A multi-dimensional renewal theorem for finite Markov chains

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1. Introduction and results

Let U, L and F be functions from $\mathbf{Z}^d$ into the set of real square matrices of finite dimension N, and let in addition $L(t)$ be positive for each t. Define the convolution $L*U$ by the formula

(1.1) $$L*U(t) = \sum_{t_1+t_2=t} L(t_1)U(t_2),$$

and put

(1.2) $$R = \sum_{n=0}^{\infty} L^{n*},$$

provided the sum converges. Here $L^{0*}=\delta$, where $\delta(0)=1$ (the identity matrix) and $\delta(t)=0$ for $t\neq 0$, and $L^{n*}=L*L^{(n-1)*}$ for $n\geqq 1$.

A solution U of the renewal equation $U-L*U=F$ is then given by $U=R*F$, provided the latter expression converges. The object of the present paper is to study the asymptotic behaviour of $R*F(t)$, as $|t|\to\infty$.

The result can be applied to first passage problems for sums of Markov dependent random variables. See Höglund 1989.

Instead of a function L defined on $\mathbf{Z}^d$ we could equally well have considered a matrix valued measure on $\mathbf{R}^d$, but our restriction will save us some labour because it makes smoothing unnecessary.

The approximation will be expressed in terms of quantities related to the matrices $\Lambda(\theta)$, $\theta\in\Theta$, where

(1.3) $$\Lambda(\theta) = \sum_t e^{\theta\cdot t} L(t)$$

and where Θ denotes the interior of the set of $\theta\in\mathbf{R}^d$ for which this sum converges. Here $\theta\cdot t$ stands for the inner product of θ and t. We shall assume that the function L is *irreducible*, by which we mean that for every i and j in $\{1, \ldots, N\}$ there is a positive integer n and a $t\in\mathbf{Z}^d$ such that $L_{ij}^{n*}(t)>0$. We shall assume that $\Theta\neq\emptyset$

and then irreducibility is equivalent to that the matrix $\Lambda(\theta)$ is irreducible for some (and hence for all) $\theta\in\Theta$.

The Laplace transform $\Lambda(\theta)$ is thus a positive and irreducible matrix whose coefficients are analytic in Θ, and hence $\Lambda(\theta)$ has a maximal positive eigenvalue $\lambda(\theta)$ corresponding to strictly positive left and right eigenvectors $e^*(\theta)=\{e_i^*(\theta)\}$ and $e(\theta)=\{e_i(\theta)\}$. This eigenvalue is simple and analytic in Θ, and $e_i^*(\theta)$, and $e_i(\theta)$ can be chosen to be analytic in Θ. Let $E(\theta)=(E_{ij}(\theta))$ stand for the eigenprojection corresponding to $\lambda(\theta)$, where

$$E_{ij}(\theta)=\frac{e_i(\theta)e_j^*(\theta)}{e(\theta)\cdot e^*(\theta)} \tag{1.4}$$

and put

$$\lambda'(\theta)=\operatorname{grad}\lambda(\theta),\quad \lambda''(\theta)=\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\lambda(\theta)\right),\quad \Delta=\{\theta\in\Theta;\ \lambda(\theta)=1\}. \tag{1.5}$$

Let

$$S_{ij}=\bigcup_{n=1}^{\infty}\{t;\ L_{ij}^{n*}(t)>0\}, \tag{1.6}$$

let G_{ij} denote the smallest subgroup of $\mathbf{Z}^d$ that contains

$$S_{ij}-S_{ij}=\{t_1-t_2;\ t_1\in S_{ij},\ t_2\in S_{ij}\} \tag{1.7}$$

and define the group G by $G=\bigcap_i\bigcap_j G_{ij}$.

The role played by G is illustrated by the following lemmas.

Note that $S_{ij}\neq\emptyset$ for all i and j if and only if L is irreducible.

Lemma 1.1. *Assume that L is irreducible. Choose for each $j\in\{1,\dots,N\}$ a $v(j)\in S_{1j}$. Then $S_{ij}\subset v(i)-v(j)+G$. G is minimal in the sense that if $w(1),\dots,w(N)$ are real numbers and H a group and $S_{ij}\subset w(i)-w(j)+H$ for some i,j, then $H\supset G$.*

Proof. The inequality

$$L_{ik}^{n_1*}(t_1)L_{kj}^{n_2*}(t_2)\leqq L_{ij}^{(n_1+n_2)*}(t_1+t_2),$$

implies $S_{ik}+S_{kj}\subset S_{ij}$. Therefore $G_{ik}+G_{kj}\subset G_{ij}$ and hence also $G_{ik}\subset G_{ij}$, $G_{kj}\subset G_{ij}$ for all i, k and j. This cannot be true unless $G_{ij}=G_{11}$ for all i and j. Let c_{ij} be an element in the coset of G that contains S_{ij}. Then $S_{ik}+S_{kj}$ is contained in the coset $c_{ik}+c_{kj}+G$, and hence $c_{ik}+c_{kj}\equiv c_{ij}$ mod G. Define $v(i)=c_{i1}$, then $c_{ik}\equiv v(i)-v(k)$ mod G.

The group H contains the set (1.7) and hence also G. ∎

Lemma 1.2. *Assume that $\Theta\neq\emptyset$. The matrix $\lambda''(\theta)$ is strictly positive definite (for all, or for some θ) if and only if* $\dim G=d$.

Proof. Theorem 1.2 of Keilson and Wishart 1964 says that if $d=1$ (and $0\in\Theta$), then $\lambda''(0)/\lambda(0)-(\lambda'(0)/\lambda(0))^2\geqq 0$ with equality if and only if there is a real α and a real sequence $\omega(1), \omega(2), \ldots$ such that $L_{ij}(t) > 0$ only when $t = \alpha + \omega(i) - \omega(j)$. Note that $\lambda''(0) = 0$ if and only if $\lambda''(0)/\lambda(0) - (\lambda'(0)/\lambda(0))^2 = 0$ and $\lambda'(0) = 0$, that is $L_{ij}(t) > 0$ only when $t = \omega(i) - \omega(j)$.

Fix $\theta \in \Theta$, $0 \neq \eta \in \mathbf{R}^d$, and let ξ be real and so small that $\theta + \xi\eta \in \Theta$. Apply the above result to the matrix $\bar{\Lambda}(\xi) = \Lambda(\theta + \xi\eta)$. The result is that $\eta \cdot \Lambda(\theta)\eta = 0$ if and only if $e^{\theta \cdot t} L_{ij}(t) > 0$ only when $\eta \cdot t = \omega(i) - \omega(j)$. Choose a sequence $w(1), w(2), \ldots$ in $\mathbf{R}^d$ such that $\omega(i) = \eta \cdot w(i)$. Then $\eta \cdot \Lambda(\theta)\eta = 0$ if and only if $L_{ij}(t) > 0$ only when $\eta \cdot (t - w(i) + w(j)) = 0$. It follows from Lemma 1.1 that this is equivalent to G beeing orthogonal to η. ▮

Lemma 1.3. *Assume that* $\dim G = d$. *Either* $\lambda'(\theta) \neq 0$ *for all* $\theta \in \Delta$, *or else* Δ *is a one-point set.*

The proof is the same as the proof of Lemma 1.1 in Höglund 1988.

We shall first consider the case when $\lambda'(\theta) \neq 0$ on Δ. In this case we are able to determine the asymptotic behaviour as t tends to infinity in the cone

$$\{\tau\lambda'(\theta);\ \tau > 0,\ \theta \in \Delta\}, \tag{1.8}$$

provided F is sufficiently regular.

Lemma 1.4. *Assume that* $\dim G = d$ *and that* $\lambda'(\theta) \neq 0$ *on* Δ. *Then the function* $\Delta \ni \theta \to \lambda'(\theta)/|\lambda'(\theta)|$ *is one to one.*

The proof is the same as the proof of Lemma 1.3 in Höglund 1988.

We shall write $\tilde{\theta}(t)$ for the solution $\theta = \tilde{\theta}(t) \in \Delta$ of the equation $\lambda'(\theta)/|\lambda'(\theta)| = t/|t|$ when t belongs to the cone (1.8).

Theorem 1.5. *Assume that L is irreducible, that* $G = \mathbf{Z}^d$, *and* $\lambda'(\theta) \neq 0$ *on* Δ. *Let* Θ_F *denote the interior of the set of* θ *for which*

$$\sum_s e^{\theta \cdot s} \|F(s)\| < \infty. \tag{1.9}$$

If $|t| \to \infty$ *in the cone* (1.8) *in such a way that* $\tilde{\theta}(t) \in \Theta_F$, *then*

$$R * F(t) = e^{-\theta \cdot t}(2\pi|t|/|\lambda'(\theta)|)^{-(d-1)/2} C(\theta)^{-1/2}\big(E(\theta)\sum_s e^{\theta \cdot s} F(s) + o(1)\big). \tag{1.10}$$

Here $\theta = \tilde{\theta}(t)$, and $C(\theta) = \lambda'(\theta) \cdot \lambda''(\theta)^{-1}\lambda'(\theta) \det \lambda''(\theta)$. *The convergence is uniform in* $t/|t|$ *as* $\tilde{\theta}(t)$ *stays within compact subsets of* $\Delta \cap \Theta_F$.

If we write $E(\theta)$ to the right of $F(s)$ in formula (1.10), we get the corresponding approximation for $V(t) = F * R(t)$, which is a solution of $V - V * L = F$.

The moment conditions on F and L are not optimal, but chosen to facilitate the proof.

An alternative to (1.10) is the approximation

$$R(t) = e^{-\theta \cdot t}(2\pi T)^{-(d-1)/2} C(\theta)^{-1/2}\big(e^{-(1/2)t \cdot \Sigma t/T} E(\theta) + o(1)\big) \tag{1.11}$$

as $T\to\infty$, which holds uniformly in θ as θ stays within compact subsets of Δ. Here

$$(1.12)\quad T=\frac{t\cdot\lambda''(\theta)^{-1}\lambda'(\theta)}{\lambda'(\theta)\cdot\lambda''(\theta)^{-1}\lambda'(\theta)},\quad t\cdot\Sigma t=\big(t-T\lambda'(\theta)\big)\cdot\lambda''(\theta)^{-1}\big(t-T\lambda'(\theta)\big).$$

The two approximations are roughly equivalent.

Note that the relative error in the approximation (1.11) equals

$$(1.13)\quad o\Big(\exp\big(\tfrac{1}{2}(t-T\lambda'(\theta))\cdot\lambda''(\theta)^{-1}(t-T\lambda'(\theta))\big)\Big)$$

and that the expression between the main brackets is minimized and equals 1 if θ is such that $\lambda'(\theta)$ and t have the same direction. This is why we let $\theta=\tilde{\theta}(t)$ in the theorem.

The condition $G=\mathbf{Z}^d$ is just a normalization. To see this, let $b_1,\ldots,b_{d'}$, $d'\leqq d$, be a basis for G, and define for $(\bar{t}_1,\ldots,\bar{t}_{d'})\in\mathbf{Z}^{d'}$

$$(1.14)\quad L_{ij}(\bar{t})=L_{ij}\big(v(i)-v(j)+\textstyle\sum_{k=1}^{d'}\bar{t}_k b_k\big).$$

Then $\bar{G}=\mathbf{Z}^{d'}$, and $R_{ij}(t)=\bar{R}_{ij}(\bar{t})$ when $t=v(i)-v(j)+\sum_{k=1}^{d'}\bar{t}_k b_k$, and $R_{ij}(t)=0$ when $t\notin v(i)-v(j)+G$.

Furthermore if B is the $d\times d'$ matrix whose columns are $b_1,\ldots,b_{d'}$, then B^TB is non-singular and $\Lambda_{ij}(\theta)=e^{\theta\cdot v(i)}\bar{\Lambda}_{ij}(\bar{\theta})e^{-\theta\cdot v(j)}$, where $\bar{\theta}=B^T\theta$. Therefore $\lambda'(\theta)=B\bar{\lambda}'(\bar{\theta})$, $\lambda''(\theta)=B\bar{\lambda}''(\bar{\theta})B^T$, and $\bar{E}(\bar{\theta})=D(\theta)^{-1}E(\theta)D(\theta)$, where $D(\theta)$ is the diagonal matrix with $D_{ii}(\theta)=e^{\theta\cdot v(i)}$. The general case follows from these identities.

The next theorem is the result that corresponds to Theorem 1.5 when $\lambda'(\theta)=0$.

Theorem 1.6. *Assume that $G=\mathbf{Z}^d$, that L is irreducible, and $\lambda'(\theta)=0$. If*

$$(1.15)\quad \textstyle\sum_s |s|^{d-2}e^{\theta\cdot s}\|F(s)\|<\infty$$

then

$$(1.16)\quad R*F(t)=e^{-\theta\cdot t}\big(t\cdot\lambda''(\theta)^{-1}t\big)^{-(d-2)/2}K(\theta)\big(E(\theta)\textstyle\sum_s e^{\theta\cdot s}F(s)+o(1)\big)$$

as $|t|=\to\infty$, provided $d\geqq 3$. Here $K(\theta)=(\det\lambda''(\theta))^{-1/2}\pi^{-1/2}\Gamma(d/2)/(d-2)$.

The counterpart to these theorems for independent random variables were given by Ney and Spitzer 1966 (Thm. 1.5), and by Spitzer 1964 (Thm. 1.6).

Concerning one-dimensional markovian renewal theory we refer to Runnenburg 1960, Orey 1961, Pyke 1961, Cinlar 1969, Jacod 1971, Iosifescu 1972, and Kesten 1974. Further one-dimensional renewal results can be found in Berbee 1979 (process with stationary increments) and Janson 1983 (m-dependent variables).

2. Proofs

We shall first show that the theorems hold under the additional assumptions that $\Lambda(\theta)$ is aperiodic and $F=\delta$ (theorems 2.1 and 2.2). In proposition 2.7 we remove the assumption $F=\delta$, and in proposition 2.8 the assumption that $\Lambda(\theta)$ is aperiodic.

Theorem 2.1. *Assume that $G=\mathbf{Z}^d$, that $\Lambda(\theta)$ is irreducible and aperiodic, and that $\lambda'(\theta)\neq 0$ on Δ. Then*

$$\sup_t \left(e^{\theta\cdot t}\|R(t)\|\right)<\infty \tag{2.1}$$

and

$$R(t)=e^{-\theta\cdot t}(2\pi T)^{-(d-1)/2}C(\theta)^{-1/2}\left(e^{-(1/2)t\cdot\Sigma t/T}+O(T^{-1/2})\right) \tag{2.2}$$

as $T\to\infty$, $\theta\in\Delta$. The error in (2.2) *is uniformly small and* (2.1) *is uniformly bounded as θ stays within compact subsets of Δ. Here T and Σ are as in* (1.12).

Theorem 2.2. *Assume that $G=\mathbf{Z}^d$, $d\geqq 3$, that L is irreducible and aperiodic, and that $\lambda'(\theta)=0$, $\theta\in\Delta$. Then*

$$\sup_t \left(e^{\theta\cdot t}\|R(t)\|\right)<\infty \tag{2.3}$$

and

$$R(t)=e^{-\theta\cdot t}(t\cdot\lambda''(\theta)^{-1}t)^{-(d-2)/2}K(\theta)(E(\theta)+O(|t|^{-1})), \tag{2.4}$$

as $|t|\to\infty$.

Proof. Define $L_\theta(t)=e^{\theta\cdot t}L(t)$ and $R_\theta(t)=e^{\theta\cdot t}R(t)$ then

$$R_\theta(t)=\sum_{n=0}^{\infty}L_\theta^{n*}(t)=\int_0^\infty P_\theta^s(t)\,ds \tag{2.5}$$

where

$$P_\theta^s(t)=\sum_{n=0}^{\infty}e^{-s}\frac{s^n}{n!}L_\theta^{n*}(t). \tag{2.6}$$

The Fourier transform of P_θ^s equals

$$\hat{P}_\theta^s(\eta)=\exp\left(s\Lambda(\theta+i\eta)-s\right)=\sum_{n=0}^{\infty}e^{-s}\frac{s^n}{n!}\Lambda(\theta+i\eta)^n. \tag{2.7}$$

We shall approximate $\Lambda(\theta+i\eta)$ by $\lambda(\theta+i\eta)E(\theta+i\eta)$, $E(\theta+i\eta)$ by $E(\theta)$, and $\lambda(\theta+i\eta)$ by $1+i\eta\cdot\lambda'(\theta)-\frac{1}{2}\eta\cdot\lambda''(\theta)\eta$. (Recall that $\lambda(\theta)=1$ when $\theta\in\Delta$.) Thus

$$\hat{P}_\theta^s(\eta)\approx\exp\left(s\lambda(\theta+i\eta)-s\right)E(\theta+i\eta)\approx\exp\left(is\eta\cdot\lambda'(\theta)-\frac{s}{2}\eta\cdot\lambda''(\theta)\eta\right)E(\theta). \tag{2.8}$$

The expression to the right is the Fourier transform of the function $\mathbf{R}^d\ni t\to Q_\theta^s(t)$, where

(2.9) $$Q_\theta^s(t) = q_\theta^s(t)E(\theta), \quad q_\theta^s(t) = \frac{\exp\left(-\frac{1}{2}(t-s\lambda'(\theta))\cdot\lambda''(\theta)^{-1}(t-s\lambda'(\theta))/s\right)}{\sqrt{2\pi s}^d\sqrt{\det\lambda''(\theta)}}.$$

We shall therefore approximate $R_\theta(t)$ by $\int_0^\infty Q_\theta^s(t)ds$. This approximation is made precise in proposition 2.5. Proposition 2.3 describes the asymptotic behaviour of $Q_\theta^s(t)$.

Given these propositions, theorems 2.1 and 2.2 follow if we show that $R_\theta(t)$ is bounded, but this follows from, for example, the local central limit theorem for L_θ^{n*}.

Proposition 2.3. *If* $\lambda'(\theta)\neq 0$ *on* Δ, *then*

(2.10) $$\int_0^\infty q_\theta^s(t)\,ds = (2\pi T)^{-(d-1)/2}C(\theta)^{-1/2}\left(e^{-(1/2)t\cdot\Sigma t/T}+O(T^{-1})\right)$$

as $T\to\infty$, $\theta\in\Delta$. *The error is uniformly small as* θ *stays within compact subsets of* Δ.

If $\lambda'(\theta)=0$, *then*

(2.11) $$\int_0^\infty q_\theta^s(t)\,ds = \left(t\cdot\lambda''(\theta)^{-1}t\right)^{-(d-1)/2}K(\theta)$$

for all $t\neq 0$.

The function

(2.12) $$G_\alpha(x) = \int_0^\infty s^{-\alpha}\exp\left[-\frac{x}{2}(s+1/s-2)\right]\frac{ds}{s}, \quad x>0$$

will appear in the proof.

Lemma 2.4. $G_\alpha(x)=\sqrt{2\pi/x}\,(1+0(1/x))$, *as* $x\to\infty$.

Proof of Lemma 2.4. Define $G_\alpha^+(x)$ as $G_\alpha(x)$ but with the domain of integration $1<s<\infty$ instead of $0<s<\infty$. Then $G_\alpha(x)=G_\alpha^+(x)+G_{-\alpha}^+(x)$, and

(2.13) $$G_\alpha^+(x) = \int_0^\infty e^{-xu}H_\alpha(du).$$

Here

(2.14) $$H_\alpha(u) = \int_\Omega s^{-\alpha}\frac{ds}{s} = \int_1^{\tau(u)} s^{-\alpha}\frac{ds}{s},$$

where

(2.15) $$\Omega = \left\{s>1;\ \frac{1}{2}\left(s+\frac{1}{s}-2\right)<u\right\} = (1,\tau(u)), \quad \tau(u) = 1+u+\sqrt{2u+u^2}.$$

Therefore

(2.16) $$H_\alpha'(u) = \tau(u)^{-\alpha}(2u+u^2)^{-1/2} = (2u)^{-1/2}-\alpha+O(u^{1/2}),$$

as $u \to 0$, and hence

(2.17) $$G_\alpha^+(x) = \Gamma\left(\tfrac{1}{2}\right)(2x)^{-1/2} - \alpha/x + O(x^{-3/2}),$$

as $x \to \infty$. ∎

Proof of Proposition 2.3. Put $m = \lambda''(\theta)^{-1/2}\lambda'(\theta)$, $\tau = \lambda''(\theta)^{-1/2}t$, $\alpha = \frac{d}{2} - 1$, and $\varkappa = (2\pi)^{-d/2}(\det \lambda''(\theta))^{-1/2}$. Then

(2.18) $$\int_0^\infty q_\theta^s(t)\,ds = \varkappa \int_0^\infty s^{-\alpha} \exp\left[-\frac{1}{2}\left(\frac{|\tau|^2}{s} + |m|^2 s - 2\tau\cdot m\right)\right]\frac{ds}{s}$$

$$= \begin{cases} \varkappa(|m|/|\tau|)^\alpha \exp[-|m|\,|\tau| + m\cdot\tau]G_\alpha(|m|\,|\tau|) & \text{when } m \neq 0,\ \tau \neq 0 \\ \varkappa(2/|\tau|^2)^\alpha \Gamma(\alpha) & \text{when } m = 0,\ \tau \neq 0,\ \alpha > 0. \end{cases}$$

Here we made the substitutions $\frac{|\tau|^2}{2s} \to s$ respectively $\frac{s|m|}{|\tau|} \to s$.

The proposition now follows from the lemma and the fact that if we define $\check\tau$ by $\tau = Tm + \check\tau$, then $m\cdot\check\tau = 0$ and

(2.19) $$|\tau|^2 = T^2|m|^2 + |\check\tau|^2, \quad |m|\,|\tau| - m\cdot\tau = \frac{|\check\tau|^2}{\sqrt{T^2 + |\check\tau|^2/|m|^2} + T}, \quad |\check\tau|^2 = t\cdot\Sigma t. \ ∎$$

Proposition 2.5. *If $\lambda'(\theta) \neq 0$ on Δ, then*

(2.20) $$\left\|R_\theta(t) - \int_0^\infty Q_\theta^s(t)\,ds\right\| = O(T^{-d/2})$$

as $T \to \infty$, $\theta \in \Delta$. The bound is uniform as θ stays within compact subsets of Δ.

If $\lambda'(\theta) = 0$, $\theta \in \Delta$, then

(2.21) $$\left\|R_\theta(t) - \int_0^\infty Q_\theta^s(t)\,ds\right\| = O(|t|^{-d+1})$$

as $|t| \to \infty$.

The essential part of the proof is the following estimate.

Lemma 2.6. *For any integer $0 \leq k \leq d+1$ and any compact $K \subset \Delta$ there is a constant C such that*

(2.22) $$\|P_\theta^s(t) - Q_\theta^s(t)\| \leq C s^{-(d+1)/2}\left|s^{-1/2}(t - s\lambda'(\theta))\right|^{-k}$$

for all $s > 0$, $t \in \mathbf{Z}^d$ and $\theta \in K$.

Proof of Proposition 2.5. It follows from the lemma that the expression on the left in (2.20) and (2.21) is dominated by

(2.23) $$c_d C \int_0^\infty (s + |t - s\lambda'(\theta)|^2)^{-(d+1)/2}\,ds$$

where c_d is a constant that depends only on d. An elementary calculation now gives the proposition. ▮

Proof of Lemma 2.6. We shall thus estimate the difference

$$\text{(2.24)}\qquad \int_{(-\pi,\pi]^d} e^{-i\eta\cdot t}\hat{P}^s_\theta(\eta)\,d\eta-\int_{\mathbf{R}^d} e^{-i\eta\cdot t}\hat{Q}^s_\theta(\eta)\,d\eta.$$

Recall that

$$\text{(2.25)}\qquad \hat{P}^s_\theta(\eta)=\exp\left(sA(\theta+i\eta)-s\right),\quad \hat{Q}^s_\theta(\eta)=\exp\left(is\eta\cdot\lambda'(\theta)-\frac{s}{2}\eta\cdot\lambda''(\theta)\eta\right)E(\theta).$$

Let $\sigma(z)$ denote the spectrum of $A(z)$. Then $\sigma(\theta+i\eta)$ is contained in the closed unit disc, and it follows from Lemma 2.2 in Höglund 1974 that $1\in\sigma(\theta+i\eta)$ if and only if $G\subset\{t;\ e^{i\eta\cdot t}=1\}$, i.e. $\eta\equiv 0 \bmod 2\pi\mathbf{Z}^d$.

Given the compact $K\subset\Delta$ choose $\varrho>0$, $\delta>0$ and $\varepsilon>0$ such that

(i) $\lambda(\theta+i\eta)$ and $E(\theta+i\eta)$ are analytic in a neighbourhood of the set $K_\delta=\{\theta+i\eta;\ \theta\in K,\ |\eta|\leqq\delta\}$.

(ii) $\log\lambda(\theta+i\eta)$ has no branching point for $\theta+i\eta\in K_\delta$.

(iii) $\left|\log\lambda(\theta+i\eta)-i\eta\cdot\lambda'(\theta)+\frac{1}{2}\eta\cdot\lambda''(\theta)\eta\right|<\frac{1}{3}\eta\cdot\lambda''(\theta)\eta$ for $\theta+i\eta\in K_\delta$.

(iv) $\operatorname{Re} w\leqq 1-3\varrho$ for $w\in\sigma(\theta+i\eta)\setminus\{\lambda(\theta+i\eta)\}$ and $\operatorname{Re}\lambda(\theta+i\eta)\geqq 1-\varrho$ when $\theta+i\eta\in K_\delta$.

(v) $\operatorname{Re} w<1-2\varepsilon$ for $w\in\sigma(\theta+i\eta)$ when $|\eta|>\delta$, $\theta\in K$.

That such a choice is possible is seen in the same way as in the proof of theorem 3.1 in Höglund 1974.

Put

$$K_s(\eta)=\exp\left[s\left(A(\theta+i\eta)-1-i\eta\cdot\lambda'(\theta)\right)\right]$$

$$\text{(2.26)}\qquad L_s(\eta)=\exp\left[s\left(\lambda(\theta+i\eta)-1-i\eta\cdot\lambda'(\theta)\right)\right]E(\theta+i\eta)$$

$$M_s(\eta)=\exp\left[-\frac{s}{2}\eta\cdot\lambda''(\theta)\eta\right]E(\theta)$$

and $y=t-s\lambda'(\theta)$. Then by repeated partial integrations

$$\text{(2.27)}\qquad (iy)^\alpha\int_{\mathbf{R}^d} e^{-i\eta\cdot y}M_s(\eta)\,d\eta=\int_{\mathbf{R}^d} M_s(\eta)\left(-\frac{\partial}{\partial\eta}\right)^\alpha e^{-i\eta\cdot y}\,d\eta$$

$$=\int_{\mathbf{R}^d} e^{-i\eta\cdot y}\frac{\partial^\alpha}{\partial\eta^\alpha}M_s(\eta)\,d\eta$$

and since the functions $\varkappa_\beta(\eta)=e^{-i\eta\cdot y}\dfrac{\partial^\beta}{\partial\eta^\beta}K_s(\eta)$, $\beta\in\mathbf{N}^d$, are functions of $\eta \bmod 2\pi\mathbf{Z}^d$

(i.e. $\varkappa_\beta(\eta_1)=\varkappa_\beta(\eta_2)$ when $\eta_1-\eta_2\in 2\pi\mathbf{Z}^d$) we also have

$$(2.28)\qquad (iy)^\alpha \int_{(-\pi,\pi]^d} e^{-i\eta\cdot y} K_s(\eta)\,d\eta = \int_{(-\pi,\pi]^d} e^{-i\eta\cdot y} \frac{\partial^\alpha}{\partial\eta^\alpha} K_s(\eta)\,d\eta.$$

Here $\alpha=(\alpha_1,\dots,\alpha_d)$ and

$$\frac{\partial^\alpha}{\partial\eta^\alpha} = \frac{\partial^{\alpha_1}}{\partial\eta_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial\eta_d^{\alpha_d}}.$$

Therefore the norm of $(iy)^\alpha(P_\theta^s(t)-Q_\theta^s(t))$ is dominated by $I_1+I_2+I_3+I_4$ where

$$I_1 = \left\|\int_{|\eta|>\delta,(-\pi,\pi]^d} e^{-i\eta\cdot y} \frac{\partial^\alpha}{\partial\eta^\alpha} K_s(\eta)\,d\eta\right\|$$

$$(2.29)\qquad I_2 = \left\|\int_{|\eta|\leqq\delta} e^{-i\eta\cdot y} \frac{\partial^\alpha}{\partial\eta^\alpha} (K_s(\eta)-L_s(\eta))\,d\eta\right\|$$

$$I_3 = \left\|\int_{|\eta|\leqq\delta} e^{-i\eta\cdot y} \frac{\partial^\alpha}{\partial\eta^\alpha} (L_s(\eta)-M_s(\eta))\,d\eta\right\|$$

$$I_4 = \left\|\int_{|\eta|>\delta} e^{-i\eta\cdot y} \frac{\partial^\alpha}{\partial\eta^\alpha} M_s(\eta)\,d\eta\right\|.$$

We are going to show that $I_j=O(s^{-(d+1-|\alpha|)/2})$ for $|\alpha|\leqq d+1$.

Put $z=\theta+i\eta$, let $\Gamma=\Gamma(z)$ be a contour surrounding $\sigma(z)$, and let $\gamma(z)$ be a contour that surrounds $\lambda(z)$ but no other point in $\sigma(z)$ when $|\eta|\leqq\delta$. Then (Kato 1966, p. 39 and p. 44)

$$(2.30)\qquad A(z)^n = \frac{1}{2\pi i}\int_{\Gamma(z)} w^n(w-A(z))^{-1}\,dw$$

for all z, and

$$(2.31)\qquad \lambda(z)^n E(z) = \frac{1}{2\pi i}\int_{\gamma(z)} w^n(w-A(z))^{-1}\,dw$$

when $|\eta|\leqq\delta$. Therefore

$$(2.32)\qquad K_s(\eta) = \frac{1}{2\pi i}\int_{\Gamma(z)} e^{s(w-1-i\eta\cdot\lambda'(\theta))}(w-A(z))^{-1}dw$$

and hence $\frac{\partial^\alpha}{\partial\eta^\alpha} K_s(\eta)=\int_\Gamma H\,dw$, where

$$(2.33)\qquad H = \frac{1}{2\pi i} e^{s(w-1)} \frac{\partial^\alpha}{\partial\eta^\alpha}\left(e^{-i\eta\cdot\lambda'(\theta)s}(w-A(\theta+i\eta))^{-1}\right).$$

In the same way we obtain $\frac{\partial^\alpha}{\partial\eta^\alpha} L_s(\eta)=\int_\gamma H\,dw$.

Let $Q(r)$ denote the contour that sorrounds the rectangle $\{w\in\mathbf{C};\ -2\leqq \operatorname{Re} w\leqq r,\ -2\leqq \operatorname{Im} w\leqq 2\}$. H is a meromorphic function the poles of which coincides with the spectrum of $\Lambda(\theta+i\eta)$. Therefore

$$\Big\|\int_\Gamma H\,dw\Big\| = \Big\|\int_{Q(1-\varepsilon)} H\,dw\Big\| \leqq \text{Const.}\, e^{-\varepsilon s}(1+s^{|\alpha|}) \tag{2.34}$$

for all $|\eta|>\delta$, $\eta\in(-\pi,\pi]$, and hence

$$I_1 \leqq \text{Const.}\, e^{-\varepsilon s}(1+s^{|\alpha|}) \leqq \text{Const.}\, s^{-h/2} \tag{2.35}$$

for all $0\leqq h\leqq d+1$.

Similarly

$$\Big\|\int_\Gamma H\,dw - \int_\gamma H\,dw\Big\| \leqq \Big\|\int_{Q(1-2\varrho)} H\,dw\Big\| \tag{2.36}$$

for $|\eta|\leqq\delta$ and hence $I_2\leqq\text{Const.}\, s^{-h/2}$ for all $0\leqq h\leqq d+1$.

In order to estimate I_3, assume that $s\geqq 1$, and make the substitution $\xi=\eta s^{1/2}$. Then

$$I_3 = s^{(|\alpha|-d)/2}\Big\|\int_{|\xi|<\delta s^{1/2}} \exp(-i\xi\cdot y s^{-1/2})\frac{\partial^\alpha}{\partial\xi^\alpha} J\,d\xi\Big\| \tag{2.37}$$

where

$$J = \exp\left(-\tfrac{1}{2}\xi\cdot\lambda''(\theta)\xi\right)\left(\exp(s\psi(\xi s^{-1/2}))E(\theta+i\xi s^{-1/2})-E(\theta)\right) \tag{2.38}$$

and $\psi(\eta)=\log\lambda(\theta+i\eta)-1-i\eta\cdot\lambda'(\theta)+\frac{1}{2}\eta\cdot\lambda''(\theta)\eta$. The functions

$$\exp(-s\psi(\xi s^{-1/2}))\frac{\partial^\beta}{\partial\xi^\beta}\exp(s\psi(\xi s^{-1/2})),\quad \beta\leqq\alpha, \tag{2.39}$$

are polynomials in the variables

$$s\frac{\partial^\gamma}{\partial\xi^\gamma}\psi(\xi s^{-1/2}) = s^{1-(1/2)|\gamma|}\psi^{(\gamma)}(\xi s^{-1/2}),\quad \gamma\leqq\beta, \tag{2.40}$$

and the latter expression is dominated by $\text{Const.}\,(1+|\xi|^3)s^{-1/2}$ for all γ. Furthermore any derivative of $E(\theta+i\xi s^{-1/2})$ is bounded for $|\xi|<\delta s^{1/2}$, and $|s\psi(\xi s^{-1/2})|<\frac{1}{3}\xi\cdot\lambda''(\theta)\xi$ for $|\xi|<\delta s^{1/2}$. The norm of I_3 is therefore dominated by

$$s^{(|\alpha|-d)/2}\,\text{Const.}\int s^{-1/2}p(\xi)\exp\left(-\tfrac{1}{6}\xi\cdot\lambda''(\theta)\xi\right)d\xi = O(s^{(|\alpha|-d-1)/2}) \tag{2.41}$$

where p is a polynomial.

It should be clear that $I_4=O(s^{-h})$ for any $h\geqq 0$.

When $s<1$ not only the difference but each term on the left in (2.22) is small. This is obviously true for Q_θ^s, and

$$P_\theta^s(t) \leqq \sum_{n=0}^\infty e^{-s}\frac{s^n}{n!}\sum_{\eta\cdot u\geqq\eta\cdot t} L_\theta^{n*}(u)e^{\eta\cdot u}e^{-\eta\cdot t} \leqq e^{-\eta\cdot t}\sum_{n=0}^\infty e^{-s}\frac{s^n}{n!}\Lambda(\theta+\eta)^n \tag{2.42}$$

for any $\eta\in\mathbf{R}^d$. But $\|A(\theta+\eta)^n\|\leqq\text{Const.}\,\lambda(\theta+\eta)^n$ and hence the norm of the sum above is dominated by

$$\text{(2.43)}\qquad \text{Const.}\exp\left[s(\lambda(\theta+\eta)-1-\eta\cdot\lambda'(\theta))-\eta\cdot(t-s\lambda'(\theta))\right].$$

The choice $\eta=\delta(t-s\lambda'(\theta))/|(t-s\lambda'(\theta))|$ shows that this expression is dominated by $\text{Const.}\exp\left[-\delta|(t-s\lambda'(\theta))|\right]$ for δ sufficiently small. ∎

We shall now remove the condition that $F=\delta$ in theorems 2.1 and 2.2.

Proposition 2.7. *Theorems* 2.1 *and* 2.2 *imply that theorems* 1.5 *and* 1.6 *hold under the extra assumption that* $A(\theta)$ *is aperiodic.*

Proof of proposition 2.7. We shall consider the case when $\lambda'(\theta)\neq 0$ on Δ. The other is similar but easier and will be omitted.

Define for $u\in\mathbf{R}^d$,

$$\text{(2.44)}\qquad T(u)=\frac{u\cdot\lambda''(\theta)^{-1}\lambda'(\theta)}{\lambda'(\theta)\cdot\lambda''(\theta)^{-1}\lambda'(\theta)},\quad \hat{u}=T(u)\lambda'(\theta),\quad \check{u}=u-\hat{u}.$$

Then $T(u)=T(\hat{u})$, and $u\cdot\Sigma u=\check{u}\cdot\lambda''(\theta)^{-1}\check{u}$. Let

$$S_1=\left\{s;\ |\hat{s}|\leqq\tfrac{1}{2}|\hat{t}|\right\},\quad S_2=\left\{s;\ |\hat{s}|>\tfrac{1}{2}|\hat{t}|\right\},$$

and write $F_\theta(t)=e^{\theta\cdot t}F(t)$, and

$$\text{(2.45)}\qquad R_\theta * F_\theta(t)=\textstyle\sum_s R_\theta(t-s)F_\theta(s)=\sum_{s\in S_1}+\sum_{s\in S_2}.$$

Assume that t and $\lambda'(\theta)$ have the same direction and that $s\in S_1$. Put

$$x=(\check{t}-\check{s})\cdot\lambda''(\theta)^{-1}(\check{t}-\check{s})/(2T(t-s)),$$

then

$$\check{t}=0,\quad T(t)=|t|/|\lambda'(\theta)|,\quad T(t-s)=T(t)(1-|\hat{s}|/|\hat{t}|),$$

and $0\leqq x\leqq\text{Const.}\,|\check{s}|^2/T(t)$. Here and below Const. depends on the compact $K\subset\Delta\cap\Theta_F$.

The inequality $0\leqq 1-e^{-x}\leqq x$ and Theorem 2.1 now yield

$$\begin{aligned}&\left\|\textstyle\sum_{S_1}-(2\pi|t|/|\lambda'(\theta)|)^{-(d-1)/2}C(\theta)^{-1/2}E(\theta)\sum_{S_1}F_\theta(s))\right\|\\ \text{(2.46)}\qquad&\leqq\text{Const.}\,|t|^{-(d-1)/2}\textstyle\sum(|\hat{s}|/|t|+|\check{s}|^2/|t|+|t|^{-1/2})\,\|F_\theta(s)\|\\ &\leqq\text{Const.}\,|t|^{-d/2}\textstyle\sum(1+|s|^2)\,\|F_\theta(s)\|.\end{aligned}$$

In order to show that this expression equals $o(|t|^{-(d-1)/2})$ uniformly on the compact $K\subset\Delta\cap\Theta_F$ it suffices to show that the last sum is uniformly bounded on K. It is a consequence of Jensen's inequality, $\exp\left(\sum_i\alpha_i\theta_i\cdot t\right)\leqq\sum_i\alpha_i\exp(\theta_i\cdot t)$, $\alpha_i\geqq 0$, $\sum_i\alpha_i=1$, that Θ_F is a convex set. Choose a compact, convex simplex $K_F\supset K$ in Θ_F. Let $\theta_1,\ldots,\theta_r$ be the corners of K_F, and let $\theta=\sum_i\alpha_i\theta_i$. Then by Jensen's inequality

$$\sum_s (1+|s|^2)\,\|F_\theta(s)\| \leqq \sum_i \alpha_i \sum_s (1+|s|^2)\,\|F_{\theta_i}(s)\| \leqq \max_i \sum_s (1+|s|^2)\,\|F_{\theta_i}(s)\| < \infty, \tag{2.47}$$

from which the uniformity follows.

Another consequence of theorem 2.1 is that $R_\theta(t)$ is bounded. Therefore

$$\left\|\sum_{S_2}\right\| \leqq \text{Const.} \sum_{S_2} \|F_\theta(s)\| \leqq \text{Const.}\,(|\hat{t}|)^{-(d-1)/2} \sum_{S_2} |s|^{(d-1)/2} \|F_\theta(s)\| = o(|t|^{-(d-1)/2}). \tag{2.48}$$

The uniformity follows in the same way as above. ∎

Proposition 2.8. *If theorems* 1.5 *and* 1.6 *hold under the extra assumption that* $A(\theta)$ *is aperiodic, then they hold as they stand.*

Proof of proposition 2.8. Let p be the period. Then there are matrices $L_1(t), \ldots, L_p(t)$ and a permutation matrix P such that

$$L(t) = P \begin{pmatrix} 0 & L_1(t) & 0 & . & 0 \\ . & 0 & L_2(t) & . & . \\ . & . & . & . & 0 \\ 0 & . & . & 0 & L_{p-1}(t) \\ L_p(t) & 0 & . & . & 0 \end{pmatrix} P^{-1}. \tag{2.49}$$

Here the zeros on the diagonal are square matrices. Assume without loss of generality that P is the identity, and put $\bar{L}=L^{p*}$. Then

$$\bar{L}(t) = \begin{pmatrix} \bar{L}_1(t) & 0 & . & 0 \\ 0 & . & . & . \\ . & . & . & 0 \\ 0 & . & 0 & \bar{L}_p(t) \end{pmatrix} \tag{2.50}$$

where

$$\begin{aligned} \bar{L}_1 &= L_1 * L_2 * \ldots * L_{p-1} * L_p \\ \bar{L}_2 &= L_2 * L_3 * \ldots * L_p * L_1 \\ &\vdots \\ \bar{L}_p &= L_p * L_1 * \ldots * L_{p-2} * L_{p-1}. \end{aligned} \tag{2.51}$$

Here the matrices $\bar{A}_k(\theta)=\sum_t e^{\theta\cdot t}\bar{L}_k(t)$ are irreducible and aperiodic square matrices for $1\leqq k\leqq p$. Also $R*F=\bar{R}*\bar{F}$, where $\bar{R}=\sum_{n=0}^{\infty}\bar{L}^{n*}$ and $\bar{F}=(I+L+\ldots+L^{(p-1)*})*F$.

In order to apply the theorems to each one of the p parts $\bar{R}_k(t)=\sum_{n=0}^{\infty}\bar{L}_k^{n*}(t)$, $1\leqq k\leqq p$, of $\bar{R}$, we must check that the groups $\bar{G}_k$ equal G, defined in Section 1 (see 1.7). Here $\bar{G}_k$ is defined as G but with L replaced by $\bar{L}_k$.

Lemma 2.9. *The groups* $\bar{G}_k$ *satisfy* $\bar{G}_k=G$ *for* $k=1, \ldots, p$.

Proof. Let X_k denote the set of indices corresponding to $\bar{L}_k$, and put for $i, j \in X_k$, $\bar{S}^k_{ij} = \bigcup_{n=1}^{\infty} \{t;\ \bar{L}^{n*}_{ij}(t) > 0\}$. Then $\bar{S}^k_{ii} = S_{ii}$, and hence $G^k_{ii} = G_{ii} = G$. ▮

We must also relate the maximal positive eigenvalue of $\bar{A}(\theta)$ to $\lambda(\theta)$.

Lemma 2.10. *The maximal positive eigenvalue of* $\bar{A}_k(\theta)$ *equals* $\lambda(\theta)^p$. *Write* $\bar{l}^k(\theta)$ *and* $\bar{r}^k(\theta)$ *for the corresponding left respectively right eigenvectors. Then there are constants* c^*, *and* c *such that* $e^*(\theta) = c^*(\bar{l}^1(\theta), \ldots, \bar{l}^p(\theta))$, *and* $e(\theta) = c(\bar{r}^1(\theta), \ldots$ $\ldots, \bar{r}^p(\theta))$.

Proof. We shall omitt the θ when convienient, and write $\bar{\lambda}_k$ for the maximal positive eigenvalue of $\bar{A}_k$. Let $A_k = \sum_t e^{\theta \cdot t} L_k(t)$, and write $e^*(\theta) = (l_1, \ldots, l_p)$, $e(\theta) = (r_1, \ldots, r_p)$ where the subvectors l_k and r_k have the same dimension as $\bar{l}_k$.

The spectrum of $\bar{A}_k$ is contained in the spectrum of A^p and hence $\bar{\lambda}_k \leqq \lambda^p$ for each $1 \leqq k \leqq p$.

We have

(2.52) $$l_1 A_1 = \lambda l_2,\ l_2 A_2 = \lambda l_3,\ \ldots,\ l_p A_p = \lambda l_1,$$

and hence

(2.53) $$l_k \bar{A}_k = \lambda^p l_k, \quad k = 1, \ldots, p.$$

But $l_k \neq 0$ and hence λ^p is an eigenvalue of $\bar{A}_k$ for each $k = 1, \ldots, p$, and hence $\bar{\lambda}_k \geqq \lambda^p$ for each $1 \leqq k \leqq p$. The remainder of the lemma follows from the fact that e^* and e are unique up to multiplicative constants. ▮

Another consequence of (2.52) is that if $\theta \in \Delta$, then

(2.54) $$\bar{E}(1 + A + \ldots + A^{p-1}) = pE.$$

Here

(2.55) $$\bar{E}(t) = \begin{pmatrix} \bar{E}_1(t) & 0 & . & 0 \\ 0 & . & . & . \\ . & . & . & 0 \\ 0 & . & 0 & \bar{E}_p(t) \end{pmatrix},$$

where $\bar{E}_k$ is the eigenprojection of $\bar{A}_k$ corresponding to the eigenvalue λ^p. Therefore

(2.56) $$\bar{E}(\theta) \textstyle\sum_s e^{\theta \cdot s} \bar{F}(s) = pE(\theta) \sum_s e^{\theta \cdot s} F(s)$$

for $\theta \in \Delta$.

Proposition 2.8 therefore follows from the following lemma.

Lemma 2.11. *If* $\lambda'(\theta) \neq 0$ *on* Δ, *then*

(2.57) $$\bar{\lambda}'(\theta) = p\lambda'(\theta), \quad \text{and} \quad \bar{C}(\theta) = p^{d+1} C(\theta).$$

for $\theta \in \Delta$.

If $\lambda'(\theta)=0$, *and* $\theta\in\Delta$, *then*

(2.58) $$\bar{\lambda}''(\theta)^{-1} = \lambda''(\theta)^{-1}/p, \quad \text{and} \quad \det\bar{\lambda}''(\theta) = p^d \det\lambda''(\theta).$$

Proof. The second statement is obvious.

Consider the first. After an orthogonal transformation we may assume that $\bar{\lambda}'(\theta)=(\bar{a}, 0, \ldots, 0)$, $\lambda'(\theta)=(a, 0, \ldots, 0)$. Write

$$\bar{\lambda}''(\theta) = \begin{pmatrix} \bar{b} & \bar{c} \\ \bar{c}^T & \bar{D} \end{pmatrix}, \quad \lambda''(\theta) = \begin{pmatrix} b & c \\ c^T & D \end{pmatrix}.$$

Here D and $\bar{D}$ are $d-1$ by $d-1$ matrices.

We have $\bar{\lambda}'(\theta)=p\lambda'(\theta)$, and $\bar{\lambda}''(\theta)=p(p-1)\lambda'(\theta)^2+p\lambda''(\theta)$ when $\theta\in\Delta$, and hence $\bar{a}=pa$, $\bar{D}=pD$. The upper left corner of $\bar{\lambda}''(\theta)^{-1}$ equals

$$\frac{\det\bar{D}}{\det\bar{\lambda}''(\theta)},$$

and hence

$$\bar{C} = \bar{a}^2 \det\bar{D} = p^{d+1} a^2 \det D. \quad \blacksquare$$

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