# *BMO* estimates for lacunary series

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**Abstract.** We prove *BMO* and $L^p$ norm inequalities in $\mathbf{R}^n$ for lacunary Walsh and generalized trigonometric series.

It is known for generalized lacunary trigonometric series $f(x)=\sum_{k=-\infty}^{k=\infty} c_k e^{ir_k x}$ with $\sum_k |c_k|^2<\infty$, $r_k$ real, $r_{-k}=-r_k$, $r_{k+1}/r_k \geqq q>1$, $k=1, 2, \ldots$, that we have constants $A(p, q)$ and $B(p, q)$ so that for any interval $I\subset \mathbf{R}$ with $|I|\geqq 4\pi/(r_1 \min(q-1, 1))$,

$$A(p, q)\left(\sum |c_k|^2\right)^{1/2} \leqq \left(\frac{1}{|I|}\int_I |f(x)|^p\,dx\right)^{1/p} \leqq B(p, q)\left(\sum |c_k|^2\right)^{1/2}$$

for $0<p<\infty$. A similar result holds for lacunary Walsh series. These $L^p$-norm inequalities can be obtained from the results of [3] by a simple change of variable.

For $p=\infty$ it is well-known that the right-hand side inequality fails. In this paper we show first that we have a norm inequality in the trigonometric case for *BMO* and in the Walsh case for *BMOd*, dyadic *BMO*, over $\mathbf{R}$. Then we turn our attention to generalize the $L^p$-norm inequalities to $\mathbf{R}^n$. Finally, as an application, we prove *BMO* norm estimate for lacunary trigonometric series and *BMOd* norm estimate for Walsh series in $\mathbf{R}^n$.

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*BMO* and *BMOd* on $\mathbf{R}^1$ are defined as follows:

Let $I_{j,l}=[(j-1)2^{-l}, j2^{-l})$, for $j=\ldots-2, -1, 0, 1, 2, \ldots$ and $l=0, 1, 2, \ldots$.

For any interval $I$, let $f_I=\frac{1}{|I|}\int_I f(y)\,dy$, and define

$$f^\#(x)=\sup_{\{I\mid x\in I\}} f_I^\# = \sup_{\{I\mid x\in I\}}\left(\frac{1}{|I|}\int_I |f(y)-f_I|^2\,dy\right)^{1/2}$$

and

$$f_d^\#(x)=\sup_{\{I_{j,l}\mid x\in I_{j,l}\}} f_{I_{j,l}}^\# = \sup_{I_{j,l}}\left(\frac{1}{|I_{j,l}|}\int_{I_{j,l}} |f(y)-f_{I_{j,l}}|^2\,dy\right)^{1/2}.$$

One then defines

$$\|f\|_{BMO} = \|f^{\#}(x)\|_{\infty},$$

$$\|f\|_{BMOd} = \|f_d^{\#}(x)\|_{\infty}.$$

Clearly, $\|f\|_{BMOd} \leqq \|f\|_{BMO}$. For more on these spaces, see e.g. [2].

Our first theorem concerns the norm inequality for the Walsh functions.

The Rademacher functions are defined as: $r_0(t)=1$ for $0 \leqq t < 1/2$; $r_0(t)=-1$ for $1/2 \leqq t < 1$; $r_0(t)=r_0(t+1)$; and $r_k(t)=r_0(2^k t)$.

The Walsh functions are then defined by $w_0(t)=1$; $w_n(t)=r_{a_1}(t) \dots r_{a_s}(t)$, where $n=2^{a_1}+2^{a_2}+\dots+2^{a_s}$, $a_1 > a_2 > \dots > a_s \geqq 0$.

**Theorem 1.** *Given a lacunary sequence $\{n_k\}$ of natural numbers with $n_1 \geqq 1$, $n_{k+1}/n_k \geqq q > 1$ and a sequence $\{c_k\}$ of complex numbers with $\sum_k |c_k|^2 < \infty$, there exist constants $A_1(q)$ and $A_2(q)$ such that for any $f(x)=c_0+\sum_{k=1}^{\infty} c_k w_{n_k}(x)$ we have:*

$$A_1(q)\left(\sum_{k\neq 0} |c_k|^2\right)^{1/2} \leqq \|f\|_{BMOd} \leqq A_2(q)\left(\sum_{k\neq 0} |c_k|^2\right)^{1/2}.$$

*Proof.* Assume $c_0=0$. The left-hand side inequality follows from the inequality:

$$\|f\|_{BMOd} \geqq \left(\int_0^1 |f(y)|^2\, dy\right)^{1/2}$$

and Bessel's inequality.

For the right-hand side inequality, we first assume that $q \geqq 2$, since in this case the $w_{n_k}$ with $n_k \geqq 2^l$ are orthogonal on $I_{j,l}$. For $n_k < 2^l$, the $w_{n_k}$ are constant on each $I_{j,l}$; we denote these constants by $w_{n_k}(I_{j,l})$. We have:

$$\begin{aligned} f_{I_{j,l}} &= 2^l \int_{I_{j,l}} \sum_{k=1}^{\infty} c_k w_{n_k}(t)\, dt \\ &= 2^l \sum_{k=1}^{\infty} c_k \int_{I_{j,l}} w_{n_k}(t)\, dt \\ &= \sum_{\{k \mid n_k < 2^l\}} c_k w_{n_k}(I_{j,l}). \end{aligned}$$

Next we calculate $f^{\#}_{I_{j,l}}$.

$$\begin{aligned} f^{\#}_{I_{j,l}} &= \left(2^l \int_{I_{j,l}} \left|\sum c_k w_{n_k}(t) - \sum_{\{k \mid n_k < 2^l\}} c_k w_{n_k}(I_{j,l})\right|^2 dt\right)^{1/2} \\ &= \left(2^l \int_{I_{j,l}} \left|\sum_{\{k \mid n_k \geqq 2^l\}} c_k w_{n_k}(t)\right|^2 dt\right)^{1/2} \\ &= \left(\sum_{\{k \mid n_k \geqq 2^l\}} |c_k|^2\right)^{1/2}. \end{aligned}$$

From this calculation we obtain

$$\|f\|_{BMOd} = \left(\sum_{k\neq 0} |c_k|^2\right)^{1/2}.$$

If we have lacunarity with $1 < q < 2$, we can find $m$ so that we have $q^m \geqq 2$. Then we may break $f(t)=\sum_k c_k w_{n_k}(t)$ into $m$ series $f_1, \dots, f_m$ with ratio of lacunarity greater than 2 and use the triangle inequality.

This theorem cannot be extended to *BMO* as the following example shows:

*Example.* Let $J_l=[1/2-2^{-l}, 1/2+2^{-l})$ for $l=2,3,4,\dots$. Then for $1\leqq k<l$ we have $r_k(t)=1$ in $[1/2, 1/2+2^{-l})$ and $r_k(t)=-1$ in $[1/2-2^{-l}, 1/2)$. Let $f(x)=\sum_{k=1}^{\infty}\frac{1}{k}r_k(x)$.

We have $f_{J_l}=0$ for each $l$ and

$$\left(\frac{1}{|J_l|}\int_{J_l}|f(y)-f_{J_l}|^2\right)^{1/2}=\left(2^{l-1}\int_{1/2-2^{-l}}^{1/2+2^{-l}}\left|\sum_{k=1}^{\infty}\frac{1}{k}r_k(t)\right|^2dt\right)^{1/2}$$

$$\geqq 2^{(l-1)/2}\left(\left(\int_{J_l}\left|\sum_{k=1}^{l-1}\frac{1}{k}r_k(t)\right|^2dt\right)^{1/2}-\left(\int_{J_l}\left|\sum_{k=l}^{\infty}\frac{1}{k}r_k(t)\right|^2dt\right)^{1/2}\right)$$

$$=\sum_{k=1}^{l-1}\frac{1}{k}-\left(\sum_{k=l}^{\infty}\frac{1}{k^2}\right)^{1/2}.$$

Therefore we have $\|f\|_{BMO}=\infty$ although $\sum|c_k|^2<\infty$.

For generalized trigonometric series a stronger result holds:

**Theorem 2.** *Let* $f(x)=\sum_{k=-\infty}^{\infty}c_ke^{ir_kx}$ *be defined on* $\mathbf{R}$ *with* $r_k$ *real,* $r_{k+1}/r_k\geqq q>1$, *and* $r_{-k}=-r_k$, *for* $k=1,2,\dots$, *and* $\sum_k|c_k|^2<\infty$.

*Then there exist* $A_1(q)$ *and* $A_2(q)$ *so that*

$$A_1(q)\left(\sum_{k\neq 0}|c_k|^2\right)^{1/2}\leqq\|f\|_{BMO}\leqq A_2(q)\left(\sum_{k\neq 0}|c_k|^2\right)^{1/2}.$$

*Proof.* Assume $c_0=0$. The $L^p$-norm inequality implies that $\sum_{-\infty}^{\infty}c_ke^{ir_kx}$ converges locally in $L_p$ norm for all $0<p<\infty$, and that there exists $A_1(q)$ so that for any interval $I\subset\mathbf{R}$ with $|I|=4\pi/d$, where $0<d\leqq r_1\min(q-1,1)$,

$$A_1(q)\left(\sum_{k\neq 0}|c_k|^2\right)^{1/2}\leqq\left(\frac{1}{|I|}\int_I|f(y)|^2dy\right)^{1/2}\leqq\|f\|_{BMO}+|f_I|.$$

To prove $A_1(q)\left(\sum_{k\neq 0}|c_k|^2\right)^{1/2}\leqq\|f\|_{BMO}$, it therefore suffices to show that if $c_0=0$, then $\lim_{d\to 0}|f_I|=0$. In fact, we prove more: under the hypotheses of the theorem,

$$c_k=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(x)e^{-ir_kx}\,dx,\quad\text{for all}\quad k.$$

For fixed $k$, $k=\dots-1,0,1,\dots$, and $n\geqq|k|$,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(\sum_{j=-n}^{n}c_je^{ir_jx}\right)\cdot e^{-ir_kx}\,dx=c_k.$$

We have:

$$\left|\frac{1}{2T}\int_{-T}^{T} f(x)e^{-ir_k x}\,dx - c_k\right| \leqq \left|\frac{1}{2T}\int_{-T}^{T}\Big(\sum_{|j|>n} c_j e^{ir_j x}\Big)e^{-ir_k x}\,dx\right|$$

$$+\left|\frac{1}{2T}\int_{-T}^{T}\Big(\sum_{j=-n}^{n} c_j e^{ir_j x}\Big)e^{-ir_k x}\,dx - c_k\right|,$$

and

$$\left|\frac{1}{2T}\int_{-T}^{T}\Big(\sum_{|j|>n} c_j e^{ir_j x}\Big)\cdot e^{-ir_k x}\,dx\right| \leqq \frac{1}{2T}\int_{-T}^{T}\Big|\sum_{|j|>n} c_j e^{ir_j x}\Big|\,dx$$

$$\leqq \left(\frac{1}{2T}\int_{-T}^{T}\Big|\Big(\sum_{|j|>n} c_j e^{ir_j x}\Big)\Big|^2 dx\right)^{1/2}$$

$$\leqq B(q)\Big(\sum_{|j|>n} |c_j|^2\Big)^{1/2} < \varepsilon,$$

if $n$ is sufficiently large. The estimate is uniform in $T>1$. Let $T\to\infty$ and get:

$$c_k = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x)e^{-ir_k x}\,dx.$$

To prove $\|f\|_{BMO} \leqq A_2(q)\big(\sum_{k\neq 0}|c_k|^2\big)^{1/2}$, it suffices to show that for any interval $I$ there exists a constant $c_I$ so that

$$\left(\frac{1}{|I|}\int_I |f(y)-c_I|^2\,dy\right)^{1/2} \leqq A_2(q)\Big(\sum_{k\neq 0}|c_k|^2\Big)^{1/2}.$$

We may assume, by Minkowski's inequality, that $f(x)=\sum_{k=1}^{\infty} c_k e^{ir_k x}$. Let $I=[a,b]$, and $c_I=\sum_{k=1}^{m-1} c_k e^{ir_k a}$ where we take

$$m = \min\{k:\ r_k(b-a) \geqq 4\pi/\min(q-1,1)\}.$$

We have

$$\left(\frac{1}{|I|}\int_I |f(y)-c_I|^2\,dy\right)^{1/2} \leqq \left(\frac{1}{b-a}\int_a^b \Big|\sum_{k=1}^{m-1} c_k\big(e^{ir_k x}-e^{ir_k a}\big)\Big|^2 dx\right)^{1/2}$$

$$+\left(\frac{1}{b-a}\int_a^b \Big|\sum_{k=m}^{\infty} c_k e^{ir_k x}\Big|^2 dx\right)^{1/2}$$

$$= J_1 + J_2.$$

Using the Schwarz inequality, we have

$$J_1 \leqq \left(\sum_{k=1}^{m-1} |c_k|^2\right)^{1/2} \left(\frac{1}{b-a}\int_a^b \sum_{k=1}^{m-1} |e^{ir_kx}-e^{ir_ka}|^2\,dx\right)^{1/2}$$

$$\leqq \left(\sum_{k=1}^{\infty} |c_k|^2\right)^{1/2}\left(\sum_{k=1}^{m-1} r_k^2\right)^{1/2}(b-a)$$

$$\leqq \left(\sum_{k=1}^{\infty} |c_k|^2\right)^{1/2}\left(\sum_{k=1}^{m-1} r_{m-1}^2 q^{2(k-m+1)}\right)^{1/2}(b-a)$$

$$\leqq r_{m-1}(b-a)\left(\sum_{k=1}^{\infty} |c_k|^2\right)^{1/2}$$

$$\leqq \frac{4\pi}{\min(q-1,\,1)}\left(\sum_{k=1}^{\infty} |c_k|^2\right)^{1/2}.$$

For $J_2$, since $r_m \min(q-1,\,1) \geqq 4\pi/(b-a)$, we have, by the $L^p$ norm inequality,

$$J_2 \leqq B(q)\left(\sum_{k\neq 0} |c_k|^2\right)^{1/2}.$$

We next generalize the $L^p$ norm estimates in [3] to $\mathbf{R}^n$. We present the results for $\mathbf{R}^2$ only since $\mathbf{R}^n$ follows similarly. We consider lacunary Walsh series first.

**Theorem 3.** *Given* $0<p<\infty$, $q_1, q_2>1$, *there exist constants* $A(p, q_1, q_2)$ *and* $B(p, q_1, q_2)$ *so that for any* $f(x, t)=\sum_{k,l} c_{k,l} w_{n_k}(x) w_{m_l}(t)$ *with* $\sum_{k,l} |c_{k,l}|^2<\infty$, $n_0, m_0=0$, $n_{k+1}/n_k \geqq q_1>1$, $m_{k+1}/m_k \geqq q_2>1$, $k, l=1, 2, \ldots$ *we have*

$$A(p, q_1, q_2)\left(\sum_{k,l} |c_{k,l}|^2\right)^{1/2} \leqq \left(\int_0^1\int_0^1 |f(x, t)|^p\,dx\,dt\right)^{1/p} \leqq B(p, q_1, q_2)\left(\sum_{k,l} |c_{k,l}|^2\right)^{1/2}.$$

*Proof.* For $p=2$ the theorem holds by orthogonality. For $p\geqq 2$, the left-hand side follows immediately from Hölder's inequality. As to the right-hand side inequality, we have:

$$\left(\int_0^1\int_0^1 \left|\sum_{k,l} c_{k,l} w_{n_k}(x) w_{m_l}(t)\right|^p dx\,dt\right)^{1/p} \leqq B(p, q_1)\left(\int_0^1 \left(\sum_k \left|\sum_l c_{k,l} w_{m_l}(t)\right|^2\right)^{p/2} dt\right)^{1/p}$$

$$\leqq B(p, q_1)\left(\sum_k \left(\int_0^1 \left|\sum_l c_{k,l} w_{m_l}(t)\right|^p dt\right)^{2/p}\right)^{1/2}$$

$$\leqq B(p, q_1, q_2)\left(\sum_{k,l} |c_{k,l}|^2\right)^{1/2}.$$

For $0<p<2$, the right-hand side inequality follows from Hölder's inequality and the result for $p\geqq 2$. To prove the left-hand side inequality, we write

$$\frac{1}{2}=\frac{(1-\theta)}{p}+\frac{\theta}{4}$$

for $0<\theta<1$, and we have, as in the one-dimensional case,

$$\left(\sum_{k,l}|c_{k,l}|^2\right)^{1/2} = \left(\int_0^1\int_0^1 |f(x,t)|^2\,dx\,dt\right)^{1/2}$$

$$\leqq \left(\int_0^1\int_0^1 |f(x,t)|^p\,dx\,dt\right)^{(1-\theta)/p}\left(\int_0^1\int_0^1 |f(x,t)|^4\,dx\,dt\right)^{\theta/4}$$

$$\leqq B^\theta(4,q_1,q_2)\left(\int_0^1\int_0^1 |f(x,t)|^p\,dx\,dt\right)^{(1-\theta)/p}\left(\sum_{k,l}|c_{k,l}|^2\right)^{\theta/2}.$$

Therefore,

$$A(p,q_1,q_2)\left(\sum_{k,l}|c_{k,l}|^2\right)^{1/2} \leqq \left(\int_0^1\int_0^1 |f(x,t)|^p\,dx\,dt\right)^{1/p}.$$

We also prove a local version of theorem 3:

**Theorem 4.** *Given* $0<p<\infty$, $q_1, q_2>1$, *there exist constants* $A(p,q_1,q_2)$ *and* $B(p,q_1,q_2)$ *so that for measurable* $E\subset[0,1]^2$ *with positive measure, there exist* $N_1=N_1(E,q_1)$ *and* $N_2=N_2(E,q_2)$ *so that for any* $f(x,t)=\sum_{k,l}c_{k,l}w_{n_k}(x)w_{m_l}(t)$ *with* $\sum_{k,l}|c_{k,l}|^2<\infty$, $n_0, m_0=0$, $n_1\geqq N_1$, $m_1\geqq N_2$ *and* $n_{k+1}/n_k\geqq q_1>1$, $m_{l+1}/m_l\geqq q_2>1$, *for* $k, l=1,2,\dots$ *we have:*

$$A(p,q_1,q_2)\left(\sum_{k,l}|c_{k,l}|^2\right)^{1/2} \leqq \left(\frac{1}{|E|}\iint_E |f(x,t)|^p\,dx\,dt\right)^{1/p}$$

$$\leqq B(p,q_1,q_2)\left(\sum_{k,l}|c_{k,l}|^2\right)^{1/2}.$$

*Proof.* We prove first the inequalities for $p=2$. Assume that $f(x,t)$ is a finite sum of terms of the form $c_{k,l}w_{n_k}(x)w_{m_l}(t)$. Let $E$ be a measurable set in $[0,1]^2$. We then have:

$$\iint_E |f(x,t)|^2\,dx\,dt$$

$$= \left(\sum_{k,l}|c_{k,l}|^2\right)|E| + \sum_{k\neq i;\,l\neq j} c_{k,l}\bar{c}_{i,j}\iint_E w_{n_k}(x)w_{n_i}(x)w_{m_l}(t)w_{m_j}(t)\,dx\,dt.$$

The second term does not exceed, in absolute value,

$$\left(\sum_{k\neq i;\,l\neq j}\left(\iint_E w_{n_k}(x)w_{n_i}(x)w_{m_l}(t)w_{m_j}(t)\,dx\,dt\right)^2\right)^{1/2}\cdot\sum_{k,l}|c_{k,l}|^2.$$

As in the proof of the one-dimensional case (see [3]), we know that if $n_1$ and $m_1$ are large enough, then the coefficient of $\sum_{k,l}|c_{k,l}|^2$ in the above expression can be made as small as we wish. Thus we have the theorem for $p=2$.

Next we prove the right-hand side inequality for $p\geqq 2$, First suppose $E=E_1\times E_2$. Then using the 1-dimensional case, we know there exist $N_1=N_1(p, q_1)$ and $N_2=N_2(p, q_2)$ such that

$$\left(\frac{1}{|E|}\left(\iint_{E_1\times E_2}\left|\sum_{k,l} c_{k,l} w_{n_k}(x) w_{m_l}(t)\right|^p dx\,dt\right)\right)^{1/p}$$

$$=\left(\frac{1}{|E_2|}\int_{E_2}\left(\frac{1}{|E_1|}\int_{E_1}\left|\sum_{k,l} c_{k,l} w_{n_k}(x) w_{m_l}(t)\right|^p dx\right) dt\right)^{1/p}$$

$$\leqq B(p, q_1)\left(\frac{1}{|E_2|}\int_{E_2}\left(\sum_k\left|\sum_l c_{k,l} w_{m_l}(t)\right|^2\right)^{p/2} dt\right)^{1/p}$$

$$\leqq B(p, q_1)\left(\sum_k\left(\frac{1}{|E_2|}\int_{E_2}\left|\sum_l c_{k,l} w_{m_l}(t)\right|^p dt\right)^{2/p}\right)^{1/2}$$

$$\leqq B(p, q_1, q_2)\left(\sum_{k\,l}|c_{k,l}|^2\right)^{1/2}$$

whenever $n_1>N_1$ and $m_1>N_2$.

Next, suppose $E=\bigcup_{\text{finite}} E_i$, where each $E_i$ is of the form $E_i=E_{i,1}\times E_{i,2}$, and the $E_i$ are disjoint. For each $E_i$, there exist $N_{i,1}$ and $N_{i,2}$ so that if $n_1>N_{i,1}$ and $m_1>N_{i,2}$,

$$\left(\frac{1}{|E_i|}\iint_{E_i}\left|\sum_{k,l} c_{k,l} w_{n_k}(x) w_{m_l}(t)\right|^p dx\,dt\right)^{1/p} \leqq B(p, q_1, q_2)\left(\sum_{k,l}|c_{k,l}|^2\right)^{1/2}.$$

Let $N_j=\max_i N_{i,j}$ for $j=1, 2$. Then when $n_1>N_1$ and $m_1>N_2$, we have

$$\left(\frac{1}{|E|}\iint_E\left|\sum_{k,l} c_{k,l} w_{n_k}(x) w_{m_l}(t)\right|^p dx\,dt\right)^{1/p}$$

$$=\left(\sum_i \frac{|E_i|}{|E|}\frac{1}{|E_i|}\iint_{E_i}\left|\sum_{k,l} c_{k,l} w_{n_k}(x) w_{m_l}(t)\right|^p dx\,dt\right)^{1/p}$$

$$\leqq \sum_i \frac{|E_i|}{|E|}\left(\frac{1}{|E_i|}\iint_{E_i}\left|\sum_{k,l} c_{k,l} w_{n_k}(x) w_{m_l}(t)\right|^p dx\,dt\right)^{1/p}$$

$$\leqq \sum_i \frac{|E_i|}{|E|} B(p, q_1, q_2)\left(\sum_{k,l}|c_{k,l}|\right)^{1/2}$$

$$= B(p, q_1, q_2)\left(\sum |c_{k,l}|^2\right)^{1/2}.$$

Next suppose $E=\bigcup_{i=1}^{\infty} E_i$, where $E_i$ are of the previous type and the $E_i$ are disjoint. We may write $E=\tilde{E}_1\cup\tilde{E}_2$, $\tilde{E}_1=\bigcup_{\text{finite}} E_i$, $\tilde{E}_1$ and $\tilde{E}_2$ are disjoint and $|\tilde{E}_2|^{1/2}\leqq|E|$. Let $N_j(E)=N_j(\tilde{E}_1)$, $j=0, 1$. Then for $f(x, t)=\sum_{k,l} c_{k,l} w_{n_k}(x) w_{m_l}(t)$

with $n_1 \geqq N_1(E)$ and $m_1 \geqq N_2(E)$,

$$\left(\frac{1}{|E|}\iint_E |f(x,t)|^p\,dx\,dt\right)^{1/p}$$

$$\leqq \left(\frac{1}{|\tilde{E}_1|}\iint_{\tilde{E}_1} |f(x,t)|^p\,dx\,dt\right)^{1/p} + \left(\frac{1}{|\tilde{E}_2|^{1/2}}\iint_{\tilde{E}_2} |f(x,t)|^p\,dx\,dt\right)^{1/p}.$$

For the second term on the right, we have

$$\frac{1}{|\tilde{E}_2|}\iint_{\tilde{E}_2} |f(x,t)|^p\,dx\,dt \leqq \left(\iint_{\tilde{E}_2} |f(x,t)|^{2p}\,dx\,dt\right)^{1/2}$$

$$\leqq \left(\int_0^1\int_0^1 |f(x,t)|^{2p}\,dx\,dt\right)^{1/2} \leqq B(2p, q_1, q_2)\left(\sum_{k,l} |c_{k,l}|^2\right)^{p/2}.$$

Therefore, combining constants,

$$\left(\frac{1}{|E|}\iint_E |f(x,t)|^p\,dx\,dt\right)^{1/p} \leqq B(p, q_1, q_2)\left(\sum_{k,l} |c_{k,l}|^2\right)^{1/2}.$$

Finally we consider a general measurable set $E$ in the unit cube in $\mathbf{R}^2$. Define $I_{k,l}=[(k-1)2^{-l}, k2^{-l}]$, $k=1, 2, \ldots, 2^l$. Define $J_{k,n,l}=I_{k,l}\times I_{n,l}$, $k, n=1, 2, \ldots, 2^l$. We want to decompose $E$. We start with $l=0$. $J_{1,1,0}=[0,1]^2$. If $|E|=|E\cap J_{1,1,0}|\geqq \frac{1}{2}|J_{1,1,0}|$ then we keep $J_{1,1,0}$ and the process stops. Otherwise, we divide $[0,1]^2$ dyadically into 4 cubes $J_{1,1,1}$, $J_{2,1,1}$, $J_{1,2,1}$ and $J_{2,2,1}$. If $|E\cap J_{k,n,1}|\geqq\frac{1}{2}|J_{k,n,1}|$ for some $k$ or $n$, then we keep that one and ignore all subsequent subdivisions of it. We subdivide the remaining cubes, and repeat the process.

In this way we obtain a sequence of disjoint intervals $J_{k_i,n_i,l_i}$. Let $F=\bigcup_i J_{k_i,n_i,l_i}$. Clearly, $|F|\leqq 2|E|$. Moreover, $F$ contains all points of density of $E$, so that $\chi_E\leqq\chi_F$ a.e. Since the theorem holds for $F$,

$$\left(\frac{1}{|E|}\iint_E |f(x,t)|^p\,dx\,dt\right)^{1/p}$$

$$\leqq \left(\frac{2}{|F|}\iint_F |f(x,t)|^p\,dx\,dt\right)^{1/p} \leqq 2^{1/p}B(p, q_1, q_2)\left(\sum |c_{k,l}|^2\right)^{1/2}.$$

This proves the right-hand side inequality for $p\geqq 2$. For $0<p<2$, the right-hand side follows from Hölder's inequality and the result for $p=2$. The left-hand side inequality follows from the convexity argument as in theorem 3.

For lacunary trigonometric series we have similar results.

**Theorem 5.** *Given* $0<p<\infty$, $q_1, q_2>1$, *there exist constants* $A(p, q_1, q_2)$ *and* $B(p, q_1, q_2)$ *such that for any* $f(x,t)=\sum_{k,l} c_{k,l}e^{ir_kx}e^{is_lt}$ *with* $\sum|c_{k,l}|^2<\infty$, *with* $r_k=-r_{-k}$, $s_l=-s_{-l}$, $r_0=s_0=0$, $r_{k+1}/r_k\geqq q_1>1$, *and* $s_{l+1}/s_l\geqq q_2>1$ *for* $k, l=1, 2, \ldots$

*and for any intervals* $I_1, I_2$ *with* $|I_1| \geqq 4\pi/(r_1 \min(q_1-1, 1))$, $|I_2| \geqq 4\pi/(s_1 \min(q_2-1, 1))$, *we have*

$$A(p, q_1, q_2)\left(\sum_{k,l} |c_{k,l}|^2\right)^{1/2} \leqq \left(\frac{1}{|I_1 \times I_2|} \int_{I_1}\int_{I_2} |f(x, t)|^p \, dx \, dt\right)^{1/p}$$

$$\leqq B(p, q_1, q_2)\left(\sum_{k,l} |c_{k,l}|^2\right)^{1/2}.$$

*Proof.* The generalization follows from the one-dimensional case as in theorem 3.

**Theorem 6.** *Given* $0<p<\infty$, $q_1, q_2>1$, *there exist constants* $A(p, q_1, q_2)$ *and* $B(p, q_1, q_2)$ *so that for measurable* $E\subset[0, 1]^2$ *with positive measure there exist* $N_1=N_1(E, q_1)$ *and* $N_2=N_2(E, q_2)$ *so that for any* $f(x, t)=\sum_{k,l} c_{k,l} e^{ir_k x} e^{is_l t}$ *with* $\sum_{k,l} |c_{k,l}|^2<\infty$, *with* $r_k=-r_{-k}$, $s_l=-s_{-l}$, $r_0=s_0=0$, $r_1>N_1$, $s_1>N_2$, $r_{k+1}/r_k \geqq q_1>1$, *and* $s_{l+1}/s_l \geqq q_2>1$ *for* $k, l=1, 2, \ldots$ *we have*

$$A(p, q_1, q_2)\left(\sum |c_{k,l}|^2\right)^{1/2} \leqq \left(\frac{1}{|E|} \iint_E |f(x, t)|^p \, dx \, dt\right)^{1/p} \leqq B(p, q_1, q_2)\left(\sum |c_{k,l}|^2\right)^{1/2}.$$

*Proof.* The generalization of the one-dimensional result follows along the same lines as the proof of theorem 4.

**Theorem 7.** *Suppose* $\{n_k\}$ *and* $\{m_l\}$ *are lacunary sequences with* $n_{k+1}/n_k>q_1>1$, $m_{l+1}/m_l>q_2>1$, *and suppose* $f(x, t)=c_0+\sum_{k,l\geqq 1} c_{k,l} w_{n_k}(x) w_{m_l}(t)$ *with* $\sum |c_{k,l}|^2<\infty$. *Then there exist constants* $A(q_1, q_2)$ *and* $B(q_1, q_2)$ *so that*

$$A(q_1, q_2)\left(\sum_{k,l\geqq 1} |c_{k,l}|^2\right)^{1/2} \leqq \|f\|_{BMOd} \leqq B(q_1, q_2)\left(\sum_{k,l\geqq 1} |c_{k,l}|^2\right)^{1/2}.$$

*Proof.* The proof follows the outline of the proof of theorem 1 except that we use theorem 3 for the left-hand side inequality.

**Theorem 8.** *Let* $f(x, t)=\sum_{k,l=-\infty}^{k,l=\infty} c_{k,l} e^{ir_k x} e^{is_l t}$ *with* $r_k$, $s_l$ *real,* $r_{k+1}/r_k>q_1>1$, $s_{l+1}/s_l>q_2>1$ *for* $k, l=1, 2, \ldots$ . *Assume also* $r_1 \geqq 4\pi/\min(q_1-1, 1)$, $s_1 \geqq 4\pi/\min(q_2-1, 1)$, $r_{-k}=-r_k$, $s_{-l}=-s_l$, *and* $\sum |c_{k,l}|^2<\infty$.
*Then there exist constants* $A(q_1, q_2)$ *and* $B(q_1, q_2)$ *so that*

$$A(q_1, q_2)\left({\sum_{k,l}}' |c_{k,l}|^2\right)^{1/2} \leqq \|f\|_{BMO} \leqq B(q_1, q_2)\left({\sum_{k,l}}' |c_{k,l}|^2\right)^{1/2},$$

*where* $\sum'$ *is the sum over all k and l except for the case where both k and l are zero.*

*Proof.* The proof follows as in the one-dimensional case, using theorem 5.

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