Non commutative Khintchine and Paley inequalities

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Introduction

Let (ε_n) be a sequence of independent ± 1-valued random variables on a probability space (D, μ) with $\mu(\varepsilon_n=1)=\mu(\varepsilon_n=-1)=1/2$ (for instance the Rademacher functions on the Lebesgue interval).

Let X be a Banach space and let (x_n) be a sequence in X. In recent years, a great deal of work has been devoted to try to find "explicit" necessary and sufficient conditions for the series

$$\sum_{n=1}^{\infty} \varepsilon_n x_n \tag{0.1}$$

to converge (in norm) almost surely, see for instance [Ka], [MP] and [LeT].

Equivalently the problem reduces to find an "explicit expression" equivalent to the norm defined by

$$\|(x_n)\| = \left(\int \left\|\sum \varepsilon_n x_n\right\|^2 d\mu\right)^{1/2} \tag{0.2}$$

considered as a norm on the set of all finitely supported sequences (x_n) in X. While a satisfactory solution seems hopeless at the moment for an *arbitrary* space X, there are cases for which the answer is known to be very simple and as complete as possible. For instance, if X is the Banach space $L_p(\Omega, \sum, m)$ $(1 \leqq p < \infty)$ the classical Khintchine inequalities (cf. [LT, I.d.6]) and Fubini's theorem imply that there is an absolute constant C such that, for all x_n in $X = L_p(\Omega, \sum, m)$, we have

$$\frac{1}{C}\|(x_n)\| \leqq \left\|\left(\sum |x_n|^2\right)^{1/2}\right\|_X \leqq C\|(x_n)\|. \tag{0.3}$$

This solves the above mentioned problem when $X = L_p(m)$. More generally, as shown by Maurey (cf. [LT] p. 50) (0.3) remains valid when X is a Banach lattice iff X is q-concave for some $q < \infty$. In this paper, we investigate what remains of (0.3) when X is a *non-commutative* L_p-space (or a non commutative analogue of a Banach lattice). In the case $1<p<\infty$, the problem has been solved in [LP 1]. This left open the more general case of C_E spaces and the case $p=1$. In that case our main result is as follows. We denote by C_1 the Banach space of all trace class operators on l_2 and we denote its norm by $\| \|_1$.

Theorem 0.1. *Let (x_n) be a finite sequence in C_1. We define*

$$|||(x_n)||| = \inf_{x_n=y_n+z_n} \{\|(\sum y_n^* y_n)^{1/2}\|_1 + \|(\sum z_n z_n^*)^{1/2}\|_1\} \tag{0.4}$$

where the infimum runs over all possible decompositions $x_n=y_n+z_n$. Then there is an absolute constant C such that for all finite sequences (x_n) in $X=C_1$ we have

$$\frac{1}{C}\|(x_n)\| \leqq |||(x_n)||| \leqq C\|(x_n)\|. \tag{0.5}$$

As an immediate consequence, we have

Corollary 0.2. *The series* (0.1) *converges a. s. in C_1 iff there is a decomposition $x_n=y_n+z_n$ with y_n, z_n in C_1 such that both series $\sum_1^\infty y_n^* y_n$ and $\sum_1^\infty z_n z_n^*$ converge in the space of compact operators and both $(\sum y_n^* y_n)^{1/2}$ and $(\sum z_n z_n^*)^{1/2}$ belong to C_1.*

In the appendix to this paper, we show that Theorem 0.1 can be viewed as dual to the form of the non-commutative Grothendieck inequality which was conjectured by Ringrose and first proved in [P 1]. Our main results are in Section II. There we prove a strengthening of Theorem 0.1 which appears as a non commutative version of Paley's inequality. Paley's inequality [Pa] says that there is a constant C such that for all functions $f=\sum_{n=0}^\infty a_n e^{int}$ in H^1 we have

$$(\sum_{k=0}^\infty |a_{2^k}|^2)^{1/2} \leqq C\|f\|_{H^1}.$$

More generally, the sequence $\{2^k\}$ can be replaced by any increasing sequence $\{n_k\}$ which is lacunary *à la Hadamard,* i. e. $\underline{\lim}\, \frac{n_{k+1}}{n_k} > 1$.

Let $\{n_k\}$ be such a sequence. It was proved in [P 2] that if (x_k) is a finite sequence in an *arbitrary* Banach space, then we have

$$\frac{1}{C}\|(x_k)\| \leqq \left(\int \|\sum e^{in_k t} x_k\|^2 dt\right)^{1/2} \leqq C\|(x_k)\| \tag{0.6}$$

where $\|(x_k)\|$ is defined in (0.2) and where C is a constant depending only on the sequence $\{n_k\}$ (essentially only on its degree of lacunarity, in fact only on its Sidon constant). Moreover, the a.s. convergence of (0.1) is equivalent to the a.s. convergence of the series $\sum_{k=0}^\infty e^{in_k t} x_k$.

For a Banach space X, the space $H^1(X)$ is defined precisely in Section I. An extension of Paley's inequality in C_1 has already been given in [BP]. Here, we will prove the following (see Theorem II.1 below)

Theorem 0.3. *Let $X=C_1$. There is a constant C such that for all functions $f=\sum_{n=0}^{\infty} x_n e^{int}$ in $H^1(X)$ (with $x_n\in X$) we have*

$$|||(x_{n_k})||| \leqq C\|f\|_{H^1(X)}. \tag{0.7}$$

In particular, we may apply (0.7) to a lacunary series $\sum_{k\geqq 0} x_{n_k} e^{in_k t}$, using (0.6) (or an elementary averaging over all choices of signs) this yields the right side of (0.5). Since the left side is very easy (see below (I.14)) we thus obtain Theorem 0.1 as an immediate consequence of Theorem 0.3. In particular, (0.7) becomes an *equivalence* when f is a lacunary series, this is an advantage over the versions of Paley's inequality considered in [BP]. Moreover, taking our appendix into account, this gives a new proof of the Ringrose conjecture mentioned above. An alternate proof was already given in [H]. Our method to prove (0.7) is very simple. It is based on the fact that every f in $H^1(C_1)$ can be written as a product $f=gh$ with g and h both in $H^2(C_2)$. This was established by Sarason in [S] while the matrix case goes back further (Helson—Lowdenslager). By (0.5), (0.6) and (0.7) we have

Corollary 0.4. *There is a constant C' such that for all $f=\sum_{n=0}^{\infty} x_n e^{int}$ in $H^1(C_1)$ we have*

$$\Big\|\sum_{k\geqq 0} x_{n_k} e^{in_k t}\Big\|_{H^1(C_1)} \leqq C'\|f\|_{H^1(C_1)}$$

and the series on the left-hand side converges in $H^1(C_1)$.

In particular if we denote by P: $H^1\to H^1$ the "orthogonal" projection onto the span of $\Lambda=\{e^{i2^k t}\}$ in H^1, then, when $X=C_1$, the operator $P\otimes \mathrm{Id}_X$ is bounded on $H^1(X)$ and $\|P\|\leqq C'$.

Remarks. (i) It is well known that for a general Banach space X, the operator $P\otimes\mathrm{Id}_X$ is not bounded on $H^1(X)$, for instance if $X=c_0$ or $C(\mathbf{T})$ hence if X is any space containing l_∞^n's uniformly, furthermore if $X=L_1/H^1$, $P\otimes\mathrm{Id}_X$ is **un**bounded on $H^1(X)$.

(ii) A variant of a proof in [BP] shows that the operator $P\otimes\mathrm{Id}_X$ is bounded on the "atomic version" of $H^1(X)$ iff X is K-convex. In that case, $P\otimes\mathrm{Id}_X$ is a fortiori bounded on $H^1(X)$. This variant of a result in [BP] was observed by the second author (see the last remarks in [BP]). It shows in particular that if $X=C_1$, $P\otimes\mathrm{Id}_X$ is *not* bounded on the *atomic* version of $H^1(X)$.

For convenience we denote by $\hat{P}$: $H^1\to\ell_2$ the operator which maps a function f to the sequence $(\hat{f}(2^k))_k$. (By Paley's inequality, $\hat{P}$ is bounded.)

In this paper, we study the range of $P\otimes\mathrm{Id}_X$: $H^1(X)\to H^1(X)$ and similarly for $\hat{P}\otimes\mathrm{Id}_X$ when $X=C_p$ $(1\leqq p<\infty)$ or more generally $X=C_E$ where E is a symmetric sequence space. The main interest is when E is 2-concave, in particular $1 \leq p \leq 2$. In that case (Section III) we obtain results similar to the preceding statements. These results can also be extended to the case when X is a general non-commutative L^1-space, i.e. the predual of a von Neumann algebra (Section II). Finally, we are able to treat more general multipliers $m\colon H^1 \to \ell_2$ in place of the operator $\hat{P}$. This is explained in Sections II, III.

We also study the boundedness of $P \otimes \mathrm{Id}_X\colon H^1(X) \to H^1(X)$ when $X = Y \hat{\otimes} Z$ is a projective tensor product of Banach spaces satisfying suitable assumptions (Part III); more precisely we give cases where

$$H^2_\Lambda(X \hat{\otimes} Y) = H^2_\Lambda(X) \hat{\otimes} Y + X \hat{\otimes} H^2_\Lambda(Y).$$

I. Definitions, notation and background

We denote by $\mathbf{T}$ the group $\mathbf{R}/2\pi\mathbf{Z}$ equipped with its normalized Haar measure dt. We denote by H^p the closure of $\{e^{int} | n \geq 0\}$ in $L_p(\mathbf{T})$, $1 \leq p < \infty$, by H^p_Λ the closed span of $\{e^{i2^k t}, k \geq 0\}$ in $L_p(\mathbf{T})$. We recall (cf. [D, Chap. 6] that an $H^1 - \ell_2$ multiplier is a sequence $m = (m_n)_{n \geq 0}$ such that there is a constant K such that for all $f = \sum_{n \geq 0} a_n e^{int}$

$$\left(\sum_{n \geq 0} |m_n a_n|^2\right)^{1/2} \leq K \|f\|_1.$$

It is known (cf. [D]) that this holds iff

$$\|m\| = \sup_{n \geq 0} \left(\sum_{k=n}^{k=2n} |m_k|^2\right)^{1/2}$$

is finite. Moreover the preceding condition is equivalent to

$$\sup_{n \geq 0} \left(\sum_{2^n \leq k \leq 2^{n+1}} |m_k|^2\right)^{1/2} < \infty. \tag{I.0}$$

We denote by $\hat{m}\colon H^1 \to H^2$ the mapping

$$\sum_{n \geq 0} a_n e^{int} \to \sum_{n \geq 0} m_n a_n e^{int}.$$

Let X be a complex Banach space with dual X^*. Let $1 \leq p < \infty$. We will denote by $H^p(X)$ (resp. $H^p_\Lambda(X)$) the closed subspace of $L^p(\mathbf{T}; X)$ spanned by $H^p \otimes X$ (resp. $H^p_\Lambda \otimes X$). We recall that $H^1_\Lambda(X)$ and $H^2_\Lambda(X)$ are canonically isomorphic.

We refer to Section 1.d in [LT] for the definitions of a p-convex or q-concave Banach lattice X. We recall that X is 2-concave iff X has cotype 2, that X is 2-convex (resp. 2-concave) iff X^* is 2-concave (resp. 2-convex), cf. [LT, Prop 1-d-4]. The 2-convexification $X^{(2)}$ of X is such that

$$\|a\|_{X^{(2)}} = \||a|^2\|_X^{1/2},$$

for example $\ell^{p(2)} = \ell^{2p}$.

We also recall ([LT], p. 47) that $X^*(\ell_2)$ is a closed norming subspace of $X(\ell_2)^*$. A symmetric sequence space E is a Banach sequence space such that either the canonical sequence is a symmetric 1-unconditional basis for E or E is the dual of such a space. The first case occurs iff E is separable. If E is not canonically isomorphic to ℓ^∞, E lies in c_0 [Si, Theorem 1.16]. We also recall [LT, Prop. 1-d-2(i)] that

$$\|ab\|_E \leqq \|a\|_{E^{(2)}} \|b\|_{E^{(2)}}.$$

Let E be a symmetric sequence space. C_E is the space of compact operators **A** on a separable Hilbert space H whose sequence of characteristic numbers $(s_n(A))_{n\geqq 1}$ belongs to E [Si] and

$$\|A\|_{C_E} = \|(s_n(A))_{n\geqq 0}\|_E.$$

When $E=\ell^p$ $(1\leqq p<\infty)$ we write C_p instead of C_{ℓ^p}. When $E=c_0$ or ℓ^∞ we write C_∞ instead of $C_{c_0}=C_{\ell^\infty}$. The space of all bounded operators on a Hilbert space H is denoted by $B(H)$. It is the dual space of C_1 if H is separable. If E is separable and $E\neq\ell^1$ the dual space of C_E is C_{E^*} [Si, Theorem 3.2]. Duality between C_E and C_{E^*} is defined by

$$\langle A, B\rangle = \operatorname{tr} AB^*.$$

If E, F, G are symmetric sequence spaces which satisfy $\|ab\|_E\leqq\|a\|_F\|b\|_G$ we have [Si, Theorem 2.8]

$$\|AB\|_{C_E} \leqq \|A\|_{C_F} \|B\|_{C_G} \tag{I.1}$$

in particular

$$\|AB\|_{C_E} \leqq \|A\|_{C_{E^{(2)}}} \|B\|_{C_{E^{(2)}}}.$$

2-concavity and 2-convexity of C_E. The spaces $C_E(\ell^2_R)$ and $C_E(\ell^2_L)$

Let $(A_k)_{k=1}^K$ be a finite sequence in $B(H)$. Let $\tilde{H}=\bigoplus_2 H$ be the Hilbertian sum of a countable number of copies of H. By a well known observation $(A_k)_{k=1}^K$ can be viewed as an operator $\mathscr{A}\colon \tilde{H}\to\tilde{H}$ $(x_1, \dots, x_K, \dots, x_n, \dots)\to(A_1(x_1), \dots, A_K(x_1), 0, \dots)$

$$\mathscr{A} = \begin{pmatrix} A_1 & 0 \dots \\ \vdots & \\ A_K & 0 \dots \\ 0 & 0 \dots \\ \vdots & \vdots \end{pmatrix}.$$

We define $\mathscr{C}_E$ exactly as C_E with $\tilde{H}$ instead of H. Hence

$$\|\mathscr{A}\|_{\mathscr{C}_E} = \||\mathscr{A}|\|_{\mathscr{C}_E} = \|(\textstyle\sum_1^K A_k^* A_k)^{1/2}\|_{C_E}. \tag{I.2}$$

This enables us to define the Banach space $C_E(\ell_R^2)$ as the completion of the finite sequences of compact operators $(A_k)_{k=1}^K$ for which the following norm is finite:

$$\|(A_k)_{k=1}^K\|_{C_E(\ell_R^2)} = \Big\|\Big(\sum_1^K A_K^* A_k\Big)^{1/2}\Big\|_{C_E}.$$

$C_E(\ell_R^2)$ is canonically isomorphic to a closed subspace of $\mathscr{C}_E$ which is also 1-complemented. $B(H)(\ell_R^2)$ is defined in the same way; $C_E(\ell_L^2)$ is defined similarly with

$$\|(A_k)_{k=1}^K\|_{C_E(\ell_L^2)} = \|(A_k^*)_{k=1}^K\|_{C_E(\ell_R^2)} = \Big\|\Big(\sum_1^K A_k A_k^*\Big)^{1/2}\Big\|_{C_E}.$$

We warn the reader that in general $\|(\sum_1^K A_k A_k^*)^{1/2}\|_{C_E}$ is different from $\|(\sum_1^K A_K^* A_k)^{1/2}\|_{C_E}$.

Clearly if E is separable and $E\neq\ell^1$ the dual space of $C_E(\ell_R^2)$ is $C_{E^*}(\ell_R^2)$ for the duality defined by

$$\langle (A_k)_1^K, (B_k)_1^K\rangle = \operatorname{tr}\sum_1^K A_k B_k^*.$$

If E, F, G are as in (I.1), (I.1) and (I.2) imply

$$\text{(I.3)}\qquad \Big\|\sum_1^K A_k B_k\Big\|_{C_E} \leqq \Big\|\Big(\sum_1^K A_k A_k^*\Big)^{1/2}\Big\|_{C_F} \Big\|\Big(\sum_1^K B_k^* B_k\Big)^{1/2}\Big\|_{C_G}$$

by writing

$$\Big\|\sum_1^K A_k B_k\Big\|_{C_E} = \left\|\begin{pmatrix} A_1 & \dots & A_K & 0 & \dots \\ 0 & & 0 & 0 & \dots \\ \vdots & & \vdots & \vdots & \end{pmatrix}\begin{pmatrix} B_1 & 0 & \dots \\ \vdots & & \\ B_K & 0 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \end{pmatrix}\right\|_{C_E}.$$

A non discrete version of (I.3) is the following: Let $(\varphi_k)_{k=1}^K \in L^2(\mu)$, $(A_k)_{k=1}^K$, $(B_k)_{k=1}^K \in C_\infty$, $A(t)=\sum_1^K \varphi_k(t) A_k$, $B(t)=\sum_1^K \varphi_k(t) B_k$

$$\text{(I.4)}\qquad \Big\|\int A(t)B(t)\,d\mu(t)\Big\|_{C_E} \leqq \Big\|\Big(\int A(t)A^*(t)\,d\mu(t)\Big)^{1/2}\Big\|_{C_F} \Big\|\Big(\int B^*(t)B(t)\,d\mu(t)\Big)^{1/2}\Big\|_{CG}$$

(approximate the φ_k's by functions with finite range).

If E is 2-convex there is a symmetric sequence space F such that $F^{(2)}=E$. Hence

$$\text{(I.5)}\qquad \|A\|_{C_E} = \|A^*A\|_{C_F}^{1/2}.$$

Let $(A_k)_{k=1}^K$ be a finite sequence in C_E. Then

$$\Big\|\Big(\sum_1^K A_k^* A_k\Big)^{1/2}\Big\|_{C_E} = \Big\|\sum_1^K A_k^* A_k\Big\|_{C_F} \leqq \Big(\sum_1^K \|A_k\|_{C_E}^2\Big)^{1/2}$$

by the triangle inequality in C_F.

$$\text{(I.6)}\qquad \Big\|\Big(\sum_1^K A_k^* A_k\Big)^{1/2}\Big\|_{C_E} \leqq \Big(\sum \|A_k\|_{C_E}^2\Big)^{1/2}$$

is the 2-convexity inequality for C_E. It holds true iff E is 2-convex. It also holds true in $B(H)$.

A non discrete version of (I.6) is the following: let $(\varphi_k)_{k=1}^K$ be an orthonormal sequence in $L^2(\mu)$, $(A_k)_{k=1}^K \in C^\infty$, $A(t)=\sum_1^K \varphi_k(t) A_k$

$$\text{(I.7)} \qquad \left\|\left(\sum_1^K A_k^* A_k\right)^{1/2}\right\|_{C_E} = \left\|\left(\int A^*(t) A(t)\, d\mu\right)^{1/2}\right\|_{C_E} \leqq \left(\int \|A(t)\|_{C_E}^2 d\mu\right)^{1/2}.$$

This also holds true in $B(H)$.

If E is 2-concave and separable, E^* is 2-convex, C_{E^*} is 2-convex. By the duality between C_E and C_{E^*} on one hand, $C_E(\ell_R^2)$ and $C_{E^*}(\ell_R^2)$ on the other hand, (I.6) implies the 2-concavity inequality for C_E

$$\text{(I.8)} \qquad \left(\sum_1^K \|A_k\|_{C_E}^2\right)^{1/2} \leqq \left\|\left(\sum_1^K A_k^* A_k\right)^{1/2}\right\|_{C_E}$$

whose non discrete version, with the same notation as in (I.7) is

$$\text{(I.9)} \qquad \left(\int \left\|\sum_1^K \varphi_k(t) A_k\right\|_{C_E}^2 d\mu\right)^{1/2} \leqq \left\|\left(\int A^*(t) A(t)\, d\mu\right)^{1/2}\right\|_{C_E} = \left\|\left(\sum_1^K A_k^* A_k\right)^{1/2}\right\|_{C_E}$$

(approximate the φ_k's by functions with finite range and apply (I.8) in order to get the first inequality which does not depend on the orthogonality of the φ_k's). Inequalities (I.6) and (I.8) are particular cases of [A], theorem 1.3.

The spaces $C_E(\ell_S^2)$, $B(H)(\ell_S^2)$, $C_E(\ell_R^2)+C_E(\ell_L^2)$

Let $B \in B(H)$, let $|B|_s$ be its symmetrized modulus:

$$|B|_s = \left(\frac{B^* B + BB^*}{2}\right)^{1/2}.$$

Let E be a 2-convex symmetric sequence space and let $E=F^{(2)}$. We recall that, if $(B_k)_{k=1}^K$ is a finite sequence of operators in C_E, $\left\|\left(\sum_1^K |B_k|_s^2\right)^{1/2}\right\|_{C_E}$ actually defines a norm on $(B_k)_{k=1}^K$, and we will define $C_E(\ell_S^2)$ as the completion of finite sequences for this norm. $B(H)(\ell_S^2)$ is defined similarly. Indeed

$$\left\|\left(\sum_1^K |B_k|_s^2\right)^{1/2}\right\|_{C_E} = \frac{1}{\sqrt{2}} \left\| \begin{pmatrix} B_1 & 0 & \dots \\ B_1^* & 0 & \dots \\ \vdots & \vdots & \\ B_K & 0 & \dots \\ B_K^* & 0 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \end{pmatrix} \right\|_{\mathcal{C}_E}.$$

We have obviously

(I.10) $$\frac{1}{\sqrt{2}}\max\{\|(B_k)_1^K\|_{C_E(\ell_R^2)}, \|(B_k)_1^K\|_{C_E(\ell_L^2)}\} \leqq \left\|\left(\sum_1^K |B_k|_s^2\right)^{1/2}\right\|_{C_E}$$

and by the 2-convexity of E and the triangle inequality in C_F

(I.11) $$\left\|\left(\sum_1^K |B_k|_s^2\right)^{1/2}\right\|_{C_E} \leqq \frac{1}{\sqrt{2}}\left(\|(B_k)_1^K\|^2_{C_E(\ell_R^2)}+\|(B_k)_1^K\|^2_{C_E(\ell_L^2)}\right)^{1/2}$$
$$\leqq \max\{\|(B_k)_1^K\|_{C_E(\ell_R^2)}, \|(B_k)_1^K\|_{C_E(\ell_L^2)}\}.$$

Similar inequalities are valid for $B(H)(\ell_S^2)$.

Let now E be a 2-concave separable symmetric sequence space. We denote by $C_E(\ell_R^2)+C_E(\ell_L^2)$ the sum of these Banach spaces:

$$\|(A_k)_1^K\|_{C_E(\ell_R^2)+C_E(\ell_L^2)} = \inf_{A_k=A_k'+A_k''}\{\|(A_k')_1^K\|_{C_E(\ell_R^2)}+\|(A_k'')_1^K\|_{C_E(\ell_L^2)}\}$$

where the infimum is taken over all decompositions for which $\|(A_k')_1^K\|_{C_E(\ell_R^2)}$ and $\|(A_k'')_1^K\|_{C_E(\ell_L^2)}$ are finite. By (I.10) and (I.11) $C_{E^*}(\ell_S^2)$ is isomorphic to a norming subspace of the dual space of $C_E(\ell_R^2)+C_E(\ell_L^2)$, namely

(I.12) $$\frac{1}{\sqrt{2}}\|(B_k)_1^K\|_{(C_E(\ell_R^2)+C_E(\ell_{LR}^2))^*} \leqq \|(B_k)_1^K\|_{C_{E^*}(\ell_S^2)} \leqq \|(B_k)_1^K\|_{(C_E(\ell_R^2)+C_E(\ell_R^2))^*}.$$

(I.8) and triangular inequality in $\ell^2(C_E)$ imply

(I.13) $$\left(\sum_1^K \|A_k\|^2_{C_E}\right)^{1/2} \leqq \|(A_k)_1^K\|_{C_E(\ell_R^2)+C_E(\ell_L^2)}.$$

(I.9) and triangle inequality in $L^2(\mu, C_E)$ imply that

(I.14) $$\left(\int\left\|\sum_1^K \varphi_k(t)A_k\right\|^2_{C_E} d\mu\right)^{1/2} \leqq \|(A_k)_1^K\|_{C_E(\ell_R^2)+C_E(\ell_L^2)}$$

where $(\varphi_k)_1^K$ is an orthogonal sequence in $L^2(\mu)$.

II. The case $X=C_1$ or X is the predual of a Von Neumann algebra

We treat the case $X=C_1$ first. We prove a more precise version of Theorem 0.3.

Theorem II.1. a) *Let P be the Paley projection: $H^1 \to H^1$, then*

(i) $\hat{P}\otimes \mathrm{Id}$: $H^1(C_1)\to C_1(\ell_R^2)+C_1(\ell_L^2)$ *is a bounded operator with norm less than* $1+\sqrt{2}$.

(ii) $P\otimes \mathrm{Id}$: $H^1(C_1)\to H^1(C_1)$ *is a bounded operator with norm less than* $1+\sqrt{2}$.

b) *Let* $m\colon H^1\to\ell^2$ *be a bounded multiplier. Then*

(i) $m\otimes\mathrm{Id}\colon H^1(C_1)\to C_1(\ell^2_R)+C_1(\ell^2_L)$ *is bounded with norm less than* $2\|m\|$.

(ii) $m\otimes\mathrm{Id}\colon H^1(C_1)\to\ell^2(C_1)$ *is bounded.*

(iii) $\hat{m}\otimes\mathrm{Id}\colon H^1(C_1)\to H^2(C_1)$ *is bounded.*

Assertion (b)(ii) already appeared in [BP, Theorem 3.3]. Before we proceed to the proof of this theorem we state an obvious consequence of (a) and (I.14), namely that $H^1_\Lambda(C_1)$ is canonically isomorphic to $C_1(\ell^2_R)+C_1(\ell^2_L)$ (take $\varphi_k(t)=e^{i2^kt}$, $k\in\mathbf{N}$ in (I.14)):

Corollary II.2 (Khintchine inequalities in C_1). *Let* $A_1,\dots,A_K\in C_1$

$$\|(A_k)_1^K\|_{C_1(\ell^2_R)+C_1(\ell^2_L)}\leqq\left(1+\sqrt{2}\right)\int\left\|\sum\nolimits_1^K e^{i2^kt}A_k\right\|_{C_1}dt\leqq\left(1+\sqrt{2}\right)\|(A_k)_1^K\|_{C_1(\ell^2_R)+C_1(\ell^2_L)}.$$

By (0.6) this implies Theorem 0.1.

Proof of Theorem II.1. Assertions (a) are a special case of assertions (b) except for the value of the constant. As ℓ^1 is 2-concave, assertions (b.ii), (b.iii) are consequences of (b.i) and (I.13), (I.14) (take $\varphi_k(t)=e^{ikt}$, $k\in\mathbf{N}$ in (I.14)). We now prove (b.i). The proof relies on the following theorem of Sarason [S, p. 198 and Theorem 4]:

Let $f\in H^1(C^1)$. Then there exist $g,h\in H^2(C_2)$ such that

(α) $f(t)=g(t)h(t)$ a.s. on $\mathbf{T}$

(β) $\|f\|_{H^1(C_1)}=\|g\|_{H^2(C_2)}\|h\|_{H^2(C_2)}$.

We now use the same method as in the scalar case. Let us first fix some notations which we will keep throughout this paper: let f, g, h be as above, $k\in\mathbf{N}$

$$\hat{f}(k)=\int g(t)h(t)e^{-ikt}\,dt=\sum\nolimits_{0\leqq q\leqq k}\hat{g}(q)\hat{h}(k-q)=A_k+B_k$$

where

$$A_k=\sum\nolimits_{0\leqq q\leqq k/2}\hat{g}(q)\hat{h}(k-q)=\int g(t)H_k(t)e^{-ikt}\,dt,\quad q\in\mathbf{N},$$

$$H_k(t)=\sum\nolimits_{k/2\leqq p\leqq k}\hat{h}(p)e^{ipt},\quad p\in\mathbf{N},\quad H_k(t)e^{-ikt}=\sum\nolimits_{0\leqq q\leqq k/2}\hat{h}(k-q)e^{-iqt}$$

and similarly

$$B_k=\sum\nolimits_{k/2<p\leqq k}\hat{g}(p)\hat{h}(k-p)=\int G_k(t)h(t)e^{-ikt}\,dt,\quad p\in\mathbf{N},$$

$$G_k(t)=\sum\nolimits_{k/2<p\leqq k}\hat{g}(p)e^{ipt},\quad G_k(t)e^{-ikt}=\sum\nolimits_{k/2<p\leqq k}\hat{g}(p)e^{i(p-k)t}.$$

In particular $A_0=\hat{g}(0)\hat{h}(0)$, $B_0=0$. As the finite sums $\sum_0^K e^{ikt}\otimes x_k$ are dense in $H^1(C_1)$ we may assume that $\hat{f}(k)=0$ for $k>K$. Let $m=(m_k)_0^\infty$ be a bounded

$H^1-\ell^2$ multiplier. We claim that

$$(\gamma) \qquad \|(m_k A_k)_0^K\|_{C_1(\ell_L^2)} = \left\|\left(\sum_0^K |m_k|^2 A_k A_k^*\right)^{1/2}\right\|_{C_1} \leqq \|m\| \|g\|_{H^2(C^2)} \|h\|_{H^2(C^2)}$$

and

$$(\delta) \qquad \|(m_k B_k)_1^K\|_{C_1(\ell_R^2)} = \left\|\left(\sum_1^K |m_k|^2 B_k^* B_k\right)^{1/2}\right\|_{C_1} \leqq \|m\| \|g\|_{H^2(C^2)} \|h\|_{H^2(C^2)}$$

which implies by (β)

$$\left\|(m_k \hat{f}(k))_0^K\right\|_{C^1(\ell_R^2)+C_1(\ell_L^2)} \leqq 2\|m\| \|f\|_{H^1(C_1)}$$

hence proves (b.i).

We now prove (γ). By Jensen's inequality in $C_1(\ell_L^2)$

$$\|(m_k A_k)_0^K\|_{C_1(\ell_L^2)} = \left\|\int (g(t) m_k H_k(t) e^{-ikt})_0^K dt\right\|_{C_1(\ell_L^2)}$$

$$\leqq \int \left\|(g(t) m_k H_k(t) e^{-ikt})_0^K\right\|_{C_1(\ell_L^2)} dt = \int \left\|\left(\sum_0^K |m_k|^2 H_k(t) H_k^*(t)\right)^{1/2} g^*(t)\right\|_{C_1} dt$$

$$\leqq \int \left\|\left(\sum_0^K |m_k|^2 H_k(t) H_k^*(t)\right)^{1/2}\right\|_{C_2} \|g^*(t)\|_{C_2} dt$$

$$\leqq \|g\|_{H^2(C_2)} \left(\int \left\|\sum_1^K |m_k|^2 H_k(t) H_k^*(t)\right\|_{C_1} dt\right)^{1/2}$$

and

$$\int \left\|\sum_0^K |m_k|^2 H_k(t) H_k^*(t)\right\|_{C_1} dt = \left\|\int \left(\sum_0^K |m_k|^2 H_k(t) H_k^*(t)\right) dt\right\|_{C_1}$$

$$= \left\|\sum_0^K |m_k|^2 \sum_{k/2 \leqq p \leqq k} \hat{h}(p) \hat{h}(p)^*\right\|_{C_1} = \left\|\sum_{p \geqq 0} \hat{h}(p) \hat{h}(p)^* \sum_{p \leqq k \leqq 2p} |m_k|^2\right\|_{C_1}$$

$$\leqq \|m\|^2 \|h\|_{H^2(C_2)}^2 .$$

This proves (γ): (δ) is proved similarly, replacing $H_k(t)$ by $G_k(t)$. This time there is no overlap between the blocks hence we actually get $\sup_{p \geqq 0} \sum_{p<k \leqq 2p} |m_k|^2$ instead of $\|m\|^2$, which gives finally the constant $1+\sqrt{2}$ for the Paley projection. This ends the proof of Theorem II.1. If we consider the lacunary sequence (3^k) instead of (2^k) there is no overlap for the blocks $H_{3^k}(t)$ and we get the constant 2 instead of $1+\sqrt{2}$.

Using known results, it is routine to extend Theorem II.1 and Corollary II.2 to the case when C_1 is replaced by the predual X of a von Neumann algebra X^*. (Such an X is called a "non-commutative L_1-space".) Indeed all the ingredients for the proof exist in the literature and they are discussed at length in the paper [HP] to which we refer the interested reader. Actually the analogue of Sarason's theorem remains true [HP, Corollary 2.5] but the weaker form [HP, Theorem 2.2] suffices. We merely make precise the definitions of $X(\ell_R^2)$ and $X(\ell_L^2)$: We may always embed X^* as a closed C^* subalgebra of $B(H)$. We then define $X^*(\ell_R^2)$ as the norm closure in $B(H)(\ell_R^2)$ of the space of finitely supported sequences (B_k) in X^* equipped with the norm

$$\|(B_k)\|_{X^*(\ell_R^2)} = \left\|\left(\sum B_k^* B_k\right)^{1/2}\right\|_{B(H)} .$$

If (A_k) is a finitely supported sequence in X we define by duality

$$\|(A_k)\|_{X(\ell_R^2)} = \sup\left\{\sum \langle A_k, B_k\rangle \mid B_k \in X^* \|(B_k)\|_{X^*(\ell_R^2)} \leqq 1\right\}.$$

Clearly the 2-convexity inequality expressed by (I.6) holds in $B(H)$ and X^*, hence by duality we have a 2-concavity inequality

$$\text{(II.1)} \qquad \left(\sum \|A_k\|_X^2\right)^{1/2} \leqq \|(A_k)\|_{X(\ell_R^2)}.$$

As a consequence of (II.1) we also have

$$\text{(II.2)} \qquad \left(\int \left\|\sum_1^K \varphi_k(t) A_k\right\|_X^2 d\mu(t)\right)^{1/2} \leqq \|(A_k)_1^K\|_{X(\ell_R^2)}$$

for any orthonormal sequence (φ_k) in $L^2(\mu)$.

We note in passing that (II.1) combined with Theorem II.1 with X replacing C_1 yields a new proof of the fact that X has cotype 2, which was first proved in [TJ 1].

III. The case $X = C_E$

We will use the notion of a "UMD-space" or the equivalent notion of an "HT-space". We refer to the survey [RF] for more details on UMD-spaces. We will say that a Banach space is HT if the Hilbert transform is bounded on $L_p(X)$ for some (or equivalently all) $1<p<\infty$. By results of Burkholder and Bourgain (see [B1], [Bu]) it is known that X is HT iff X is UMD. It is apparently not known whether "E UMD" implies "C_E UMD". However, it is easy to see that if C_E is HT, then $C_{E^{(2)}}$ also is HT. Indeed, this can be shown by essentially the same proof as for C_p (cf. [RF] proof of Proposition 3) as follows: if $\mathscr{H}$ denotes the vector valued Hilbert transform acting on $L_p(C_p)$ we have the classical identity

$$\mathscr{H}(f)^2 = f^2 + \mathscr{H}(f\mathscr{H}(f) + \mathscr{H}(f)f),$$

hence if $M(E,p)$ denotes the norm of $\mathscr{H}\colon L^p(C_E) \to L^p(C_E)$ we obtain

$$M(E^{(2)}, 2p) \leqq 2(1 + M(E,p)^2)^{1/2} \quad (1 < p < \infty).$$

This is a classical trick going back to an idea of M. Riesz exploited by Cotlar [Co]; the non-commutative version of this trick has been known for a long time (the second author learnt it from Paul Muhly back in 1976). Incidentally, the interested reader will find presented in ([LT], pp. 154—155) a version of the "Cotlar trick" adapted to martingale transforms in the scalar case.

This trick combined with interpolation and duality implies that C_p $(1<p<\infty)$ ([RF, Proposition 3]) and C_{pq} $(1<p<\infty,\ 1<q<\infty)$ ([A, Corollary 2.10]) are HT hence UMD.

We will prove a generalization of Theorem 0.3 and Corollary 0.4 as follows.

Theorem III.1. *Let E be a symmetric sequence space such that $C_{E^{(2)}}$ is* HT. *Let $m=(m_k)_{k\geq 0}$ be an $H^1-\ell^2$ multiplier (in particular $m=\hat{P}$). Then*

(i) $m\otimes \mathrm{Id}\colon H^1(C_E)\to C_E(\ell^2_R)+C_E(\ell^2_L)$ *is bounded.*

(ii) $P\otimes \mathrm{Id}\colon H^1(C_E)\to H^1(C_E)$ *is bounded.*

(iii) *If moreover E is 2-concave (for example $C_E=C_p$ $1\leq p\leq 2$) $m\otimes \mathrm{Id}\colon H^1(C_E)\to H^1(C_E)$ is bounded.*

The case E 2-convex will be considered in Remark III.5 below.

Proof. (iii) is an immediate consequence of (i) and (I.14) applied for $\varphi_k(t)=e^{i2^k t}$.

In order to prove (i) and (ii) we keep the notation of the proof of Theorem II.1, replacing C_1 and C_2 by C_E and $C_{E^{(2)}}$.

Sarason's theorem [S, Theorem 4 and proof p. 204] together with (I.1) implies that for every $f\in H^1(C_E)$ there exist $g, h\in H^2(C_{E^{(2)}})$ such that

$$(\alpha)\qquad f(t) = g(t)h(t) \quad \text{a.s. on } \mathbf{T}$$

$$(\beta)\qquad \|f\|_{H^1(C_E)} = \|g\|_{H^2(C_{E^{(2)}})}\|h\|_{H^2(C_{E^{(2)}})}.$$

(i) The proof is analogous to the proof of Theorem II.1 up to the majoration of $\int \left\|\sum_0^K |m_k|^2 H_k(t)H_k^*(t)\right\|_{C_E} dt$. We claim that there exists a constant $C>0$ such that

$$\text{(III.1)}\qquad \int \left\|\sum_0^K |m_k|^2 H_k(t)H_k^*(t)\right\|_{C_E} dt \leq C\,\|m\|^2\|h\|^2_{H^2(C_{E^{(2)}})}$$

and similarly for the G_k's which concludes the proof as in Theorem II.1.

In order to prove this claim, let $I_0=\{0\}$ and $I_j=[2^{j-1}, 2^j[$ $(j\geq 1)$ be the dyadic intervals in $\mathbf{N}$. We recall that $\|m\|^2$ is equivalent to $\sup_{j\geq 0}\sum_{k\in I_j}|m_k|^2$. In particular $(2^{-1}\|m\|^{-2}|m_k|^2)_0^{2^K}$ lies in the unit ball of the space $(\oplus_{0\leq j\leq K}\ell^1(I_j))_\infty$. The extreme points of the convex set of positive sequences in this ball are sequences $(r_k)_0^{2^K}$ such that $r_k\in\{0, 1\}$ for every k and $r_k=1$ for at most one k in each I_j $(0\leq j\leq K)$. We call such a sequence (r_k) a Marcinkiewicz multiplier and we denote by $(k_\ell)_0^L$ the increasing sequence of integers such that $r_k=1$ iff $k\in(k_\ell)_0^L$. Inequality (III.1) will be proved for any $H^1-\ell^2$ multiplier m as soon as we prove it for Marcinkiewicz multipliers.

As $E^{(2)}$ is 2-convex, (I.7) applied for $\varphi_\ell=\varepsilon_\ell$ gives

$$\text{(III.2)}\qquad \left\|\left(\sum_0^K |r_k|^2 H_k(t)H_k^*(t)\right)^{1/2}\right\|_{C_{E^{(2)}}} = \left\|\left(\sum_0^L H_{k_\ell}(t)H_{k_\ell}^*(t)\right)^{1/2}\right\|_{C_{E^{(2)}}}$$

$$\leq \left(\int \left\|\sum_0^L \varepsilon_\ell H_{k_\ell}(t)\right\|^2_{C_{E^{(2)}}} d\mu\right)^{1/2}$$

for almost all $t\in\mathbf{T}$.

For any choice of signs (ε_ℓ) we compute

$$\sum_0^L \varepsilon_\ell H_{k_\ell}(t) = \sum_{\ell=0}^L \varepsilon_\ell \sum_{k_\ell/2}^{k_\ell} \hat{h}(p)e^{ipt} = \sum_{p\geqq 0} a_p \hat{h}(p)e^{ipt}$$

where $a_p = \sum_{p\leqq k_\ell \leqq 2p} \varepsilon_\ell$ $(p\in\mathbf{N})$.

If $2^{\ell-1}\leqq p<2^\ell$ there exists a partition of $[2^{\ell-1}, 2^\ell[$ in at most M (M an absolute constant) disjoint intervals on which a_p is a constant integer, with values in $[-2,2]$. As $C_{E^{(2)}}$ is UMD and (a_p) satisfies the assumptions of [B2, Theorem 4] (a version of the Marcinkiewicz multiplier theorem) (see also [MC, Theorem 1.3]) we get that for any ε_ℓ $(\varepsilon_\ell=\pm 1)$

$$\text{(III.3)} \qquad \left\|\sum_0^L \varepsilon_\ell H_{k_\ell}(t)\right\|_{H^2(C_{E^{(2)}})} \leqq C\|h\|_{H^2(C_{E^{(2)}})}$$

where C is a positive constant. Combined with (III.2) this implies our claim (III.1) for Marcinkiewicz multipliers, hence for every $H^1-\ell^2$ multiplier.

(ii) Let $\hat{f}(2^k)=A_{2^k}+B_{2^k}$. We have

$$\int \left\|\sum_0^K e^{i2^kt}A_{2^k}\right\|_{C_E} dt = \int \left\|\sum_0^K e^{i2^kt}\int g(u)H_{2^k}(u)e^{-i2^ku}\,du\right\|_{C_E} dt$$
$$\leqq \iint \|g(u)\|_{C_{E^{(2)}}} \left\|\sum_0^K e^{i2^k(t-u)}H_{2^k}(u)\right\|_{C_{E^{(2)}}} du\,dt$$
$$\leqq \|g\|_{H^2(C_{E^{(2)}})} \left(\iint \left\|\sum_0^K e^{i2^ks}H_{2^k}(u)\right\|^2_{C_{E^{(2)}}} ds\,du\right)^{1/2}.$$

By (0.6) the last term is less than $C\left(\iint \left\|\sum_0^K \varepsilon_k H_{2^k}(u)\right\|^2_{C_{E^{(2)}}} du\,d\mu\right)^{1/2}$. By the proof of (i) above this is majorized by $C'\|h\|_{H^2(C_{E^{(2)}})}$. The computation is similar for the $B_k's$ and the triangle inequality in $H^1(C_E)$ ends the proof.

The Paley projection can be replaced by any Marcinkiewicz multiplier. But we do not know if (iii) holds true under the only assumption that $C_{E^{(2)}}$ is UMD.

Remark III.2. In order to prove (i) it could seem more natural to compute $\|(m_kA_k)_0^K\|_{C_E(\ell_L^2)} = \left\|\int (\bar{m}_k e^{ikt}H_k^*(t)g^*(t))_0^K dt\right\|_{C_E(\ell_R^2)}$ by using (I.4) in C_E with

$$A(t) = \begin{pmatrix} \bar{m}_0 H_0^*(t) & 0 \dots \\ \vdots & \\ \bar{m}_K e^{iKt} H_K^*(t) & 0 \dots \\ 0 & 0 \dots \\ \vdots & \vdots \end{pmatrix}$$

but we could not majorize $\left\|\left(\int A(t)A^*(t)dt\right)^{1/2}\right\|_{C_{E^{(2)}}}$ properly.

Remark III.3. Theorem III.1(i) and (I.13) imply that $m\otimes \mathrm{Id}\colon H^1(C_E)\to\ell^2(C_E)$ is bounded when $C_{E^{(2)}}$ is UMD and E is 2-concave. This is proved in [BP] for any Banach space X such that X^* has type 2.

As in Part II, Theorem III.1 and (I.14) imply the following:

Corollary III.4. (Khintchine inequalities). *Let $C_E=C_p$ $(1<p<2)$ or more generally let E be 2-concave and $C_{E^{(2)}}$ be* UMD. *There exists a positive constant M such that for every finite sequence $(A_k)_1^K$ in C_E*

$$\|(A_k)_1^K\|_{C_E(\ell_R^2)+C_E(\ell_L^2)} \leqq M\Big(\int \Big\|\sum_1^K e^{i2^k t}A_k\Big\|_{C_E}^2 dt\Big)^{1/2} \leqq M\|(A_k)_1^K\|_{C_E(\ell_R^2)+C_E(\ell_L^2)}.$$

By (I.13) we obtain as a consequence that C_E has cotype 2: this was proved in [TJ1] for C_p $(1<p<2)$ and in [TJ2] for all 2-concave E's. The inequalities in this corollary were already proved in [LP1] for C_p $(1<p<2)$ by a different method, with a constant M_p, $M_p\to+\infty$ as $p\to 1$.

Theorem IV.4 in the appendix to this paper, combined with the main result of [LP2] gives another proof of Corollary III.4 for a large class of 2-concave E's.

Remark III.5. When E is 2-convex, the following result is better than Theorem III.1(i):

Let E be a 2-convex symmetric sequence space. Then for every $H^1-\ell^2$ multiplier m, $m\otimes\mathrm{Id}\colon H^1(C_E)\to C_E(\ell_S^2)$ is bounded, with norm less than $2\|m\|$.

Actually every $f\in H^1(C_E)$ can be written as $f=gh$ a.s. on $\mathbf{T}$, where g is scalar valued, and $\|f\|_{H^1(C_E)}=\|g\|_{H^2}\|h\|_{H^2(C_E)}$. With the notation of Part II and Remark III.2,

$$\|(m_k B_k)_0^K\|_{C_E(\ell_S^2)} \leqq \int \Big(\sum_0^K |m_k|^2 |G_k(t)|^2\Big)^{1/2} \||h(t)|_s\|_{C_E} dt \leqq \|m\|\|g\|_{H^2}\|h\|_{H^2(C_E)}$$

$$\|(m_k A_k)_0^K\|_{C_E(\ell_L^2)} = \Big\|\int A(t)\bar g(t)\,dt\Big\|_{\mathscr{C}_E} \leqq \|g\|_{H^2}\Big\|\Big(\int A^*(t)A(t)\,dt\Big)^{1/2}\Big\|_{\mathscr{C}_E}.$$

As E is 2-convex let F be such that $E=F^{(2)}$. By the same computation as in Theorem II.1

$$\Big\|\Big(\int A^*(t)A(t)\,dt\Big)^{1/2}\Big\|_{\mathscr{C}_E} = \Big\|\int \Big(\sum_0^K |m_k|^2 H_k(t)H_k^*(t)\Big)dt\Big\|_{C_F}^{1/2} \leqq \|m\|\|h\|_{H^2(C_E)}.$$

The computation is similar for $\|(m_k A_k)_0^K\|_{C_E(\ell_R^2)}$, which gives the result by (I.11).

We recall that if E is 2-convex the identity mapping: $\ell^2(C_E)\to C_E(\ell_S^2)$ has norm 1 by (I.7) and (I.11), as well as the canonical mapping: $H^2(C_E)\to C_E(\ell_S^2)$, $f\to(\hat f(k))_{k\geqq 0}$. In general $C_E(\ell_S^2)$ cannot be replaced by $H^2(C_E)$ in the assertion of this remark ($P\otimes\mathrm{Id}\colon H^1(C^\infty)\to H^1(C^\infty)$ is not bounded) nor by $\ell^2(C_E)$ (if $\hat P\otimes\mathrm{Id}\colon H^1(C_E)\to\ell^2(C_E)$ is bounded C_E has cotype 2 hence E is 2-concave).

Let us now restrict to the case $m=\hat P$. By the assertion above if E is 2-convex $\hat P\otimes\mathrm{Id}\colon H^1(C_E)\to C_E(\ell_S^2)$ is bounded. If moreover $C_E(\ell_S^2)$ is canonically isomorphic to $H_\Lambda^2(C_E)$ we get that $P\otimes\mathrm{Id}\colon H^1(C_E)\to H^1(C_E)$ is bounded. C_E is K-convex iff $H_\Lambda^2(C_E)$ is in norming duality with $H_\Lambda^2(C_{E^*})$. If moreover the Khintchine inequalities as in Corollary III.4 hold in $H_\Lambda^2(C_{E^*})$, then $H_\Lambda^2(C_E)$ and $C_E(\ell_S^2)$ are canonically isomorphic. Actually (see the introduction and [BP]) it was already known that $P\otimes\mathrm{Id}\colon H^1(C_E)\to H^1(C_E)$ is bounded if C_E is K-convex. We recall that a Banach space is K-convex iff it has a type >1 [P4]. C_E has type 2 if E is 2-convex and q-concave for a finite q [TJ 2, Proposition 2].

Remark III.6. The spaces C_E in the above results can be replaced by the non commutative $L^p(M, \tau)$ spaces $(1<p<\infty)$ where M is a Von Neumann algebra and τ a semi-finite normal faithful trace. We refer to [N] for their definition and properties. The extension of Sarason's theorem is proved in [X, Théorème 2.1]; $L^{2p}(M, \tau)$ is UMD [BGM, § 6]; the extensions of (I.7), (I.14), (I.13) are proved in the same way as in Part I: $L^p(M, \tau)(\ell^2_R)$ is defined similarly as $C_p(\ell^2_R)$, it can be identified with a 1-complemented subspace of $L^p(\tilde{M}, \tilde{\tau})$ where M is viewed as a w^* closed subalgebra of $B(H)$, $\tilde{H}$ is a Hilbertian sum of countable copies of H, $\tilde{M}=\{(B_{ij})_{i,j\geq 1} | \forall i,j,\ B_{ij}\in M,\ \|(B_{ij})_{ij}\|_{B(\tilde{H})}<+\infty\}$ and $\tilde{\tau}((B_{ij})_{ij})=\sum_{i\geq 1}\tau(B_{ii})$.

Finally we give an extension of Theorem III.1(ii) in a different direction. We consider the projective tensor product $X\hat{\otimes} Y$ of two Banach spaces X and Y. Then Theorem III.1(ii) can be generalized as soon as we have a substitute for Sarason's theorem.

Theorem III.7. *Let X and Y be* HT *Banach spaces such that the natural product map*

$$H^2(X)\hat{\otimes} H^2(Y) \to H^1(X\hat{\otimes} Y)$$

is surjective. Then

a) $$H^2_A(X\hat{\otimes} Y) = H^2_A(X)\hat{\otimes} Y + X\hat{\otimes} H^2_A(Y),$$

b) $$P\otimes \mathrm{Id}\colon H^1(X\hat{\otimes} Y) \to H^1(X\hat{\otimes} Y) \quad \textit{is bounded.}$$

By [P5, Theorem 3.1] the natural product map is surjective if X and Y have type 2.

For any Banach space Z let $\mathrm{Rad}\, Z$ denote the closed span of $\{\varepsilon_k\otimes z | k\in \mathbf{Z},\ z\in Z\}$ in $L^2(D, \mu, Z)$. By (0.6) the assertion (a) above is equivalent to the following identity (with equivalent norms)

$$\mathrm{Rad}\,(X\hat{\otimes} Y) = \mathrm{Rad}\,(X)\hat{\otimes} Y + X\hat{\otimes}\,\mathrm{Rad}\,(Y).$$

Note that assertion (a) applied for $X=Y=\ell^2$ implies Corollary II.2. The space $C_1(\ell^2_R)$ can be identified isometrically with $\ell^2\hat{\otimes}\ell^2(\ell^2)$ or equivalently with $\ell^2\hat{\otimes} H^2_A(\ell^2)$ and similarly for $C_1(\ell^2_L)$. This is easy to verify by considering the dual norms.

Proof. Let us first assume that $f(t)=g(t)\otimes h(t)$ a.e. on $\mathbf{T}$ and $\|f\|_{H^1(X\hat{\otimes} Y)}=\|g\|_{H^2(X)}\|h\|_{H^2(Y)}$. We define $A_k=\int g(t)\otimes H_k(t)e^{-ikt}\,dt$, $B_k=\int G_k(t)\otimes h(t)e^{-ikt}\,dt$, for $k\geq 0$, where G_k, H_k are defined as in the proof of Theorem II.1. Replacing

C_E by $X\hat{\otimes} Y$ and $C_{E^{(2)}}$ by X or Y in the proof of Theorem III.1(ii) we obtain

$$\left\|\sum_0^K e^{i2^k t} A_k\right\|_{X\hat{\otimes} H^2_A(Y)} = \left\|\int g(u)\otimes \sum_0^K e^{i2^k(t-u)} H_k(u)\, du\right\|_{X\hat{\otimes} H^2_A(Y)}$$

$$\leqq \int \left\|g(u)\otimes \sum_0^K e^{i2^k(t-u)} H_k(u)\right\|_{X\hat{\otimes} H^2_A(Y)} du = \int \|g(u)\|_X \left\|\sum_0^K e^{i2^k(t-u)} H_k(u)\right\|_{H^2_A(Y)} du$$

$$\leqq \|g\|_{H^2(X)} \left(\iint \left\|\sum_0^K e^{i2^k(t-u)} H_k(u)\right\|_Y^2 dt\, du\right)^{1/2}.$$

By the HT assumption on Y, (0.6) and [B2, Theorem 4] the last term is less than $C\|h\|_{H^2(Y)}$ by the same argument as in Theorem III.1(ii). The majoration of $\left\|\sum_0^K e^{i2^k t} B_k\right\|_{H^2_A(X)\hat{\otimes} Y}$ is similar.

By assumption every F in $H^1(X\hat{\otimes} Y)$ lies in the closed convex hull of f's as above with $\|f\|_{H^1(X\hat{\otimes} Y)} \leqq \|F\|_{H^1(X\hat{\otimes} Y)}$ which proves that

$$P\otimes \mathrm{Id}_{X\hat{\otimes} Y}\colon H^1(X\hat{\otimes} Y) \to X\hat{\otimes} H^2_A(Y) + H^2_A(X)\hat{\otimes} Y$$

is bounded. As the identity: $X\hat{\otimes} H^2_A(Y) + H^2_A(X)\hat{\otimes} Y \to H^2_A(X\hat{\otimes} Y)$ obviously has norm 1 we have proved (a) and (b).

Remark III.7. For the "probability oriented" reader, we should mention that the preceding theorem as well as Theorems 0.1 and 0.2 remain valid if we replace the sequence (ε_n) or the sequence $(e^{i2^n t})$ by a sequence of independent normal Gaussian random variables. This follows from well-known results (cf. Proposition 3.2 in [P4]).

Appendix

We will prove that the Khintchine inequalities for C_E (E 2-concave) as stated above are equivalent to a factorization theorem for bounded operators: $C_{E^*} \to H$ (H a Hilbert space).

We first recall a weak form of Grothendieck's theorem:

Theorem IV.1 (cf. [P3], Theorem 5.4]). *Let $C(K)$ be the space of continuous functions on a compact set K. Let T be a bounded operator: $C(K)\to H$. Then there exists a probability measure μ on K such that*

$$\forall \varphi\in C(K) \quad \|T(\varphi)\|_H \leqq \sqrt{\frac{\pi}{2}}\, \|T\| \|\varphi\|_{L^2(\mu)}.$$

The proof in [P3, Theorem 5.4] uses Khintchine Gaussian inequalities in L^1 spaces. Other proofs are given in [P3].

This theorem was generalized as follows:

Theorem IV.2 (cf. [P1] and [P3, Theorem 9.4]). *Let A be a C^* algebra and let T be a bounded operator: $A \to H$. Then there exists a state f on A such that*

$$\forall x \in A \quad \|T(x)\|_H \leqq 2\|T\| \left\langle f, \frac{x^*x+xx^*}{2} \right\rangle^{1/2}.$$

We wish to replace in the preceding statements a $C(K)$ space by a 2-convex Banach lattice and a C^* algebra A by a space C_E with E 2-convex.

Proposition IV.3. *Let X be a Banach lattice, or a C^*-algebra or $X=C_E$, let $X^{(2)}$ denote respectively the 2-convexification of X, or X, or $C_{E^{(2)}}$. For any Banach space Z and any bounded operator $T\colon X^{(2)} \to Z$ the following assertions are equivalent:*

(i) *there exist $C>0$ and $f \in X^*, f \geqq 0, \|f\|_{X^*}=1$ such that*

$$\forall x \in X^{(2)} \quad \|T(x)\| \leqq C\|T\| \left\langle f, \frac{x^*x+xx^*}{2} \right\rangle^{1/2};$$

(ii) *for every finite sequence (x_i) in X*

$$\sum_1^n \|T(x_i)\|^2 \leqq C^2\|T\|^2 \left\| \sum_1^n \frac{x_i^*x_i+x_ix_i^*}{2} \right\|_X.$$

The proof is an application of the Hahn—Banach theorem, it is similar to the proof of [P1, Proposition 1.1].

When X is a Köthe function space on $(\Omega, \sum, \mu)$ [LT, 1.b.7] and Z is a Hilbert space, Theorem IV.1 easily implies inequality (ii) above (cf. [M, proof of Theorem 28]): put $x_i=h_i(\sum_1^n |x_i|^2)^{1/2}$ $(1 \leqq i \leqq n)$ hence $\sum_1^n |h_i|^2=1$, consider the operator $\tilde{T}\colon L^\infty(\mu) \to H$, $h \to T(h(\sum_n^1 |x_i|^2)^{1/2})$ and apply Theorems IV.1 and IV.3 to $\tilde{T}$.

This argument does not seem to work for C_E spaces. The main result of this appendix is the following:

Theorem IV.4. *Let X be as in Proposition IV.3. Let Y be either the dual space of $X^{(2)}$ or its predual if $X^{(2)}$ is a dual space. The following assertions are equivalent:*

(i) *there exists a constant $C>0$ such that for every bounded operator $T\colon X^{(2)} \to H$ (Hilbert space) and every finite sequence (x_i) in $X^{(2)}$*

$$\sum_1^n \|T(x_i)\|^2 \leqq C^2\|T\|^2 \left\| \sum_1^n \frac{x_ix_i^*+x_i^*x_i}{2} \right\|_X;$$

(ii) *The Khintchine inequalities hold in Y, i.e. there exists a constant $M>0$ such that for every finite sequence (y_i) in Y*

$$\sup \left\{ \sum_1^n \langle x_i, y_i \rangle \,\middle|\, \left\|\sum_1^n |x_i|_S^2\right\|_X = 1 \right\} \leqq M \left(\int \left\| \sum_1^n \varepsilon_i y_i \right\|_Y^2 d\mu \right)^{1/2} = M\|(y_i)\|.$$

In particular Theorems IV.2 and IV.4 imply Corollary II.2. On the other hand Corollary III.4 together with Theorem IV.4 and Proposition IV.3 imply for example:

Corollary IV.5. *There exists a constant* $C>0$ *such that for every bounded operator* $T\colon C_{2p}\to H$ $(1<p<\infty)$ *there exists* $f\geqq 0$, $\|f\|_{C_{p'}}=1$ $\left(\frac{1}{p}+\frac{1}{p'}=1\right)$ *such that*

$$\forall x\in C_{2p}\quad \|T(x)\|\leqq C\|T\|\langle f, |x|_S^2\rangle^{1/2}.$$

Such a factorization theorem is proved directly in [LP 2] for a large class of spaces $C_{E^{(2)}}$.

Proof of Theorem IV.4. (ii)⇒(i) is proved by duality:

$$\begin{aligned}\sum_1^n \|T(x_i)\|_H^2 &= \sup\left\{\sum_1^n \langle T(x_i), z_i\rangle \,\Big|\, \sum_1^n \|z_i\|_H^2 \leqq 1\right\}\\ &= \sup\left\{\sum_1^n \langle T(x_i), z_i\rangle \,\Big|\, \int \Big\|\sum_1^n \varepsilon_i z_i\Big\|_H^2 d\mu \leqq 1\right\}\\ &\leqq \sup\left\{\sum_1^n \langle x_i, T^*(z_i)\rangle \,\Big|\, \int \Big\|\sum_1^n \varepsilon_i T^*(z_i)\Big\|_{X^{(2)*}}^2 d\mu \leqq \|T\|^2\right\}\\ &\leqq \sup\left\{\sum_1^n \langle x_i, y_i\rangle \,\Big|\, \int \Big\|\sum_1^n \varepsilon_i y_i\Big\|_{X^{(2)*}}^2 d\mu \leqq \|T\|^2\right\}\\ &= \sup\left\{\sum_1^n \langle x_i, y_i\rangle \,\Big|\, \int \Big\|\sum_1^n \varepsilon_i y_i\Big\|_Y^2 d\mu \leqq \|T\|^2\right\}\\ &\leqq \sup\left\{\sum_1^N \langle x_i, y_i\rangle \,\Big|\, y_i\in Y, \|(y_i)_1^n\|_{X^{(2)}(\ell_S^2)^*} \leqq M\|T\|\right\} \quad \text{by (ii)}\\ &= M\|T\|\,\|(x_i)_1^n\|_{X^{(2)}(\ell_S^2)}\end{aligned}$$

which proves (i) with $C\leqq M$.

(i)⇒(ii): let $(y_i)_1^n$ be a finite sequence in Y and let

$$u\colon \ell_n^2\to Y\quad e_i\to y_i\quad (1\leqq i\leqq n)$$

where $(e_i)_1^n$ denotes the canonical basis of ℓ_n^2.

By the Pietsch factorization theorem for 2-summing operators there exist a Hilbert space H and two linear operators $S\colon \ell_n^2\to H$, $R\colon H\to Y$ such that

$$u=R\circ S;\quad \|R\|_{H\to Y}\leqq 1;\quad \pi_2(S)\leqq \pi_2(u)$$

where π_2 denotes the 2-summing norm.

Let $(x_i)_1^n$ be a sequence in $X^{(2)}$. Then, by (i) applied to R^*,

$$\begin{aligned}\Big|\sum_1^n \langle x_i, y_i\rangle\Big| &= \Big|\sum_1^n \langle x_i, u(e_i)\rangle\Big| = \Big|\sum_1^n \langle R^*(x_i), S(e_i)\rangle\Big|\\ &\leqq \Big(\sum_1^n \|R^*(x_i)\|_H^2\Big)^{1/2}\Big(\sum_1^n \|S(e_i)\|_H^2\Big)^{1/2} \leqq C\|R^*\|\,\|(x_i)_1^n\|_{X^{(2)}(\ell_S^2)}\,\pi_2(S)\\ &\leqq C\pi_2(u)\,\|(x_i)_1^n\|_{X^{(2)}(\ell_S^2)}.\end{aligned}$$

Finally, by [P3, pp. 35—36] if Y has cotype 2 there is a constant such that

$$\pi_2(u) \leqq K\Big(\int \big\|\sum_1^n \varepsilon_i y_i\big\|_Y^2 \, d\mu\Big)^{1/2}.$$

This implies (ii) with $M \leqq CK$.

The only case not already mentioned is the case $X^{(2)} = C_{E^{(2)}}$. By [TJ2] if F is a 2-concave symmetric sequence space C_F has Steinhaus cotype 2, hence cotype 2 by [P2] and so has C_F^{**}. Let F be the dual of $E^{(2)}$ if $E^{(2)}$ is separable or its predual if $E^{(2)}$ is a dual space. Hence $Y = C_F$ or $Y = C_F^{**}$ has cotype 2, which proves our claim and concludes the proof.

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