, t)\,ds\,dt = \lim_{\mu\to\infty}\alpha_{m_\mu n_\mu kl}\,a_{kl} = a_{kl}$$

in view of (2.1). Similarly for b_{kl}, c_{kl} and d_{kl}. Therefore $h^0=f^0$, whence $f^0\in L_{\Psi}$.

Everywhere in the sequel we denote by P the set of all double trigonometric polynomials t_{kl}.

Theorem 2.2. *If the method A satisfies the conditions* (2.1) *and* (2.2), *then in order that $f^0\in L_{\Phi}$ it is necessary and sufficient that the double sequence $(\sigma_{mn}f)$ is boundedly convergent in L_{Φ} in norm.*

Proof. Necessity. Let $f^0\in L_{\Phi}$. Then by theorem 2.1

$$\|\sigma_{mn}f\|_{\Phi} = O(1). \tag{2.5}$$

We show that $\sigma_{mn}\colon L_{\Phi}\to L_{\Phi}$ are continuous linear operators for any m, n. In fact, the function e belongs to L_{Ψ} (see for example [16], p. 145). Setting

$$G = \max\{\|e\|_{\Phi}, \|e\|_{\Psi}\},$$

we obtain that the Fourier coefficients

$$a_{kl} = \pi^{-2}\iint_Q f(s, t)\cos ks\cos lt\,ds\,dt$$

(see for example [23], p. 176) are continuous linear functionals on L_{Φ}, because by Hölder's inequality (see for example [15], p. 74, [26], theorem 3)

$$|a_{kl}| \leqq \pi^{-2}\iint_Q |f(s, t)|\,ds\,dt \leqq \pi^{-2}\|f\|_{\Phi}G \leqq G\|f\|_{\Phi}.$$

Hence, and also by analogous formulas for b_{kl}, c_{kl} and d_{kl}, we obtain

$$\|A_{kl}\|_\Phi \leqq 4G^2 \|f\|_\Phi. \tag{2.6}$$

Consequently σ_{mn} are continuous linear operators from L_Φ into itself.

By the principle of uniform boundedness ([27], p. 135, theorem 1), from (2.5) it follows that there exists a constant $\gamma > 0$, such that

$$\|\sigma_{mn} f\|_\Phi \leqq \gamma \|f\|_\Phi. \tag{2.7}$$

The set P of all trigonometric polynomials is dense in L_Φ (see [27], p. 128, ex. 8). Therefore there exists a double sequence $(t_{mn}) \subseteqq P$ such that for any $\varepsilon > 0$ there exists a number $N > 0$ such that

$$\gamma \|f - t_{\mu\nu}\|_\Phi < \varepsilon/3 \tag{2.8}$$

for each $\mu, \nu \geqq N$. Choose $\mu = \nu = N$ and denote

$$t_{NN} = \sum\nolimits_{i,j \leqq N} A_{ij}^N, \quad \tau = \|t_{NN}\|_\Phi,$$

we obtain

$$\sigma_{mn} t_{NN} = \sum\nolimits_{k,l \leqq N} \alpha_{mnkl} A_{kl}^N.$$

In view of (2.1) there exist indices N_{kl} such that for any $k, l = 1, \ldots, N$, each $m, n \geqq N_{kl}$ and arbitrary positive integers i, j we have

$$|\alpha_{mnkl} - \alpha_{m+i,n+j,kl}| < \varepsilon/(12G^2 \tau N^2). \tag{2.9}$$

Let $N_0 = \max\{N_{kl}: k, l = 1, \ldots, N\}$. Then (2.9) is true for all $m, n \geqq N_0$ and for all positive integers i, j if $k, l \leqq N$. Consequently (2.9) and (2.6) imply that

$$\begin{aligned}
\|\sigma_{mn} t_{NN} - \sigma_{m+i,n+j} t_{NN}\|_\Phi &= \Big\|\sum\nolimits_{k,l \leqq N} (\alpha_{mnkl} - \alpha_{m+i,n+j,kl}) A_{kl}^N\Big\|_\Phi \\
&\leqq \sum\nolimits_{k,l \leqq N} |\alpha_{mnkl} - \alpha_{m+i,n+j,kl}| \, \|A_{kl}^N\|_\Phi \\
&< 4G^2 \tau N^2 \varepsilon/(12G^2 \tau N^2) = \varepsilon/3.
\end{aligned}$$

From this, (2.7) and (2.8) we obtain for all positive integers i, j:

$$\begin{aligned}
\|\sigma_{mn} f - \sigma_{m+i,n+j} f\|_\Phi &\leqq \|\sigma_{mn} f - \sigma_{mn} t_{NN}\|_\Phi \\
&\quad + \|\sigma_{mn} t_{NN} - \sigma_{m+i,n+j} t_{NN}\|_\Phi \\
&\quad + \|\sigma_{m+i,n+j} t_{NN} - \sigma_{m+i,n+j} f\|_\Phi \\
&< 2\gamma \|f - t_{NN}\|_\Phi + \varepsilon/3 < 2\varepsilon/3 + \varepsilon/3 = \varepsilon
\end{aligned}$$

for all $n, n \geqq N_0$. Since L_Φ is a complete space, then, taking into account (2.5), the double sequence $(\sigma_{mn} f)$ boundedly converges to some function $h \in L_\Phi$ (see [5], p. 11, theorem 5, and compare [18], p. 255), i.e.

$$b\text{-}\lim_{m,n} \|\sigma_{mn} f - h\|_\Phi = 0. \tag{2.10}$$

We will prove that $h^0 = f^0$. From (2.10) it follows that $(\sigma_{mn} f)$ converges weakly to h. Employing the general form of a continuous linear functional in L_Φ (see [15], p. 128, or [27], p. 138 and p. 142), we obtain from (2.10) and (2.1) that

$$\pi^{-2} \iint_Q h(s, t) e_1(s, t)\, ds\, dt = \lim_{m,n} \pi^{-2} \iint_Q (\sigma_{mn} f)(s, t) e_1(s, t)\, ds\, dt$$

$$= \lim_{m,n} \alpha_{mnkl} a_{kl} = a_{kl}.$$

Analogously for the Fourier coefficients b_{kl}, c_{kl} and d_{kl}. Therefore $h^0 = f^0$, whence from (2.10)

$$b\text{-}\lim_{m,n} \|\sigma_{mn} f - f^0\|_\Phi = 0. \tag{2.11}$$

Sufficiency. Let the double sequence $(\sigma_{mn} f)$ boundedly converge in the norm in L_Φ to some limit h. Since L_Φ is a Banach space, then $h \in L_\Phi$, and hence (2.10) is true. From (2.10) and what we proved above, it follows that $h^0 = f^0$. Consequently $f^0 \in L_\Phi$.

For simple Fourier series theorem 2.1 and theorem 2.2 were proved by Tynnov ([24], pp. 67—70) and previously by Goes ([7], pp. 377—378) for the method of arithmetical means.

From theorems 2.1 and 2.2, putting $\Phi(u) = cu^p$, where $0 < c = \text{const}$, we conclude the following:

Corollary 2.3. *If the method A satisfies the conditions* (2.1) *and* (2.2), *then a necessary and sufficient condition for*

a) $f^0 \in L^p$ *with* $p > 1$ *is* $\|\sigma_{mn} f\|_p = O(1)$,
b) $f^0 \in M$ *is* $\|\sigma_{mn} f\|_M = O(1)$,
c) $f^0 \in L^p$ *with* $p \geqq 1$ *is that* $(\sigma_{mn} f)$ *boundedly converges in* L^p.

For simple Fourier series corollary 2.3 is known (see [11], pp. 215—217).

Theorem 2.4. *If the method A satisfies the conditions* (2.1) *and* (2.2), *then in order that $f^0 \in C$ it is necessary and sufficient that*

$$b\text{-}\lim_{m,n} \|f - \sigma_{mn} f\|_C = 0.$$

Proof. See [3], p. 159, proof of lemma 3.1, and [24], p. 71, proof of theorem 3.

3. On A-complementary spaces

Let the function g have the double Fourier series expansion

$$g^0(s, t) = \sum_{k,l} E_{kl}(s, t),$$

where

$$E_{kl}(s, t) = 2^{-\varkappa}(\alpha_{kl} \cos ks \cos lt + \beta_{kl} \sin ks \cos lt + \gamma_{kl} \cos ks \sin lt + \delta_{kl} \sin ks \sin lt).$$

Let X be some space of double series f^0. Following Goes ([6], p. 348; [7], p. 373; [8], p. 151) and Tynnov ([24], p. 75) the space of all trigonometric series g^0 for which the double numerical series

$$\langle f^0, g^0 \rangle := \sum_{k,l} 2^{-\varkappa}(\alpha_{kl} a_{kl} + \beta_{kl} b_{kl} + \gamma_{kl} c_{kl} + \delta_{kl} d_{kl}) \tag{3.1}$$

is boundedly A-summable for every f^0 in X, is called *A-complementary to X.* This space is denoted by $(X \to A)$, i.e.

$$(X \to A) = \{g^0 \colon \langle f^0, g^0 \rangle \in A_b' \ \forall f^0 \in X\},$$

where A_b' is the set of all boundedly A-summable double series. If $a_{kl}=0$ for all f^0 in X, we also assume that $\alpha_{kl}=0$ for all g^0 in $(X \to A)$. The conventions with regard to the coefficients b_{kl}, c_{kl} and d_{kl} are analogous.

Denote the *A-means* of the series (3.1) by h_{mn}, i.e.

$$h_{mn} = \sum_{k,l \leqq m,n} 2^{-\varkappa} \alpha_{mnkl}(\alpha_{kl} a_{kl} + \beta_{kl} b_{kl} + \gamma_{kl} c_{kl} + \delta_{kl} d_{kl}). \tag{3.2}$$

Now we can write

$$(X \to A) = \{g^0 \colon (h_{mn}) \in bc \ \forall f^0 \in X\},$$

where bc is the space of all boundedly convergent double sequences. In [2] we required $(h_{mn}) \in rc$ instead of $(h_{mn}) \in bc$, where rc is the space of all regular (completely) convergent double sequences ([2], p. 42).

Theorem 3.1. *If the method A satisfies conditions* (2.1) *and* (2.2), *then the A-complementary space to L_Φ is L_Ψ.*

Proof. We wish to prove that $(L_\Phi \to A) = L_\Psi$. For each g in L_Ψ we associate a function $f \in L_\Phi$ and define the functionals $\varphi_{mn} \colon L_\Phi \to \mathbf{R}$ by the formula

$$\varphi_{mn} f = h_{mn}.$$

Expressing the Fourier coefficients of the double series f^0 by integrals (cf. for example [23], p. 176), we obtain

$$h_{mn} = \pi^{-2} \iint_Q f(s, t)(\sigma_{mn} g)(s, t) \, ds \, dt. \tag{3.3}$$

From this, taking into account the general form and upper bound of the norm of linear continuous functionals in L_Φ (cf. [26], p. 155, or [27], p. 138, theorem 2), we see that (φ_{mn}) is a double sequence of continuous linear functionals, which converges by virtue of condition (2.1) on the dense set P in L_Φ, and its norms satisfy the inequality

$$\|\sigma_{mn}g\|_\Psi/2 \leqq \|\varphi_{mn}\| \leqq \|\sigma_{mn}g\|_\Psi. \tag{3.4}$$

Consequently, by theorem 2.1 we have

$$\|\varphi_{mn}\| = O(1), \tag{3.5}$$

because conditions (2.1) and (2.2) are fulfilled. According to the Banach—Steinhaus theorem (see Kull [17], p. 10, theorem III) the double sequence of functionals (φ_{mn}) is pointwise boundedly convergent on L_Φ, that is the double numerical sequence (h_{mn}) boundedly converges for any $f^0 \in L_\Phi$. This means that $g^0 \in (L_\Phi \to A)$.

We will prove the opposite inclusion $(L_\Phi \to A) \subseteq L_\Psi$. Let $g^0 \in (L_\Phi \to A)$, i.e. let the double series (3.1) be boundedly A-summable for any $f^0 \in L_\Phi$, which means that the double sequence $(\varphi_{mn} f)$ boundedly converges for any $f^0 \in L_\Phi$. Then by the Banach—Steinhaus theorem, (3.5) is true, and from the left inequality in (3.4) we obtain $\|\sigma_{mn}g\|_\Psi = O(1)$, which by theorem 2.1 means that $g^0 \in L_\Psi$.

Theorem 3.2. *If the method A satisfies conditions* (2.1) *and* (2.2), *then the A-complementary space to L_Ψ is L_Φ.*

Proof. We need to prove that $(L_\Psi \to A) = L_\Phi$. For each $g \in L_\Phi$ we associate a function $f \in L_\Psi$ and define functionals $\varphi_{mn}\colon L_\Psi \to \mathbf{R}$ and $\varphi\colon L_\Psi \to \mathbf{R}$ by the formulas

$$\varphi_{mn} f = h_{mn}, \quad \varphi f = \pi^{-2} \iint_Q f(s,t)\, g(s,t)\, ds\, dt. \tag{3.6}$$

Taking into account the general form of linear continuous functionals on L_Ψ (cf. [27], p. 142, remark 2), we see that φ_{mn} and φ are continuous linear functionals on L_Ψ. In view of (3.3) and (3.6) we have by Hölder's inequality (see for example [27], p. 82)

$$|\varphi_{mn} f - \varphi f| \leqq \pi^{-2} \|f\|_\Psi \|\sigma_{mn} g - g\|_\Phi.$$

But, since $g^0 \in L_\Phi$, then condition (2.11) is satisfied for g by theorem 2.2. Hence, the double sequence of functionals (φ_{mn}) is pointwise convergent on L_Φ to the functional φ, that is the double numerical sequence (h_{mn}) is boundedly convergent for any $f^0 \in L_\Psi$. Hence $g^0 \in (L_\Psi \to A)$.

We shall prove the opposite inclusion $(L_\Psi \to A) \subseteq L_\Phi$. Let the double series (3.1) be boundedly A-summable for any $f^0 \in L_\Psi$, that is by the first equation of (3.6) the double sequence $(\varphi_{mn} f)$ is boundedly convergent for any $f \in L_\Psi$. Then by the Banach—Steinhaus theorem, (3.5) is true. In view of (3.3) and the inequality (cf.

[27], p. 142, Remark 2)

$$\|\sigma_{mn}g\|_{\Phi}/2 \leqq \|\varphi_{mn}\|,$$

we get $\|\sigma_{mn}g\|_{\Phi}=O(1)$. From the proof of the necessity part of theorem 2.2, it follows that (2.11) is true for g, that is $(\sigma_{mn}g)$ boundedly converges on L_{Φ}. By theorem 2.2 we obtain that $g^0 \in L_{\Phi}$.

For simple series theorems 3.1 and 3.2 were proved by Tynnov ([24], pp. 72—73), and earlier by Goes ([7], pp. 377—378; see also [30], p. 178) for arithmetical means.

Let V be the space of all functions of two variables having bounded variation in the sense of Vitali (cf. [29], p. 220, or [5], p. 220) on Q. We denote by dV the space of all double Fourier—Stieltjes series of functions in V, i.e. double Fourier series f^0 with coefficients a_{kl} defined by the double Riemann—Stieltjes integral

$$a_{kl} = \pi^{-2}\iint_Q \cos ks \cos lt F(ds\,dt)$$

for suitable $F \in V$ (cf. [31], p. 313). The coefficients b_{kl}, c_{kl} and d_{kl} are given by analogous formulas.

We now prove (cf. [7], p. 383, and [24], pp. 74—75) the following

Theorem 3.3. *If the method A satisfies conditions* (2.1) *and* (2.2), *then the A-complementary space to C is dV.*

Proof. We wish to prove that $(C \to A)=dV$. If $f^0 \in dV$, then for some F we have $F^0 \in V$. For any function $g \in C$, with double Fourier series g^0 we have

$$\pi^{-2}\iint_Q (\sigma_{mn}g)(s,t)F(ds\,dt) = h_{mn}. \tag{3.7}$$

Since A satisfies (2.1) and (2.2), by Theorem 2.4 the double sequence $(\sigma_{mn}g)$ is boundedly convergent uniformly on Q for each $g^0 \in C$. In view of (3.7) the double sequence (h_{mn}) boundedly converges for any $g^0 \in C$, that is $f^0 \in (C \to A)$.

We will prove the opposite inclusion $(C \to A) \subseteq dV$. Let $f^0 \in (C \to A)$, i.e. let the double series (3.1) be boundedly A-summable for any $g^0 \in C$, this means that (h_{mn}) is boundedly convergent for any $g^0 \in C$. We define a double sequence of continuous linear functionals $\psi_{mn}\colon C \to \mathbf{R}$ by the formula

$$\psi_{mn}g = \pi^{-2}\iint_Q g(s,t)(\sigma_{mn}f)(s,t)\,ds\,dt. \tag{3.8}$$

Then the limiting functional ψ is also linear and continuous on C (cf. [17], p. 12, Theorem IV). By the Riesz—Markov theorem (see [19], p. 129, or [20], p. 246) on the general form of continuous linear functionals on C, the functional ψ is given by

$$\psi g = \lim_{m,n} \psi_{mn} g = \pi^{-2}\iint_Q g(s,t)F_1(ds\,dt) \tag{3.9}$$

with $F_1 \in V$. Hence choosing $g=e_1$ and taking into account (2.1) and (3.8), we obtain

$$\psi e_1 = \pi^{-2} \iint_Q e_1(s, t) F_1(ds\, dt) = \pi^{-2} \lim_{m,n} \iint_Q e_1(s, t)(\sigma_{mn} f^0)(s, t)\, ds\, dt$$

$$= \lim_{m,n} \alpha_{mnkl} a_{kl} = a_{kl}.$$

Analogously, choosing $g=e_2$ we obtain $\psi e_2 = b_{kl}$ and so on. Since $F_1 \in V$, we obtain that $f^0 \in dV$.

4. Finding conditions for multipliers

A double sequence $\varepsilon = (\varepsilon_{mn})$ is called a *multiplier of class* (X, Y) if the double series

$$(Tf)^0(s, t) = \sum_{k,l} \varepsilon_{kl} A_{kl}(s, t)$$

is the double Fourier series of a function from Y, whenever f^0 belongs to X.

Let also $B = (\beta_{mnkl})$ be a triangular matrix summability method in the series-to-sequence form. Denote the B-means of the double series f^0 by $\tau_{mn} f$, i.e.

$$(\tau_{mn} f)(s, t) = \sum_{k,l \leq m,n} \beta_{mnkl} A_{kl}(s, t).$$

The numbers ε_{mn} are called *summability factors of type* (A_b, B_b) if for each boundedly A-summable double series $\sum u_{kl}$ the double series $\sum \varepsilon_{kl} u_{kl}$ is boundedly B-summable.

We shall prove (cf. [9], p. 143, [25], p. 94)

Theorem 4.1. *If the numbers ε_{mn} are summability factors of type (A_b, B_b), then the double sequence ε is a multiplier of the class $((X \to A), (X \to B))$ and of the class $(X, (X \to A) \to B)$ for any space X and any methods A and B.*

Proof. Let $g^0 \in (X \to A)$ be arbitrary. Then the double sequence (h_{mn}), where $h_{mn} = \langle f^0, \sigma_{mn} g \rangle$ are defined by (3.2), is boundedly convergent for each $f^0 \in X$, according to the definition of A-complementary space. As long as ε_{mn} are summability factors of type (A_b, B_b), the double sequence (k_{mn}), where

$$k_{mn} = \sum_{k,l \leq m,n} 2^{-\varkappa} \beta_{mnkl} \varepsilon_{kl} (\alpha_{kl} a_{kl} + \beta_{kl} b_{kl} + \gamma_{kl} c_{kl} + \delta_{kl} d_{kl}),$$

or briefly, $k_{mn} = \langle f^0, \tau_{mn}(Tg) \rangle$, boundedly converges for any $f^0 \in X$. Hence the double series $(Tg)^0 \in (X \to B)$ and consequently ε is a multiplier of the class $((X \to A), (X \to B))$. Since also $k_{mn} = \langle \tau_{mn}(Tf), g^0 \rangle$ and the double sequence (k_{mn}) boundedly converges for each $g^0 \in (X \to A)$, then $(Tf)^0 \in ((X \to A) \to B)$, and consequently ε is a multiplier of the class $(X, (X \to A) \to B)$.

If in theorem 4.1 we take X to be one of the spaces L_Φ, L_Ψ and C, then we obtain from theorems 3.1, 3.2 and 3.3 the following:

Theorem 4.2. *Let* $A=(\alpha_{mnkl})$ *and* $B=(\beta_{mnkl})$ *be two summability methods so that*

$$b\text{-}\lim_{m,n} \alpha_{mnkl} = b\text{-}\lim_{m,n} \beta_{mnkl} = 1,$$

and their double sequences of Lebesgue constants are bounded. If ε_{mn} *are summability factors of type* (A_b, B_b), *then* ε *is a multiplier of the class* (L_Φ, L_Φ), *of the class* (L_Ψ, L_Ψ) *and of the class* (dV, dV).

The case $L_\Phi = L^p$ of theorem 4.2 is contained in theorem 3.2 of [3], and for $p>1$ in theorem 4 of [2].

For example, if we take A and B in theorem 4.2 to be the Cesàro method $C^{\alpha,\beta}$ of order $\alpha, \beta>0$, then condition (2.1) is satisfied. That condition (2.2) holds follows from Nikolskiĭ's theorem (cf. [4], p. 5) or can be obtained by direct computation (cf. [30], p. 94), because the Cesàro method is factorable. Applying theorem 2 from [1] and the supplement to theorem 6 from [13], we obtain

Corollary 4.3. *If for* $\alpha, \beta>0$ *the conditions*

$$\varepsilon_{mn} = O(1), \quad \lim_n \Delta_m^{\alpha+1} \varepsilon_{mn} = \lim_m \Delta_n^{\beta+1} \varepsilon_{mn} = 0,$$

$$\sum\nolimits_{m,n} (m+1)^\alpha (n+1)^\beta \, |\Delta_{mn}^{\alpha+1,\beta+1} \varepsilon_{mn}| < \infty$$

are satisfied, then ε *is a multiplier of all the classes of theorem* 4.2.

5. Some topological properties of A-complementary spaces

A Banach space X of numerical sequences is called a *BK-space* whenever convergence in norm of any sequence in X implies its coordinate-wise convergence (cf. [28], p. 466). In particular, each of its coordinates is a continuous linear functional in the space. We can also regard sets of double sequences of quadruples of numbers

$$(a_{kl}, b_{kl}, c_{kl}, d_{kl})$$

as *BK*-spaces. For example, the set of all quadruples of Fourier coefficients of some Banach space of functions of two variables is a *BK*-space, if the coordinate-wise convergence of any sequence follows from its convergence in norm. This space is called the *space of double Fourier series.*

Let h_{mn} be defined as in (3.2). As we saw above the equalities

$$h_{mn} = \langle f^0, \sigma_{mn} g \rangle = \langle \sigma_{mn} f, g^0 \rangle \tag{5.1}$$

are known for the summability method A. We prove the following (compare [6], p. 351, Satz 2.1, [7], p. 373, [24], p. 76, theorem 12)

Theorem 5.1. *If the space X of double Fourier series is a BK-space and A satisfies condition* (2.1), *then the A-complementary space* $(X\to A)$ *is also a BK-space with norm*

$$(5.2)\qquad \|f^0\|_{(X\to A)} = \sup_{m,n} \sup_{\|g^0\|_X \leqq 1} |h_{mn}|.$$

Proof. The set $(X\to A)$ is a vector space. It is important to obtain that if $f^0\in(X\to A)$, then $\|f^0\|_{(X\to A)}<\infty$ in (5.2). In fact, defining for $g^0\in X$

$$(5.3)\qquad \varphi_{mn} g^0 = h_{mn},$$

we see that (φ_{mn}) is a double sequence of continuous linear functionals on X (compare [28], p. 471, Satz 4.4a). If $f^0\in(X\to A)$, then according to the definition of $(X\to A)$ the double sequence $(h_{mn})\in bc$ for each $g^0\in X$, and by the Banach—Steinhaus theorem (see [17], p. 10, theorem III) we obtain (3.5) and therefore $\|f^0\|_{(X\to A)}<\infty$, because

$$(5.4)\qquad \sup_{m,n} \sup_{\|g^0\|_X\leqq 1} |h_{mn}| = \sup_{m,n} \|\varphi_{mn}\| < \infty.$$

In view of (5.1) it is clear that (5.2) satisfies the axioms of a norm.

Let us prove that $(X\to A)$ is a complete space. Let (f_i^0) be a Cauchy sequence in $(X\to A)$, where

$$f_i^0 = (a_{kl}^i, b_{kl}^i, c_{kl}^i, d_{kl}^i).$$

Then for any $\varepsilon>0$ there exists a number $N=N(\varepsilon)$ such that for any $i\geqq N$ and any natural j the equality

$$(5.5)\qquad \|f_i^0-f_{i+j}^0\|_{(X\to A)} = \sup_{m,n} \sup_{\|g^0\|_X\leqq 1} |\langle f_i^0-f_{i+j}^0, \sigma_{mn} g\rangle| < \varepsilon$$

holds. If we put $g=\beta_{kl}e_2$, then we obtain

$$2^{-\varkappa}|\alpha_{mnkl}\beta_{kl}(b_{kl}^i-b_{kl}^{i+j})| \leqq \|f_i^0-f_{i+j}^0\|_{(X\to A)} < \varepsilon$$

for $i\geqq N$ and any j. By (2.1) (and analogous reasoning for the other coefficients), it follows that there exists

$$(5.6)\qquad f^0 = (a_{kl}, b_{kl}, c_{kl}, d_{kl})$$

such that $a_{kl}^{i+j}\to a_{kl}, \ldots, d_{kl}^{i+j}\to d_{kl}$ for $j\to\infty$. Since the inequality (5.5) is valid for any j, the inequality

$$(5.7)\qquad \sup_{\|g^0\|_X\leqq 1} |\langle f_i^0-f^0, \sigma_{mn} g\rangle| \leqq \varepsilon$$

holds for all $i\geqq N$ independently of m, n. Taking sup over m, n in (5.7), we obtain

$$(5.8)\qquad \|f_i^0-f^0\|_{(X\to A)} \leqq \varepsilon$$

for all $i \geqq N$. From (5.8) it follows that $f^0 \in (X \to A)$, that is $(h_{mn}) \in bc$ for all $g^0 \in X$, because, in view of (5.1), the divergence of the double sequence $(\langle \sigma_{mn} f, g_1^0 \rangle)$ even for some $g_1^0 \in X$ with $\|g_1^0\|_X \leqq 1$ is impossible. In fact, if $i \geqq N$, then for each m, n, by virtue of (5.7) and (5.1), we have

$$\varepsilon \geqq \sup_{\|g^0\|_X \leqq 1} |\langle \sigma_{mn}(f_i - f), g^0 \rangle| \geqq |\langle \sigma_{mn}(f_i - f), g_1^0 \rangle|$$

$$\geqq \big| |\langle \sigma_{mn} f_i, g_1^0 \rangle| - |\langle \sigma_{mn} f, g_1^0 \rangle| \big|$$

and the double sequence $(\langle \sigma_{mn} f_i, g_1^0 \rangle) \in bc$ for all $i > N$, because $f_i^0 \in (X \to A)$. Thus $(X \to A)$ is a Banach space.

It remains to prove the coordinate-wise convergence in $(X \to A)$. In fact, if, for example, we take $g_2 = e_2 / \|e_2\|_X$, we obtain from (5.1) and (5.2) that

$$2^{-\varkappa} |b_{kl}^i - b_{kl}| \, |\alpha_{mnkl}| = |\langle \sigma_{mn}(f_i^0 - f^0), g_2 \rangle| \cdot \|e_2\|_X$$

$$\leqq \|f_i^0 - f^0\|_{(X \to A)} \cdot \|e_2\|_X,$$

whence by (2.1)

$$2^{-\varkappa} |b_{kl}^i - b_{kl}| \leqq \|f_i^0 - f^0\|_{(X \to A)} \cdot \|e_2\|_X,$$

which yields that $b_{kl}^i \to b_{kl}$ for $i \to \infty$. An analogous argument applies to the coordinates a_{kl}, c_{kl} and d_{kl}. Consequently, $(X \to A)$ is a BK-space.

Corollary 5.2. *If X is any of the spaces C, M, L^p, L_Φ, L_Ψ or V and the summability method A satisfies condition* (2.1), *then the complementary space $(X \to A)$ is a BK-space.*

Proof. Since the above-mentioned spaces X of functions f are Banach spaces, the corresponding spaces X of Fourier coefficients (5.6) with norm $\|f^0\|_X = \|f\|_X$ are BK-spaces, and by theorem 5.1 the A-complementary spaces $(X \to A)$ are BK-spaces with the norm (5.2).

The following corollary holds true (compare [6], Satz 2.3, [8], Satz 3′, and [24], theorem 13):

Corollary 5.3. *If the space X of double Fourier series is a BK-space, A satisfies condition* (2.1) *and $X \cap P$ is dense in X, then $f^0 \in (X \to A)$ if and only if*

$$\|f^0\|_{(X \to A)} < \infty. \tag{5.9}$$

Proof. The fact that $f^0 \in (X \to A)$ implies (5.9) is contained in the proof of Theorem 5.1. We shall prove the converse. Assume (5.9). Then (5.4) is true for the double sequence (φ_{mn}) of continuous linear functionals, defined by (5.3). By the Banach—Steinhaus theorem it remains to prove that $(h_{mn}) \in bc$ on a set dense in X. If $g^0 \in X \cap P$, then g^0 is some t_{kl} and therefore

$$(5.10) \qquad (\langle \sigma_{mn} f, g^0 \rangle) \in bc$$

in view of (2.1) and (5.1). Since $X \cap P$ is dense in X, it follows that (5.10) is valid for any $g^0 \in X$, that is $f^0 \in (X \to A)$.

6. Identical classes of multipliers

Now we require the following definition (see [10], p. 34, [21], p. 202).

Let X be a normed space with dual X'. A closed subset $\Gamma \subset X'$ is called a *norm-determining manifold for* X if for any $x \in X$ we have

$$\sup \{|\varphi x|: \ \|\varphi\|_{X'} \leqq 1, \ \varphi \in \Gamma\} = \|x\|_X.$$

For example, the dual X' itself is a norm-determining manifold for X (see [21], p. 186, theorem 4.3—B, or [10], p. 30, theorem 2.7.4). Consequently, the space M is a norm-determining manifold for $L := L^1$, because $L' = M$ (see [14], p. 191). It is advantageous to assume that $C' = dV$, and then dV is a norm-determining manifold for C. This is possible because in view of (5.1) and (3.9) any continuous linear functional φ on C can be represented by the formula (cf. [9], p. 140)

$$\varphi f = \lim_{m,n} \langle f^0, \sigma_{mn} g \rangle$$

with $f \in C$ and $g^0 \in (C \to A) = dV$ by Theorem 3.3, if A satisfies conditions (2.1) and (2.2).

Now we can prove (cf. [9], Satz 6, [25], theorem 4.3) the following

Theorem 6.1. *Let the summability method A satisfy* (2.1). *Let X and Y be BK-spaces of double Fourier series, P be dense in X, and let Γ and Λ be norm-determining manifolds for Y. Then*

$$(6.1) \qquad (\Gamma, (X \to A)) = (\Lambda, (X \to A)).$$

Proof. By the definition of multipliers and corollary 5.3, it follows that ε is a multiplier of the class $(\Gamma, (X \to A))$ if and only if $\|(Tf)^0\|_{(X \to A)} < \infty$ for each $f^0 \in \Gamma$. Let $T_{mn}: Y' \to X'$ be continuous linear operators determined by the formula

$$T_{mn} f^0 = \sigma_{mn}(Tf).$$

Let the continuous linear functionals $f_{mn} = T_{mn} f^0 \in X'$ be defined by the formula

$$f_{mn} g^0 = \langle T_{mn} f^0, g^0 \rangle.$$

Then by (5.2) and (5.1) we obtain

$$\|(Tf)^0\|_{(X\to A)} = \sup_{m,n} \sup_{\|g^0\|_X \leqq 1} |f_{mn} g^0| = $$
$$= \sup_{m,n} \|f_{mn}\|_{X'} = \sup_{m,n} \|T_{mn} f^0\|_{X'}.$$

Therefore ε is a multiplier of the class $(\Gamma, (X\to A))$ if and only if

(6.2)
$$\sup_{m,n} \|T_{mn} f^0\|_{X'} < \infty \quad \forall f^0 \in \Gamma.$$

By the uniform boundedness theorem ([10], p. 26, theorem 2.5.5), condition (6.2) is equivalent to

(6.3)
$$\|T_{mn}\|_{\mathscr{L}(\Gamma, X')} = O(1).$$

In view of (5.1)

$$\langle T_{mn} f^0, g^0\rangle = \langle f^0, T_{mn} g^0\rangle,$$

that is T_{mn} are self-adjoint operators ([21], p. 214). Therefore from the previous calculations we obtain that

$$\|T_{mn}\|_{\mathscr{L}(\Gamma, X')} = \sup_{\substack{\|f^0\|_{Y'} \leqq 1 \\ f^0 \in \Gamma}} \sup_{\|g^0\|_X \leqq 1} |\langle f^0, T_{mn} g^0\rangle|$$
$$= \sup_{\|g^0\|_X \leqq 1} \|T_{mn} g^0\|_Y = \|T_{mn}\|_{\mathscr{L}(X, Y)},$$

because Γ is a norm-determining manifold for Y. Since Λ is also a norm-determining manifold for the same Y, it also follows that

$$\|T_{mn}\|_{\mathscr{L}(\Lambda, X')} = \|T_{mn}\|_{\mathscr{L}(X, Y)}$$

and consequently

$$\|T_{mn}\|_{\mathscr{L}(\Gamma, X')} = \|T_{mn}\|_{\mathscr{L}(\Lambda, X')},$$

whence in view of the equivalence of (6.2) and (6.3) the equation (6.1) follows.

From theorem 6.1 we obtain (cf. [9], p. 141) the following corollaries:

Corollary 6.2. *If the summability method A satisfies* (2.1) *and X is any one of the spaces L_Φ, L^p or C, then*

(6.6)
$$(C, (X \to A)) = (M, (X \to A)).$$

Proof. One must prove that the Banach space C is a norm-determining manifold for the Banach space L. In fact, $C \subset M = L'$ and for each $g \in C$ there exists a functional $\varphi \in L'$ such that

$$\varphi x = \iint_Q x(s, t) g(s, t)\, ds\, dt$$

(see [14], p. 191) for all $x \in L$, whence $\sigma \leqq \|x\|_L$, where

$$\sigma = \sup \{|\varphi x| \colon \|\varphi\|_M \leqq 1,\ g \in C\}.$$

On the other hand for each $x\in L$ there exists the function $g_0\in M$ with $g_0(s,t)=\operatorname{sgn} x(s,t)$. According to Lusin's theorem ([12], p. 106) for each $\delta>0$ there exists a subset $E_\delta\subset Q$ with Lebesgue measure $mE_\delta>4\pi^2-\delta$ and a function $g_\delta\in C$ such that $g_\delta(s,t)=g_0(s,t)$ on E_δ and $|g_\delta(s,t)|\leqq 1$. Designating $e_\delta=Q\setminus E_\delta$ and

$$\varphi_\delta x=\iint_Q x(s,t)\,g_\delta(s,t)\,ds\,dt,$$

by the absolute continuity of the Lebesgue integral for any $\varepsilon>0$ there exists $\delta>0$ such that

$$\begin{aligned}\sigma\geqq|\varphi_\delta x| &= \left|\iint_{E_\delta}+\iint_{e_\delta}\right|\geqq\left|\iint_{E_\delta}\right|-\left|\iint_{e_\delta}\right|\\ &=\|x\|_L-\iint_{e_\delta}|x(s,t)|\,ds\,dt-\iint_{e_\delta}x(s,t)g_\delta(s,t)\,ds\,dt\\ &\geqq\|x\|_L-2\iint_{e_\delta}|x(s,t)|\,ds\,dt>\|x\|_L-2\varepsilon\end{aligned}$$

and hence $\sigma=\|x\|_L$.

Now the set P is dense in C (see [22], p. 14) and in L_Φ (see [27], p. 128) and hence in L^p. It remains to adapt theorem 6.1 with $Y=L$, $\Gamma=C$ and $\Lambda=M$.

Corollary 6.3. *If the summability method A satisfies* (2.1) *and X is any one of the spaces L_Φ, L^p or C, then*

$$(L,(X\to A))=(dV,(X\to A)).$$

Proof. The Banach spaces L and dV are norm-determining manifolds for C. In fact, we have seen that $dV=C'$. Further, $L'\subset L\subset C'$ and from the theorem on sufficiently many continuous functionals (see [10], p. 30, theorem 2.7.4), it follows that for each $x_0\in C\subset L$ there exists a functional $\varphi_0\in L'$ with $\|\varphi_0\|_{L'}=1$ and $\varphi_0x_0=\|x_0\|_C$. Consequently

$$\sup\{|\varphi x_0|:\ \|\varphi\|_{C'}\leqq 1,\ \varphi\in L\}\geqq\sup\{|\varphi x_0|:\ \|\varphi\|_{L'}\leqq 1,\ \varphi\in L'\}\geqq\|x_0\|_C.$$

The inverse inequality follows from $|\varphi x|\leqq\|\varphi\|_{C'}\|x\|_C$. Thus L is also a norm-determining manifold for C. Because P is dense in the considered spaces, it remains to adapt theorem 6.1 with $Y=C$, $\Gamma=L$ and $\Lambda=dV$.

From corollaries 6.2 and 6.3 with the help of theorems 3.1 and 3.3 we obtain:

Corollary 6.4. *The following identities between classes of multipliers hold:*

1) $(C,L_\Psi)=(M,L_\Psi)$, 1a) $(C,L^p)=(M,L^p)$ *with* $p>1$,

1b) $(C,M)=(M,M)$ 2) $(C,dV)=(M,dV)$,

3) $(L,L_\Psi)=(dV,L_\Psi)$, 3a) $(L,L^p)=(dV,L^p)$ *with* $p>1$,

3b) $(L,M)=(dV,M)$, 4) $(L,dV)=(dV,dV)$.

The following is also true (cf. [9], p. 143, Satz 12, [25], p. 93, theorem 4.5):

Theorem 6.5. *For any summability method A and arbitrary spaces X and Y, the inclusion*

$$(X,Y) \subseteqq ((Y \to A), (X \to A))$$

holds.

Proof. Let ε be a multiplier of class (X, Y), that is $(Tf)^0 \in Y$ for any $f^0 \in X$. Let $g^0 \in (Y \to A)$. Since

$$(Y \to A) = \{g^0 \colon \langle g^0, h^0 \rangle \in A_b' \ \forall h^0 \in Y\}$$

and $h^0 = (Tf)^0 \in Y$, it follows that $\langle g^0, (Tf)^0 \rangle \in A_b'$ for any $f^0 \in X$. Since (5.1) means hat $\langle g^0, (Tf)^0 \rangle = \langle (Tg)^0, f^0 \rangle$, one obtains $\langle (Tg)^0, f^0 \rangle \in A_b'$ for any $f^0 \in X$, that is $(Tg)^0 \in (X \to A)$.

Corollary 6.6. *The following identities between classes of multipliers hold:*

1) $(L_\Phi, L_\Phi) = (L_\Psi, L_\Psi)$, 1a) $(L^p, L^p) = (L^q, L^q)$, $p > 1$,

1b) $(L, L) = (M, M)$, 2) $(L_\Phi, M) = (L, L_\Psi)$,

2a) $(L^p, M) = (L, L^q)$, $p > 1$, 3) $(L_\Phi, L) = (M, L_\Psi)$,

3a) $(L^p, L) = (M, L^q)$, $p > 1$, 4) $(L_\Psi, L) = (M, L_\Phi)$,

where $p^{-1} + q^{-1} = 1$.

Proof. For example 2) follows from theorem 6.5, putting $X = L_\Phi$, $Y = M$, and A satisfying (2.1) and (2.2), with the help of theorems 3.1 and 3.2, since

$$(L_\Phi, M) \subseteqq ((M \to A), (L_\Phi \to A)) = (L, L_\Psi) \subseteqq ((L_\Psi \to A), (L \to A)) = (L_\Phi, M).$$

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