Long time small solutions to nonlinear parabolic equations

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Abstract

A sharp result on global small solutions to the Cauchy problem

$$u_t = \Delta u + f(u, Du, D^2 u, u_t) \quad (t > 0), \quad u(0) = u_0$$

in $\mathbf{R}^n$ is obtained under the the assumption that f is $\mathbf{C}^{1+r}$ for $r > 2/n$ and $\|u_0\|_{\mathbf{C}^2(\mathbf{R}^n)} + \|u_0\|_{\mathbf{W}^2_1(\mathbf{R}^n)}$ is small. This implies that the assumption that f is smooth and $\|u_0\|_{\mathbf{W}^k_1(\mathbf{R}^n)} + \|u_0\|_{\mathbf{W}^k_2(\mathbf{R}^n)}$ is small for k large enough, made in earlier work, is unnecessary.

0. Introduction

Let $f \in \mathbf{C}^{1+r}$ in a neighbourhood of the origin and consider the global existence of small solutions to the Cauchy problem

$$\text{(0.1)} \qquad \begin{aligned} &u_t = \Delta u + f(u, Du, D^2 u, u_t) \quad &\text{in} \quad \mathbf{R}^n \times \mathbf{R}_+, \\ &u(0) = u_0 &\text{in} \quad \mathbf{R}^n. \end{aligned}$$

Here $n \geqq 1$, $f(w) = O(|w|^{1+r})$ for small $w \in R^{n^2+n+2}$, $r \geqq 1$ and $r > 2/n$.

It is well known (see [1] and [5]) that the problem

$$\text{(0.2)} \qquad \begin{aligned} &u_t = \Delta u + u^{1+r} \quad &\text{in} \quad \mathbf{R}^n \times \mathbf{R}_+, \quad r > 0, \\ &u(0) = u_0 &\text{in} \quad \mathbf{R}^n, \end{aligned}$$

has a unique global solution provided $r > 2/n$ and u_0 is small. In addition, if $r \leqq 2/n$, then the solution of (0.2) may possibly blow up in a finite time no matter how small u_0 is.

It is the purpose of the paper to deduce the following.

Theorem 0.1. *Let* $\theta \in (0, 1)$,

$$\text{(0.3)} \qquad r \in [1, \infty), \quad r > 2/n, \quad f(w) = O(|w|^{1+r}) \quad \text{as} \quad w \to 0$$

for $w\in R^{n^2+n+2}$, *and* $f\in \mathbf{C}^{1+r}$ *near the origin. Then there exists a constant* $\delta>0$ *such that whenever*

$$\|u_0\|_{C^2(R^n)}+\|u_0\|_{W_1^2(R^n)}<\delta,$$

(0.1) *admits a unique classical solution u such that*

$$\sup_{t>0}\big((t+1)^{(1/2)n}(\|u(t)\|_{\mathbf{C}^2(\mathbf{R}^n)}+\|u_t(t)\|_{\mathbf{C}(\mathbf{R}^n)}+t^{(1/2)\theta}([u(t)]_{\mathbf{C}^{2+\theta}(\mathbf{R}^n)}+[u_t(t)]_{\mathbf{C}^\theta(\mathbf{R}^n)}))$$
$$+\|u(t)\|_{\mathbf{W}_1^2(\mathbf{R}^n)}+\|u_t(t)\|_{\mathbf{L}_1(\mathbf{R}^n)}+t^{(1/2)\theta}([u(t)]_{\mathbf{B}_{1,\infty}^{2+\theta}(\mathbf{R}^n)}+[u_t(t)]_{\mathbf{B}_{1,\infty}^\theta(\mathbf{R}^n)})\big)<\infty.$$

As a consequence, we have the following by-products.

Corollary 0.1. (Klainerman [2].) *Assume*

$$r\in N,\quad (1+1/r)/r>n/2,$$

f does not depend on u_t *and f is smooth with* $f(w)=O(|w|^{1+r})$ *for small w. There exists an integer k and a small* $\delta>0$, *such that if*

$$\|u_0\|_{\mathbf{W}_1^k(\mathbf{R}^n)}+\|u_0\|_{\mathbf{W}_2^k(\mathbf{R}^n)}<\delta,$$

then there is a unique global classical solution u of (0.1) *such that*

$$t^{(1+\varepsilon)/r}\|u(t)\|_{\mathbf{L}_\infty(\mathbf{R}^n)}+\|u(t)\|_{\mathbf{L}_1(\mathbf{R}^n)}<\infty\quad \text{as}\quad t\to\infty$$

for some small $\varepsilon>0$.

Corollary 0.2. (Ponce [4].) *If f does not depend on* u_t *and is a smooth linear function with respect to second derivatives in a neighbourhood of the origin and* (0.3) *with integer r is valid, then there is an integer* $k>2+n/2$ *and a constant* $\delta>0$ *such that for any* u_0 *with*

$$\|u_0\|_{\mathbf{W}_1^k(\mathbf{R}^n)}+\|u_0\|_{\mathbf{W}_2^k(\mathbf{R}^n)}<\delta,$$

(0.1) *has a unique global classical solution u satisfying*

$$\sup_{t>0} t^{(1/2)n}\|u(t)\|_{\mathbf{C}^2(\mathbf{R}^n)}<\infty.$$

Corollary 0.3. (Zheng-Chen [6], Li-Chen [3].) *Assume that f is smooth and satisfies* (0.3) *with integer r. Then for every integer* $k\geqq n+5$, *there is a small* $\delta>0$, *such that for* u_0 *with*

$$\|u_0\|_{\mathbf{W}_1^k(\mathbf{R}^n)}+\|u_0\|_{\mathbf{W}_2^{k+1}(\mathbf{R}^n)}<\delta,$$

(0.1) *has a unique classical solution u such that*

$$\sup_{t>0}(t+1)^{(1/2)n}\big(\|u(t)\|_{\mathbf{C}^2(\mathbf{R}^n)}+\|u_t(t)\|_{\mathbf{L}_\infty(\mathbf{R}^n)}\big)<\infty.$$

The outline of the paper is as follows. The first section contains the notation. The second section describes a decay estimate for a linear heat equation. Section 3 contains the proof of Theorem 0.1, which is obtained by means of a fixed point theorem.

1. Notation

We use the following notation in the paper.

$\mathbf{R}_+=(0, \infty)$. N is the set of all nonnegative integers. $[s]$ is the integer part of $s\in\mathbf{R}$. $n\in\mathbf{N}\setminus\{0\}$.

$$x = (x_1, \dots, x_n)\in\mathbf{R}^n, \quad D_i = \partial/\partial x_i, \quad D^\alpha = D_1^{\alpha_1}\dots D_n^{\alpha_n}$$

with $\alpha=(\alpha_1, \dots, \alpha_n)\in\mathbf{N}^n$ and $|\alpha|=\alpha_1+\dots+\alpha_n$. D^k denotes the vector $(D^\alpha, \dots, D^\beta)$ for all $\alpha, \dots, \beta\in\mathbf{N}^n$ with $|\alpha|=\dots=|\beta|=k$. $D^1=D$, $u_t=\partial u/\partial t$ and Δ denotes the Laplacian.

All functions in the paper are real. c denotes a positive constant which may be different from formula to formula, but is always independent of the variables and functions occuring in a given place. Especially, it does not depend on the "time" variable $t\in\mathbf{R}_+$.

$\mathbf{W}_p^k(\mathbf{R}^n)$, for $k\in\mathbf{N}$, $p\in[1, \infty)$, is the space of functions u on $\mathbf{R}^n$ such that

$$\|u\|_{\mathbf{W}_p^k(\mathbf{R}^n)} = \|u\|_{k,p} = \sum_{|\alpha|\le k} \|D^\alpha u\|_{\mathbf{L}_p(\mathbf{R}^n)} < \infty.$$

$\mathbf{C}(\mathbf{R}^n)$ denotes the set of all bounded and uniformly continuous functions on $\mathbf{R}^n$.

For $k\in\mathbf{N}$,

$$\mathbf{C}^k(\mathbf{R}^n) = \{u\,|\,D^\alpha u\in\mathbf{C}(\mathbf{R}^n) \text{ for all } |\alpha| \le k\}$$

endowed with the norm

$$\|u\|_{\mathbf{C}^k(\mathbf{R}^n)} = \|u\|_{k,0} = \|u\|_k = \sum_{|\alpha|\le k} \|D^\alpha u\|_{\mathbf{L}_\infty(\mathbf{R}^n)},$$

$\mathbf{C}^0(\mathbf{R}^n)=\mathbf{C}(\mathbf{R}^n)$ and $\|\cdot\|_0=\|\cdot\|$.

$\mathbf{C}^s(\mathbf{R}^n)$, for $s\in\mathbf{R}_+\setminus\mathbf{N}$, is the space of all functions $u\in\mathbf{C}^{[s]}(\mathbf{R}^n)$ with

$$[u]_s = \sup_{x\ne y,\, x, y\in\mathbf{R}^n} \frac{|D^{[s]}u(x)-D^{[s]}u(y)|}{|x-y|^{s-[s]}} < \infty.$$

Of course, $\mathbf{C}^s(\mathbf{R}^n)$ is endowed with the norm

$$\|u\|_{\mathbf{C}^s(\mathbf{R}^n)} = \|u\|_s = \|u\|_{[s]}+[u]_s.$$

$\mathbf{B}^s_{1,\infty}(\mathbf{R}^n)$, for $s\in\mathbf{R}_+\setminus\mathbf{N}$, is the space of all functions $u\in\mathbf{W}^s_1(\mathbf{R}^n)$ such that

$$[u]_{s,1} = \sup_{y\in\mathbf{R}^n, y\neq 0} \frac{\int_{\mathbf{R}^n} |D^{[s]}u(x+y)-D^{[s]}u(x)|\,dx}{|y|^{s-[s]}} < \infty.$$

Similarly, $\mathbf{B}^s_{1,\infty}(\mathbf{R}^n)$ is endowed with the norm

$$\|u\|_{\mathbf{B}^s_{1,\infty}(\mathbf{R}^n)} = \|u\|_{s,1} = \|u\|_{[s],1}+[u]_{s,1}.$$

$U(t)$, $t>0$, is the Gauss—Weierstrass semigroup written in the form

$$U(t)u(x) = (4\pi t)^{-(1/2)n}\int_{\mathbf{R}^n} \exp(-|x-y|^2/4t)u(y)\,dy.$$

For $\theta\in(0,1)$, $A_0=C(\mathbf{R}^n)$ and $A_1=\mathbf{L}_1(\mathbf{R}^n)$, we denote by $D^i_\Delta(\theta,\infty)$ with $i=0, 1$ the Banach space of all functions $u\in A_i$ such that

$$|[u]|_{\theta,i} = \sup_{t>0}\|t^{1-\theta}\Delta U(t)u\|_{A_i} < \infty$$

endowed with the norm

$$\|u\|_{D^i_\Delta(\theta,\infty)} = \|u\|_{A_i}+|[u]|_{\theta,i}.$$

2. An a priori estimate

This section is devoted to deducing a basic a priori estimate for solutions to the linear heat equation

$$(2.1)\qquad \begin{aligned} u_t &= \Delta u+f \quad \text{in} \quad \mathbf{R}^n\times\mathbf{R}_+,\\ u(0) &= u_0 \quad \text{in} \quad \mathbf{R}^n. \end{aligned}$$

Let us begin with two basic lemmas.

Lemma 2.1. *Let* $\theta\in(0,1)$, $k\in N$ *and* $t\in\mathbf{R}_+$. *Then we have*

(i) $\|U(t)u\| \leqq c(t+1)^{-(1/2)n}(\|u\|+\|u\|_{0,1})$ *for* $u\in\mathbf{L}_1(\mathbf{R}^n)\cap\mathbf{C}(\mathbf{R}^n)$,

(ii) $\|D^k U(t)u\| \leqq ct^{-(1/2)k}\|u\|$ *for* $u\in\mathbf{C}(\mathbf{R}^n)$,

(iii) $\|D^k U(t)u\|_{0,1} \leqq ct^{-(1/2)k}\|u\|_{0,1}$ *for* $u\in\mathbf{L}_1(\mathbf{R}^n)$.

It should be noted that the result is well-known and trivial. It can be deduced immediately from the definition of $U(t)$.

Lemma 2.2. *Let* $\theta\in(0,1)$. *Then we have*

(i) $|[u]|_{(1/2)\theta,0} \leqq c[u]_\theta$ *for* $u\in\mathbf{C}^\theta(\mathbf{R}^n)$,

(ii) $\|[u]\|_{(1/2)\theta,1} \leqq c[u]_{\theta,1}$ *for* $u \in \mathbf{B}^{\theta}_{1,\infty}(\mathbf{R}^n)$,

(iii) $[u]_{\theta,1} \leqq c\|[u]\|_{(1/2)\theta,1}$ *for* $u \in D^1_\Delta\left(\frac{1}{2}\theta, \infty\right)$,

(iv) $[u]_\theta \leqq c\|[u]\|_{(1/2)\theta,0}$ *for* $u \in D^0_\Delta\left(\frac{1}{2}\theta, \infty\right)$.

Proof. The first two inequalities are proved in the same way. For example, for $u \in C^\theta(R^n)$, $x \in \mathbf{R}^n$ and $t \in R_+$, we have

$$|\Delta U(t)u(x)| \leqq ct^{-1-(1/2)n}\int_{\mathbf{R}^n} \exp(-|y|^2/6t)|u(x+y)-u(x)|\,dy$$

$$\leqq ct^{-(1/2)n+(1/2)\theta-1}\int_{\mathbf{R}^n} \exp(-|y|^2/8t)\,dy[u]_\theta$$

$$\leqq ct^{(1/2)\theta-1}[u]_\theta .$$

For the proof of (iii), we set

$$R(t, \Delta)u(x) = \int_0^\infty \exp(-st)U(s)u(x)\,ds \quad \text{for} \quad t>0, \quad x\in\mathbf{R}^n, \quad u\in D^1_\Delta\left(\tfrac{1}{2}\theta, \infty\right).$$

By lemma 2.1 and the property $\Delta U(t)u=(U(t)u)_t$, we obtain the following

$$t^{(1/2)\theta}\|tR(t, \Delta)u+u\|_{0,1} = t^{(1/2)\theta}\|\Delta R(t, \Delta)u\|_{0,1} \leqq c,$$

$$t^{1/2}\|DR(t, \Delta)u\|_{0,1} \leqq c\|u\|_{0,1} \quad \text{and} \quad \left(R(t, \Delta)u\right)_t = R^2(t, \Delta)u.$$

Therefore, for $x \in \mathbf{R}^n$ and $t>s>0$, we have

$$\int_{\mathbf{R}^n} |u(x+y)-u(y)|\,dy$$

$$\leqq 2\int_{\mathbf{R}^n} |tR(t, \Delta)u(y)+u(y)|\,dy + \int_{\mathbf{R}^n} |sR(s, \Delta)u(x+y)-sR(s, \Delta)u(y)|\,dy$$

$$+\int_s^t \int_{\mathbf{R}^n} |\Delta R^2(\tau, \Delta)u(x+y)-\Delta R^2(\tau, \Delta)u(y)|\,dy\,d\tau$$

$$\leqq ct^{-(1/2)\theta}\|[u]\|_{(1/2)\theta,1}+cs^{1/2}|x|\,\|u\|_{0,1}+c|x|\int_s^t \tau^{-1/2}\|\Delta R(\tau, \Delta)u\|_{0,1}\,d\tau$$

$$\leqq c(t^{-(1/2)\theta}\|[u]\|_{(1/2)\theta,1}+s^{1/2}|x|\,\|u\|_{0,1}+t^{1/2-(1/2)\theta}|x|\cdot\|[u]\|_{(1/2)\theta,1}).$$

Letting $s \to 0$ and setting $t=|x|^{-2}$, we have (iii).

Similarly, we obtain (iv) and complete the proof.

With the use of the above preparations, we proceed to the proof of the following.

Proposition 2.1. *Let* $\theta\in(0,1)$, $n\geqq 1$, $rn>2$, *and let* u *be the solution of* (2.1). *Then we have*

$$\sup_{t>0}\Big((t+1)^{(1/2)n}\big(\|u(t)\|_2+\|u_t(t)\|+t^{(1/2)\theta}\big(\textstyle\sum_{k\leqq 2}[u(t)]_{k+\theta}+[u_t(t)]_\theta\big)\big)$$
$$+\|u(t)\|_{2,1}+\|u_t(t)\|_{0,1}+t^{(1/2)\theta}\big(\textstyle\sum_{k\leqq 2}[u(t)]_{k+\theta,1}+[u_t(t)]_{\theta,1}\big)\Big)$$
$$\leqq c\Big(\|u_0\|_2+\|u_0\|_{2,1}+\sup_{t>0}(t+1)^{(1/2)nr}\big(\|f(t)\|+\|f(t)\|_{0,1}\big)\Big)$$
$$+c\sup_{t>0}(t+1)^{(1/2)nr}t^{(1/2)\theta}\big([f(t)]_\theta+[f(t)]_{\theta,1}\big)$$

provided the right-hand side is finite.

Proof. Since u is the solution of (2.1), we have, for $t>0$,

$$\|u_t(t)\|_{0,1}+t^{(1/2)\theta}[u_t(t)]_{\theta,1}+(t+1)^{(1/2)n}\big(\|u_t(t)\|+t^{(1/2)\theta}[u_t(t)]_\theta\big)$$
$$\leqq \|u(t)\|_{2,1}+t^{(1/2)\theta}[u(t)]_{2+\theta,1}+(t+1)^{(1/2)n}\big(\|u(t)\|_2+t^{(1/2)\theta}[u(t)\|_{2+\theta}\big)$$
$$+\|f(t)\|_{0,1}+t^{(1/2)\theta}[f(t)]_{\theta,1}+(t+1)^{(1/2)nr}\big(\|f(t)\|+t^{(1/2)\theta}[f(t)]_\theta\big).$$

Hence it suffices to prove the estimate without the u_t terms.

Equivalently, the solution u can be written

$$u(t)=U(t)u_0+\int_0^t U(t-s)f(s)\,ds. \tag{2.2}$$

It follows from lemmas 2.1 and 2.2 that

$$(t+1)^{(1/2)n}\|U(t)u_0\|_2+\|U(t)u_0\|_{2,1}\leqq c(\|u_0\|_2+\|u_0\|_{2,1}),$$

and, for $k=0,1,2$,

$$(t+1)^{(1/2)n}t^{(1/2)\theta}[U(t)u_0]_{k+\theta}+t^{(1/2)\theta}[U(t)u_0]_{k+\theta,1}$$
$$\leqq ct^{(1/2)\theta}\Big((t+1)^{(1/2)n}\sup_{s>0}s^{1-(1/2)\theta}\|\Delta U(s)D^kU(t)u_0\|$$
$$+\sup_{s>0}s^{1-(1/2)\theta}\|\Delta U(s)D^kU(t)u_0\|_{0,1}\Big)$$
$$\leqq ct^{(1/2)\theta}\sup_{s>0}s^{1-(1/2)\theta}(s+t)^{-1}(\|D^ku_0\|+\|D^ku_0\|_{0,1})$$
$$\leqq c(\|u_0\|_2+\|u_0\|_{2,1}).$$

Consequently, it remains to estimate the integral term of (2.2).

Set

$$M(f)=\sup_{t>0}(t+1)^{(1/2)nr}\big(\|f(t)\|+\|f(t)\|_{0,1}+t^{(1/2)\theta}([f(t)]_\theta+[f(t)]_{\theta,1})\big).$$

We calculate, for $t, s \in R_+$ and $k=0, 1, 2$,

$$\|\Delta U(s) D^k \int_0^{(1/2)t} U(t-\sigma) f(\sigma)\, d\sigma\|$$

$$= \left\|\int_0^{(1/2)t} U((t+s-\sigma)/2) \Delta D^k U((t+s-\sigma)/2) f(\sigma)\, d\sigma\right\|$$

$$\leqq c\int_0^{(1/2)t} (t+s-\sigma+1)^{-(1/2)n}(t+s-\sigma)^{-1-(1/2)k}\big(\|f(\sigma)\|+\|f(\sigma)\|_{0,1}\big)\, d\sigma,$$

by lemma 2.1,

$$\leqq c(t+1)^{-(1/2)n}(t+s)^{-1-(1/2)k}\int_0^{(1/2)t} (\sigma+1)^{-(1/2)nr}\, d\sigma M(f)$$

$$\leqq c(t+1)^{-(1/2)n} s^{(1/2)\theta-1} t^{-(1/2)\theta} t^{-(1/2)k}\big(1-(t+1)^{1-(1/2)nr}\big)M(f)$$

$$\leqq c(t+1)^{-(1/2)n} t^{-(1/2)\theta} s^{(1/2)\theta-1} M(f),$$

$$\left\|\Delta U(s) D^k \int_0^{(1/2)t} U(t-\sigma) f(\sigma)\, d\sigma\right\|_{0,1}$$

$$= \left\|\int_0^{(1/2)t} \Delta D^k U(t+s-\sigma) f(\sigma)\, d\sigma\right\|_{0,1}$$

$$\leqq c\int_0^{(1/2)t} (t+s-\sigma)^{-1-(1/2)k}\|f(\sigma)\|_{0,1}\, d\sigma,$$

by lemma 2.1,

$$\leqq cs^{(1/2)\theta-1} t^{-(1/2)\theta} M(f).$$

Moreover, note that

$$(t+1)^{-(1/2)nr}\int_0^t (s+\sigma)^{(1/2)\theta-1-(1/2)k}\, d\sigma \leqq cs^{(1/2)\theta-1}(t+1)^{-(1/2)nr} t^{1-(1/2)k}$$

$$\leqq cs^{(1/2)\theta-1}, \quad \text{provided} \quad k=0, 1,$$

$$(t+1)^{-(1/2)nr}\int_0^t (s+\sigma)^{(1/2)\theta-1-(1/2)k}\, d\sigma \leqq \int_0^t (s+\sigma)^{(1/2)\theta-2}\, d\sigma$$

$$\leqq cs^{(1/2)\theta-1}, \quad \text{provided} \quad k=2,$$

$$(t+1)^{-(1/2)nr}\int_0^t (s+\sigma+1)^{-(1/2)n}(s+\sigma)^{(1/2)\theta-1-(1/2)k}\, d\sigma$$

$$\leqq (t+1)^{-(1/2)nr} s^{(1/2)\theta-1}\int_0^t (s+\sigma+1)^{-(1/2)n}\, d\sigma$$

$$\leqq c(t+1)^{-(1/2)n} s^{(1/2)\theta-1}, \quad \text{provided} \quad k=0,$$

and

$$(t+1)^{-(1/2)nr}\int_0^t (s+\sigma+1)^{-(1/2)n}(s+\sigma)^{(1/2)\theta-1-(1/2)k}\, d\sigma$$

$$\leqq (t+1)^{-(1/2)nr}\int_0^t (s+\sigma)^{(1/2)\theta-2}\, d\sigma$$

$$\leqq c(t+1)^{-(1/2)n} s^{(1/2)\theta-1}, \quad \text{provided} \quad k=1, 2.$$

We have, for $k=0, 1, 2$,

$$\left\|\Delta U(s) D^k \int_{(1/2)t}^{t} U(t-\sigma)f(\sigma)\, d\sigma\right\|_{0,1}$$

$$= \left\|\int_{(1/2)t}^{t} D^k U((t+s-\sigma)/2)\Delta U((t+s-\sigma)/2)f(\sigma)\, d\sigma\right\|_{0,1}$$

$$\leqq c\int_{(1/2)t}^{t} (t+s-\sigma)^{-(1/2)k}\|\Delta U((t+s-\sigma)/2)f(\sigma)\|_{0,1}\, d\sigma,$$

by lemma 2.1,

$$\leqq c\int_{(1/2)t}^{t} (t+s-\sigma)^{(1/2)\theta-1-(1/2)k}(\sigma+1)^{-(1/2)nr}\sigma^{-(1/2)\theta}\, d\sigma M(f),$$

by lemma 2.2,

$$\leqq c(t+1)^{-(1/2)nr}t^{-(1/2)\theta}\int_0^t (s+\sigma)^{(1/2)\theta-1-(1/2)k}\, d\sigma M(f)$$

$$\leqq cs^{(1/2)\theta-1}t^{-(1/2)\theta}M(f),$$

and

$$\left\|\Delta U(s) D^k \int_{(1/2)t}^{t} U(t-\sigma)f(\sigma)\, d\sigma\right\|$$

$$= \left\|\int_{(1/2)t}^{t} U((t+s-\sigma)/3) D^k U((t+s-\sigma)/3)\Delta U((t+s-\sigma)/3)f(\sigma)\, d\sigma\right\|$$

$$\leqq c\int_{(1/2)t}^{t} (t+s-\sigma+1)^{-(1/2)n}(t+s-\sigma)^{(1/2)\theta-1-(1/2)k}([f(\sigma)]_{\theta,1}+[f(\sigma)]_\theta)\, d\sigma$$

$$\leqq cM(f)\int_{(1/2)t}^{t} (t+s-\sigma+1)^{-(1/2)n}(t+s-\sigma)^{(1/2)\theta-1-(1/2)k}(\sigma+1)^{-(1/2)nr}\sigma^{-(1/2)\theta}\, d\sigma$$

$$\leqq cM(f)(t+1)^{-(1/2)nr}t^{-(1/2)\theta}\int_0^t (s+\sigma+1)^{-(1/2)n}(s+\sigma)^{(1/2)\theta-1-(1/2)k}\, d\sigma$$

$$\leqq cM(f)s^{(1/2)\theta-1}t^{-(1/2)\theta}(t+1)^{-(1/2)n}.$$

In view of lemma 2.2, we thus have shown that

$$t^{(1/2)\theta}\left(\sum_{k\leqq 2}[u(t)]_{k+\theta,1}+(t+1)^{(1/2)n}\left(\sum_{k\leqq 2}[u(t)]_{k+\theta}+[u_t(t)]_\theta\right)\right)\leqq cM(f).$$

Further, let us show that

$$(t+1)^{(1/2)n}\|D^2u(t)\|+\|D^2u(t)\|_{0,1}\leqq cM(f).$$

Note that, for $t>0$ and $v\in D_\Delta^i(\frac{1}{2}\theta, \infty)$ with $i=0, 1$,

$$\|D^2U(t)v(\cdot)\|_{0,i}\leqq ct^{-1-(1/2)n}\int_{R^n}\exp(-|y|^2/6t)\|v(\cdot+y)-v(\cdot)\|_{0,i}\, dy$$

$$\leqq ct^{(1/2)\theta-1}[v]_{\theta,i}.$$

We have

$$\left\|D^2\int_{(1/2)t}^{t} U(t-s)f(s)\,ds\right\|_{1,i} \leqq c\int_{(1/2)t}^{t} (t-s)^{(1/2)\theta-1}[f(s)]_{\theta,i}\,ds$$
$$\leqq c(t+1)^{-(1/2)nr}t^{-(1/2)\theta}\int_0^t (t-s)^{(1/2)\theta-1}\,ds\,M(f)$$
$$\leqq c(t+1)^{-(1/2)n}M(f),$$

and, similarly,

$$\left\|D^2\int_0^{(1/2)t} U(t-s)\,f(s)\,ds\right\|_{0,1}+(t+1)^{(1/2)n}\left\|D^2\int_0^{(1/2)t} U(t-s)f(s)\,ds\right\|$$
$$\leqq c\int_0^{(1/2)t}(t-s)^{-1}\big(1+(t+1)^{(1/2)n}(t-s+1)^{-(1/2)n}\big)\big(\|f(s)\|_{0,1}+\|f(s)\|\big)\,ds$$
$$\leqq ct^{-1}\int_0^t (s+1)^{-(1/2)nr}\,dsM(f) \leqq cM(f).$$

Finally, we show that

$$\|u(t)\|_{0,1}+(t+1)^{(1/2)n}\|u(t)\| \leqq cM(f).$$

Indeed, following the above, we have

$$\left\|\int_0^t U(t-s)f(s)\,ds\right\|_{0,1}+(t+1)^{(1/2)n}\left\|\int_0^{(1/2)t} U(t-s)f(s)\,ds\right\| \leqq cM(f),$$

and, since $rn>2$ and $r\geqq 1$, we get

$$\left\|\int_{(1/2)t}^{t} U(t-s)f(s)\,ds\right\| \leqq c\int_{(1/2)t}^{t}(t-s+1)^{-(1/2)n}\big(\|f(s)\|+\|f(s)\|_{0,1}\big)\,ds$$
$$\leqq c(t+1)^{-(1/2)nr}\int_0^t (t-s+1)^{-(1/2)n}\,dsM(f)$$
$$\leqq c(t+1)^{-(1/2)n}M(f).$$

The proof is complete.

3. Proof of Theorem 0.1

With the use of the Banach fixed point theorem, Theorem 0.1 is, in fact, a simple consequence of Proposition 2.1.

Proof of Theorem 0.1. Let B denote the unit ball of $\mathbf{R}^{n^2+n+2}$. Without loss of generality, we suppose that $f\in \mathbf{C}^{1+r}(\bar{B})$.

In order to apply the fixed point theorem, we need the following notation.

X is the Banach space of all functions u on $\mathbf{R}^n\times\mathbf{R}_+$ such that the norm

$$\|u\|_{\mathbf{X}} = \sup_{t>0}\big((t+1)^{(1/2)n}(\|u(t)\|_2+\|u_t(t)\|+t^{(1/2)\theta}(\textstyle\sum_{k\leq 2}[u(t)]_{k+\theta}+[u_t(t)]_\theta))$$

$$+\|u(t)\|_{2,1}+\|u_t(t)\|_{0,1}+t^{(1/2)\theta}(\textstyle\sum_{k\leq 2}[u(t)]_{k+\theta,1}+[u_t(t)]_{\theta,1})\big)$$

is finite.

Y is the Banach space of all functions $u\in\mathbf{C}^2(\mathbf{R}^n)$ such that the norm

$$\|u\|_Y = \|u\|_2+\|u\|_{2,1}$$

is finite.

For $E\in(0, 1)$,

$$X(E) = \{u\in\mathbf{X}\,|\,\|u\|_{\mathbf{X}}\leqq E\},\quad\text{and}\quad Y(E) = \{u\in\mathbf{Y}\,|\,\|u\|_{\mathbf{Y}}\leqq E^2\}.$$

For $u\in X(E)$, $u_0\in Y(E)$ and $t>0$, we set

$$u^* = (u, Du, D^2u, u_t),\quad\text{and}\quad T_{u_0}u(t) = U(t)u_0+\int_0^t U(t-s)f(u^*(s))\,ds.$$

Recall that $f\in\mathbf{C}^{1+r}(\bar{B})$ and $f(w)=O(|w|^{1+r})$ for small $w\in B$. Hence there is a constant $E\in(0, 1)$ such that for $u, v\in X(E)$ and $u_0\in Y(E)$, the following estimates hold.

$$\begin{aligned}
[f(u^*(t))]_{\theta,1} &\leqq c[u^*(t)]_{\theta,1}\|u^*(t)\|^r\\
&\leqq c(\textstyle\sum_{k\leqq 2}[u(t)]_{k+\theta,1}+[u_t(t)]_{\theta,1})(\|u(t)\|_2+\|u_t(t)\|)^r\\
&\leqq ct^{-(1/2)\theta}(t+1)^{-(1/2)nr}\|u\|_{\mathbf{X}}^{1+r},
\end{aligned}$$

$$\begin{aligned}
\|f(u^*(t)\|_{0,1} &\leqq c(\|u(t)\|_{2,1}+\|u_t(t)\|_{0,1})(\|u(t)\|_2+\|u_t(t)\|)^r\\
&\leqq c(t+1)^{-(1/2)nr}\|u\|_{\mathbf{X}}^{1+r},
\end{aligned}$$

$$\begin{aligned}
t^{(1/2)\theta}[f(u^*(t))]_\theta+\|f(u^*(t))\| &\leqq ct^{(1/2)\theta}[u^*(t)]_\theta\|u^*(t)\|^r+\|u^*(t)\|^{1+r}\\
&\leqq c(t+1)^{-(1/2)nr}\|u\|_{\mathbf{X}}^{1+r},
\end{aligned}$$

$$\begin{aligned}
&[f(u^*(t))-f(v^*(t))]_{\theta,1}\\
&\leqq c(\|u^*(t)\|+\|v^*(t)\|)^r[u^*(t)-v^*(t)]_{\theta,1}\\
&\quad+c([u^*(t)]_\theta+[v^*(t)]_\theta)(\|u^*(t)\|+\|v^*(t)\|)^{r-1}\|u^*(t)-v^*(t)\|_{0,1}\\
&\leqq ct^{-(1/2)\theta}(t+1)^{-(1/2)nr}(\|u\|_{\mathbf{X}}+\|v\|_{\mathbf{X}})^r\|u-v\|_X,
\end{aligned}$$

$$\begin{aligned}
\|f(u^*(t))-f(v^*(t))\|_{0,1} &\leqq c(\|u^*(t)\|+\|v^*(t)\|)^r\|u^*(t)-v^*(t)\|_{0,1}\\
&\leqq c(t+1)^{-(1/2)nr}(\|u\|_{\mathbf{X}}+\|v\|_{\mathbf{X}})^r\|u-v\|_X,
\end{aligned}$$

and

$$\begin{aligned}
&(t+1)^{(1/2)nr}(t^{(1/2)\theta}[f(u^*(t))-f(v^*(t))]_\theta+\|f(u^*(t))-f(v^*(t))\|)\\
&\leqq c(\|u\|_{\mathbf{X}}+\|v\|_{\mathbf{X}})^r\|u-v\|_{\mathbf{X}}.
\end{aligned}$$

From the above estimates and proposition 2.1, we have

$$\|T_{u_0}(u)\|_X \leqq c(\|u_0\|_Y + \|u\|_X^{1+r}) \leqq cE^2 \leqq E$$

and

$$\|T_{u_0}(u) - T_{u_0}(v)\|_X \leqq c(\|u\|_X + \|v\|_X)^r \|u-v\|_X$$
$$\leqq cE\|u-v\|_X \leqq \frac{1}{2}\|u-v\|_X,$$

provided $u, v \in X(E)$, $u_0 \in Y(E)$ with $E \in (0, 1)$ sufficiently small. Taking into account the Banach theorem, we have that the operator T_{u_0} has a unique fixed point $u \in X(E)$ provided $u_0 \in Y(E)$. The proof is complete.

Acknowledgement. The author is grateful to the referee for his valuable criticisms and kind suggestions.

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Received May 16, 1989

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