Realization of the invariant inner products on the highest quotients of the composition series

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§ 1. Introduction

Let Z be an irreducible JB^*-algebra of finite dimension d and rank r. Let D be the open unit ball of Z; it is a bounded symmetric domain of tube type. We denote by $G=\mathrm{Aut}(D)_0$ the connected component of the identity in the group $\mathrm{Aut}(D)$ of all biholomorphic automorphisms of D. The isotropy subgroup at the origin

$$K = \{\varphi\in G;\ \varphi(0) = 0\} = G\cap GL(Z)$$

is a maximal compact subgroup.

In the recent work [FK] Faraut and Koranyi describe the Hilbert spaces of analytic functions on D which are invariant under the unitary action of G given by

$$U^{(\lambda)}(\varphi) = f\circ\varphi\cdot(J\varphi)^{\lambda/p},\quad \varphi\in G.$$

Here $J\varphi=\det(\varphi')$ is the complex Jacobian of φ, p is the genus of D (to be defined below) and λ ranges over a permissible set of non negative real numbers called the Wallach set. The formulas for the invariant inner products in [FK] are in terms of a certain orthogonal expansion of the functions (called the Peter—Weyl decomposition), which refines the homogeneous expansion.

The purpose of this work is to provide more concrete formulas (in terms of integrals of certain derivatives) for the invariant inner products, in the special cases of the highest quotients associated with points λ is the discrete part of the Wallach set. The main results are Theorems 12, 14, and 19 below.

Our formulas for the invariant inner products exhibit the invariant Hilbert spaces as certain Besov spaces. They can be used effectively to define duality in the

(1) Research partially supported by an NSF grant No. DMS 8702371.

invariant Banach spaces of analytic functions on D. Our results extend some known facts concerning the Dirichlet space in the context of the unit disk in the complex plane.

We fix some notation and terminology. For more details see [FK], [U1] and the references therein. Fix a frame $\{e_1, e_2, \ldots, e_r\}$ of minimal orthogonal tripotents, let $e=e_1+\ldots+e_r$. Let $Z=\sum_{1\leqq i\leqq j\leqq r}\oplus Z_{i,j}$ be the Peirce decomposition of Z relative to $\{e_r\}_{j=1}^r$ and let

$$Z_j = \sum_{1\leqq i\leqq l\leqq j}\oplus Z_{i,l}, \quad j = 1, 2, \ldots, r.$$

Z_j is a JB^*-algebra with unit $e_1+\ldots+e_j$. Let N_j be the determinant ("norm") polynomial of Z_j, $j=1, \ldots, r$. We denote $N_r=N$.

For $1\leqq i<j\leqq r$ let $a=\dim(Z_{i,j})$. It is known that a is independent of the particular choice of i, j with $1\leqq i<j\leqq r$. Thus $d=r+r(r-1)\frac{a}{2}$. The *genus* of D is $p=(r-1)a+2=2d/r$.

For two polynomials p, q let

$$(p, q)_F = \partial_p(q^*)(0)$$

be the *Fischer inner product*. Here $\partial p=p\left(\frac{\partial}{\partial z}\right)$ and $q(z^*)=\overline{q^*(z)}$. It is known that the Fischer inner product is given by

$$(p, q) = c\int_Z p(z)\overline{q(z)}\, e^{-\|z\|^2}\, dV(z) = \frac{1}{\pi^d}\int_Z p(z)\overline{q(z)}\, e^{-\|z\|^2}\, dz$$

where $dV(z)$ is the usual Lebesgue's volume measure and $\|z\|$ is the unique K-invariant inner product on Z, normalized so that the norm of a minimal tripotent is 1. The Fischer inner product is called the "Fock inner product" by some authors. In particular, the Fischer inner product is K-invariant:

$$(p\circ k, q\circ k)_F = (p, q)_F; \quad k\in K.$$

Let $S=\{k(e); k\in K\}$ be the *Shilov boundary* of D and let $d\sigma$ be the unique K-invariant probability measure on S. The Hardy space $H^2(S)$ is the completion of the space $P=P(Z)$ of analytic polynomials with respect to the inner product of $L^2(S, \sigma)$.

A *signature* is an r-tuple $\underline{m}=(m_1, m_2, \ldots, m_r)$ of integers satisfying $m_1\geqq m_2\geqq\ldots\geqq m_r\geqq 0$. The *conical polynomial* associated with the signature $\underline{m}$ is

$$N_{\underline{m}}(z) = N_1^{m_1-m_2}(z)\, N_2^{m_2-m_3}(z)\ldots N_r^{m_r}(z).$$

Let

$$P_{\underline{m}} := \operatorname{span}\{N_{\underline{m}}\circ k;\ k\in K\}.$$

It is known (see [S] and [U1]) that the spaces $\{P_{\underline{m}}\}_{\underline{m}}$ are precisely the irreducible K-orbits in the space P of analytic polynomials on D. Moreover, P admits a direct sum decomposition

$$P = \sum_{\underline{m}} \oplus P_{\underline{m}} \quad \text{(sum over all signatures)},$$

called the "Peter—Weyl decomposition", see [U1], which is orthogonal with respect to the Fischer inner product.

Let $L=\{k\in K;\ k(e)=e\}$. For each signature $\underline{m}$ consider the *spherical polynomial*

$$\varphi_{\underline{m}}(z) = \int_L N_{\underline{m}}\big(l(z)\big)\,dl$$

(dl is the Haar measure of L). $\varphi_{\underline{m}}$ is the unique L-invariant polynomial in $P_{\underline{m}}$ satisfying $\varphi_{\underline{m}}(e)=1$.

For $\lambda\in\mathbf{C}$ and a signature $\underline{m}$ we define

$$(\lambda)_{\underline{m}} = \prod_{j=1}^{r} \prod_{v=0}^{m_j-1} \left(\lambda+v-(j-1)\frac{a}{2}\right).$$

It is known [FK, Theorem 3.6] that

$$\|\varphi_{\underline{m}}\|_F^2 = \left(\frac{d}{r}\right)_{\underline{m}} \Big/ \dim P_{\underline{m}}.$$

The *Bergman kernel* $K(z,w)$ is the reproducing kernel of the space $L^2_a(D)$ of analytic functions in $L^2(D)=L^2(D,dV)$. Notice that $K(z,0)=1$ for all $z\in D$. The *Wallach set* $W(D)$ of D is the set of all $0\leqq\lambda$ so that $K(z,w)^{\lambda/p}$ is positive definite, i.e. $\sum_{i,j=1}^{n} a_i\bar a_j K(z_j,z_i)^{\lambda/p}\geqq 0$ for all finite sequences $\{z_j\}_{j=1}^n$ in D and $\{a_j\}_{j=1}^n$ in $\mathbf{C}$. It is known (see [FK]) that the Wallach set consists of a *discrete part* $W_d(D)=\{(v-1)\frac{a}{2}\}_{v=1}^r$ and a *continuous part* $W_c(D)=((r-1)\frac{a}{2},\infty)$. For $\lambda\in W(D)$ one defines an inner product $(\cdot,\cdot)_\lambda$ on $\mathscr{H}_\lambda^{(0)}:=\operatorname{span}\{K(\cdot,w)^{\lambda/p};\ w\in D\}$ via

$$\big(K(\cdot,w)^{\lambda/p}, K(\cdot,z)^{\lambda/p}\big)_\lambda = K(z,w)^{\lambda/p}$$

and let $\mathscr{H}_\lambda$ denote the completion of $\mathscr{H}_\lambda^{(0)}$. Define an action $U^{(\lambda)}$ of G on analytic functions on D by

$$U^{(\lambda)}(\varphi)f = (f\circ\varphi)\cdot(J\varphi)^{\lambda/p}$$

where $(J\varphi)(z)=\det\big(\varphi'(z)\big)$ is the complex Jacobian of φ at z, $p=(r-1)a+2$ is the genus of D and we used the principal branch of the power function. By the transformation rule

$$J\varphi(z)K\big(\varphi(z),\varphi(w)\big)\overline{J\varphi(w)} = K(z,w);\quad z,w\in D,\ \varphi\in G$$

one sees that for each $\varphi \in G$, $U^{(\lambda)}(\varphi)$ is a unitary operator on $\mathscr{H}_\lambda$. Unless λ/p is an integer the map $\varphi \mapsto U^{(\lambda)}(\varphi)$ is not continuous, nor is it an anti-homomorphism. There is however a natural way to extend $U^{(\lambda)}$ to an *anti-representation of the covering group* $\tilde{G}$ of G (see for instance [B], [U2]). This yields the important formula

$$U^{(\lambda)}(\varphi\psi) = c(\varphi, \psi, \lambda) \cdot U^{(\lambda)}(\psi) U^{\lambda}(\phi), \quad \varphi, \psi \in G,$$

where $c(\varphi, \psi, \lambda)$ is a unimodular number. Clearly, $U^{(\lambda)}(\text{id}) = I$ where "id" is the identity function on D.

The classification of the irreducible bounded symmetric domains up to biholomorphic isomorphism, due to E. Cartan, is the following.

$$D(\mathrm{I}_{n,m}) = \{z \in M_{n,m}(\mathbf{C});\ zz^* < I_n\}, \quad 1 \leqq n \leqq m;$$

$$D(\mathrm{II}_n) = \{z \in D(\mathrm{I}_{n,n});\ z^T = -z\}, \quad 5 \leqq n;$$

$$D(\mathrm{III}_n) = \{z \in D(\mathrm{I}_{n,n});\ z^T = z\}, \quad 2 \leqq n;$$

$$D(\mathrm{IV}_n) = \Big\{z \in \mathbf{C}^n;\ \Big[\Big(\sum_{j=1}^n |z_j|^2\Big)^2 - \Big|\sum_{j=1}^n z_j^2\Big|^2\Big]^{1/2} < 1 - \sum_{j=1}^n |z_j|^2\Big\}, \quad 5 \leqq n;$$

$$D(\mathrm{V}) = \{z \in M_{1,2}(O);\ \|z\| < 1\};$$

$$D(\mathrm{VI}) = \{z \in M_{3,3}(O);\ z^* = z,\ \|z\| < 1\}.$$

Here z^T is the transpose of the matrix z, O is the complex 8-dimensional Cayley algebra. The domains of types I—IV are *classical.* $D(\mathrm{V})$ and $D(\mathrm{VI})$ are the *exceptional* 16 and 27 dimensional domains. The domains of *tube type* are $D(\mathrm{I}_{n,n})$, $D(\mathrm{II}_n)$ for n even, $D(\mathrm{III}_n)$, $D(\mathrm{IV}_n)$, and $D(\mathrm{VI})$. The parameters of these domains are given in the following table:

parameter \ type	$\mathrm{I}_{n,n}$	II_n ($6 \leqq n$ even)	III_n ($2 \leqq n$)	IV_n ($4 \leqq n$)	VI
d = division	n^2	$n(n-1)/2$	$n(n+1)/2$	n	27
r = rank	n	$n/2$	n	2	3
$a = \dim Z_{i,j}$, $1 \leqq i \leqq j \leqq r$	2, if $2 \leqq n$; 0, if $n = 1$	4	1	$n-2$	8
p = genus	$2n$	$2(n-1)$	$n+1$	n	18

Acknowledgement. The author benefited from several conversations with H. Upmeier on subjects related to this paper. He is also grateful to Z. Yan and A. Koranyi for sending him their unpublished results ([Y] and [K] respectively) after the first draft of the paper was written.

§2. Analysis of the differential operator ∂_N.

Proposition 1. *$N(k(z))=\chi(k)N(z)$, $k\in K$, where $\chi\colon k\to\mathbf{T}$ is a multiplicative homomorphism.*

This is well-known.

Notation. $\underline{m}'=(m_1, m_2, \dots, m_{r-1}, 0)$.

If $\underline{m}$ is a signature and l is a non-negative integer, we denote

$$\underline{m}+l = (m_1+l, m_2+l, \dots, m_r+l).$$

Proposition 2. *Let $\underline{m}$ be any signature, $l\geqq 0$. $P_{\underline{m}}N^l=P_{\underline{m}+l}$. In particular, $P_{\underline{m}'}N^l=P_{\underline{m}'+l}$.*

Proof.

$$\begin{aligned} P_{(m_1+l,\dots,m_r+l)} &= \operatorname{span}\{((N_{\underline{m}}N^l)\circ k)\colon k\in K\} \\ &= \operatorname{span}\{(N_{\underline{m}}\circ k)N^k;\ k\in K), \quad \text{by Proposition 1} \\ &= \operatorname{span}\{N_{\underline{m}}\circ k;\ k\in K\}\cdot N^l \\ &= P_{\underline{m}}N^l \end{aligned}$$ ∎

Lemma 3. $\partial_N(N_{\underline{m}'}N^l)\in P_{\underline{m}'}N^{l-1}$.

Proof. Clearly, $f=\partial_N(N_{\underline{m}'}N^l)\in\mathscr{P}_s$ the space of homogeneous polynomials of degree s, where $s=|m'|+(l-1)r=\sum_{j=1}^{r-1}m_j+(l-1)r$. Let $g\in\mathscr{P}_s$. Then $\partial_g f$ is constant and so

$$\partial_g f = \partial_{gN}(N_{\underline{m}'}N^l) = \partial_{gN}(N_{\underline{m}'}N^l)(0) = (gN, N_{\underline{m}'}N^l)_F.$$

If $g=\sum_{|\underline{\sigma}|=s}g_{\underline{\sigma}}$, $g_{\underline{\sigma}}\in P_{\underline{\sigma}}$, then

$$(g_{\underline{\sigma}}N, N_{\underline{m}'}N^l)_F = 0$$

unless $\underline{\sigma}=(m_1+l-1, \dots, m_{r-1}+l-1, l-1)$.

So

$$f\in P_{(m_1+l-1,\dots,m_{r-1}+l-1,l-1)} = P_{\underline{m}'}N^{l-1}.$$ ∎

Notation. $\partial_N(N_{\underline{m}'}N^l)=F_{\underline{m}',l}N^{l-1}$, $F_{\underline{m}',l}\in P_{\underline{m}'}$.

In the proof of Lemma 4 below we use the fact that $N^*=N$ and that for every polynomials p, q

(†) $$(p^*, q^*)_F = (q, p)_F$$

and

$$(\dagger\dagger) \qquad (\partial_N p, q)_F = (p, Nq)_F.$$

The first formula follows easily from the integral form of the Fischer inner product, while the second follows from the definition (i.e. the differential form of the Fischer inner product).

Lemma 4. $\partial_N(g\circ k)=((\partial_N g)\circ k)\chi(k),\ k\in K.$

Proof. It is enough to assume that $g\in P_{\underline{m}'}N^l$. Then $\partial_N(g\circ k)\in P_{\underline{m}'}N^{l-1}$. Let $h\in P_{\underline{m}'}N^{l-1}$. Then $\partial_h(\partial_N(g\circ k))=\partial_{hN}(g\circ k)$ is constant. Hence

$$\begin{aligned}(\partial_h(\partial_N(g\circ k))) &= \partial_{hN}(g\circ k)(0)\\ &= (g\circ k, h^*N)_F, && \text{by } (\dagger),\\ &= (g, (h^*\circ k^{-1})N)_F\chi(k), && \text{by } K\text{-invariance},\\ &= (\partial_N g, h^*\circ k^{-1})_F\chi(k), && \text{by } (\dagger\dagger),\\ &= ((\partial_N g)\circ k, h^*)_F\chi(k), && \text{by } K\text{-invariance},\\ &= \partial_h(\partial_N(g)\circ k)\chi(k), && \text{by } (\dagger).\end{aligned}$$

Hence $\partial_N(g\circ k)=(\partial_N(g))\circ k\cdot\chi(k)$ as desired. ∎

Lemma 5. $\partial_N((N_{\underline{m}'}\circ k)N^l)=(F_{\underline{m}',l}\circ k)N^{l-1}$ *for any* $k\in K$.

Proof.
$$\begin{aligned}\partial_N((N_{\underline{m}'}\circ k)N^l) &= \partial_N((N_{\underline{m}'}N^l)\circ k)/\chi(k)^l\\ &= ((\partial_N(N_{\underline{m}'}N^l))\circ k)/\chi(k)^{l-1}, \quad \text{by Lemma 4},\\ &= (F_{\underline{m}',l}\circ k)N^{l-1}.\end{aligned}$$ ∎

We define linear operators $\{T_l\}_{l=1}^{\infty}$, $T_l\colon P_{\underline{m}'}\to P_{\underline{m}'}$, as follows.

$$T_l\left(\textstyle\sum_\nu c_\nu(N_{\underline{m}'}\circ k_\nu)\right) = \textstyle\sum_\nu c_\nu F_{\underline{m}',l}\circ k_\nu$$

and extend T_l to the space $\mathscr{H}(Z):=\sum_{\underline{m}'}\oplus P_{\underline{m}'}$ of harmonic polynomials by linearity. By Lemma 5, for every harmonic polynomial $q\in\mathscr{H}(Z)$

$$T_l(q) = \partial_N(qN^l)/N^{l-1}.$$

Lemma 6. *Let* $q\in\mathscr{H}(Z)$, $l\geqq 1$. *Then* $T_l(q\circ k)=T_l(q)\circ k;\ \forall k\in K.$

Proof. Let $q\in P_{\underline{m}'}$. Then

$$\begin{aligned}T_l(q\circ k) &= \partial_N((q\circ k)N^l)/N^{l-1} = \partial_N((qN^l)\circ k)/(N^{l-1}\cdot\chi(k)^l)\\ &= \frac{((\partial_N(qN^l))\circ k)\cdot\chi(k)}{N^{l-1}\chi(k)^l} = \left(\frac{\partial_N(qN^l)}{N^{l-1}}\right)\circ k = T_l(q)\circ k.\end{aligned}$$ ∎

Corollary 7. *There are numbers* $c_{\underline{m}',l}$ *such that*

$$T_l(q) = c_{\underline{m}',l}\, q \quad for \quad q \in P_{\underline{m}'}.$$

Proof. T_l: $P_{\underline{m}} \to P_{\underline{m}'}$ commutes with the action of the group K. However, $P_{\underline{m}'}$ is K-irreducible. So by Schur's lemma in representation theory $T_l|_{P_{\underline{m}'}} = c_{\underline{m}',l} I_{P_{\underline{m}'}}$. ▮

Lemma 8.

$$c_{\underline{m}',l} = \|\varphi_{\underline{m}'} N^l\|_F^2 / \|\varphi_{\underline{m}'} N^{l-1}\|_F^2 = \Pi_{j=1}^r \left(\frac{d}{r} + m_j + l - 1 - (j-1)\frac{a}{2}\right)$$

where $a = \dim(Z_{ij})$ *for* $1 \leqq i < j \leqq r$, $(m_r = 0)$.

Proof. Since N is L-invariant and $N(e) = 1$, for every signature $\underline{m}$, $\varphi_{\underline{m}} N^l = \varphi_{\underline{m}+l}$. Also

$$\partial_N(\varphi_{\underline{m}'+l}) = \partial_N(\varphi_{\underline{m}'} N^l) = c_{\underline{m},l}\, \varphi_{\underline{m}'} N^{l-1} = c_{\underline{m}',l}\, \varphi_{\underline{m}',l}\, \varphi_{\underline{m}'+l+1}.$$

Hence

$$c_{\underline{m},l} = (\partial_N(\varphi_{\underline{m}'+l}), \varphi_{\underline{m}'+l+1})_F / \|\varphi_{\underline{m}'+i-1}\|_F^2 = \|\varphi_{\underline{m}'+l}\|_F^2 / \|\varphi_{\underline{m}'+l-1}\|_F^2$$

by (††) above. But

$$\|\varphi_{\underline{m}}\|_F^2 = \left(\frac{d}{r}\right)_{\underline{m}} \Big/ \dim(P_{\underline{m}})$$

(see Section 1) and $\dim(P_{\underline{m}'+l}) = \dim(P_{\underline{m}'+l-1}) = \dim(P_{\underline{m}'})$ by Proposition 2. Thus

$$c_{\underline{m}',l} = \left(\frac{d}{r}\right)_{\underline{m}+l} \Big/ \left(\frac{d}{r}\right)_{\underline{m}'+l-1} = \Pi_{j=1}^r \left(\frac{d}{r} + m_j + l - 1 - (j-1)\frac{a}{2}\right).$$

Corollary 9. *Let* $\underline{m}$ *be any signature with* $m_r \geqq 1$. *Let*

$$c_{\underline{m}} = \Pi_{j=1}^r \left(\frac{d}{r} + m_j - 1 - (j-1)\frac{a}{2}\right).$$

Let $c_{\underline{m}} = 0$ *if* $m_r = 0$. *Let* $f = \sum_{\underline{m}} f_{\underline{m}}$ *be analytic in a neighborhood of* $\bar{D}$, $f_{\underline{m}} \in P_{\underline{m}}$. *Then*

$$\partial_N f = \partial_N \left(\textstyle\sum_{\underline{m}} f_m\right) = \textstyle\sum_{m_r \geqq 1} c_{\underline{m}} f_{\underline{m}} / N.$$

Proof. If $m_r \geqq 1$ and $f \in P_{\underline{m}}$, then $f = g N^{m_r}$, with $g \in P_{(m_1 - m_r, \ldots, m_{r-1} - m_r, 0)}$. By Lemma 8

$$\partial_N f = c_{(m_1 - m_r, \ldots, m_{r-1} - m_r, 0)\, m_r} f / N = c_{\underline{m}} f / N.$$

If $m_r = 0$ then $P_{\underline{m}}$ consists of harmonic polynomials, and hence $\partial_N f = 0$ for all $f \in P_{\underline{m}}$ (see [U]). ▮

Corollary 10. *For every non-negative integer* s

$$N^s \partial_{N^s} \textstyle\sum_{\underline{m}} f_{\underline{m}} = \textstyle\sum_{m_r \geqq s} b_{\underline{m},s} f_{\underline{m}}$$

with $b_{\underline{m},s}=c_{\underline{m}}\cdot c_{(m_1-1,\dots,m_r-1)}\cdots c_{(m_1-s+1,\dots,m_r-s+1)}=\dfrac{\left(\frac{d}{r}\right)_{\underline{m}}}{\left(\frac{d}{r}\right)_{\underline{m}-s}}$ for $m_r\geqq s$ and $b_{\underline{m},s}=0$ if $m_r<s$.

Proof. Since $\partial_{N^s}=(\partial_N)^s$, we get by Corollary 9 for any signature $\underline{m}$ with $m_r\geqq s$ and $f_{\underline{m}}\in P_{\underline{m}}$

$$\begin{aligned}\partial_{N^s}f_{\underline{m}} &= (\partial_N)^s f_{\underline{m}}\\ &= c_{\underline{m}}(\partial_N)^{s-1}f_{\underline{m}}/N, &&\text{with } f_{\underline{m}}/N\in P_{(m_1-1,\dots,m_r-1)}\\ &= c_{\underline{m}}\cdot c_{(m_1-1,\dots,m_r-1)}(\partial_N)^{s-2}(f_{\underline{m}}/N^2) &&\text{with } f_{\underline{m}}/N^2\in P_{(m_1-2,\dots,m_r-2)}\\ &\cdots\cdots\cdots\\ &= c_{\underline{m}}\cdot c_{(m_1-1,\dots,m_r-1)}\cdots c_{(m_1-s+1,\dots,m_r-s+1)}(f_{\underline{m}}/N^s).\end{aligned}$$

If $m_r<s$ then $b_{\underline{m},s}=0$ and the same proof yields

$$\partial_{N^s}f_{\underline{m}} = 0 = b_{\underline{m},s}f_{\underline{m}}/N^s. \quad \blacksquare$$

Corollary 10′.

$$N^s\partial_{N^s}\left(\textstyle\sum_{\underline{m}}f_{\underline{m}}\right) = \textstyle\sum_{m_r\geqq s}\prod_{i=1}^{s}\prod_{j=1}^{r}\left(\dfrac{d}{r}+m_j-i-(j-1)\dfrac{a}{2}\right)f_{\underline{m}}.$$

§ 3. Characterization of the invariant inner products on the highest quotients by integration over the Shilov boundary

Fix $\lambda=\lambda_\nu=(\nu-1)\frac{a}{2}$, $\nu=1,2,\dots,r$, be a point in $W_d(D)$. For a signature $\underline{m}$ let $q(\lambda,\underline{m})$ be the multiplicity of λ as a root of the polynomial

$$(\xi)_{\underline{m}} = \prod_{j=1}^{r}\prod_{l=0}^{m_r-1}\left(\xi+l-(j-1)\frac{a}{2}\right)$$

(We set $q(\lambda,\underline{m})=0$ if λ is not a root of $(\xi)_{\underline{m}}$). Set $q(\lambda)=\sup_{\underline{m}}q(\lambda,\underline{m})$. Clearly $q(\lambda)\leqq r$. More precisely

$$q(\lambda)=\begin{cases} r-\nu+1; & a \text{ even}\\ \left[\dfrac{r-\nu}{2}\right]+1: & a=1,\ D=D(\mathrm{III}_r)\\ 1; & D=D(\mathrm{IV}_n),\ n\,\text{odd};\end{cases}$$

Consider the action $U^{(\lambda)}$ of G on analytic functions on D defined by

$$U^{(\lambda)}(\varphi)f=(f\circ\varphi)(J\varphi)^{\lambda/p}.$$

Let $P^{(\lambda)}=\operatorname{span}\{U^{(\lambda)}(\varphi)P;\ \varphi\in G\}$. For $j=0, 1, \ldots, q(\lambda)$ define

$$M_j^{(\lambda)} = \Big\{f\in P^{(\lambda)};\ f = \sum_{\substack{\underline{m} \\ q(\lambda,\underline{m})\leq j}} f_{\underline{m}},\ f_{\underline{m}}\in P_{\underline{m}}\Big\}.$$

Clearly,

(*) $$\{0\} \subsetneqq M_0^{(\lambda)} \subsetneqq M_1^{(\lambda)} \subsetneqq \ldots \subsetneqq M_{q(\lambda)}^{(\lambda)} = P^{(\lambda)}.$$

According to [FK] (see also [Ø] for the special case of domain of type $I_{n,n}$) the spaces $M_j^{(\lambda)}$ are $U^{(\lambda)}$-invariant and the quotients $M_j^{(\lambda)}/M_{j-1}^{(\lambda)}$ are irreducible (where $M_{-1}^{\lambda}:=\{0\}$). Thus (*) is a *composition series* of $P^{(\lambda)}$. It is not hard to see that in fact every $U^{(\lambda)}$-invariant subspace of $P^{(\lambda)}$ is one of the spaces in (*). Moreover, if we define for $\xi\in\mathbf{C}$ and $f, g\in P^{(\lambda)}$

$$(f, g)_\xi := \textstyle\sum_{\underline{m}} (f_{\underline{m}}, g_{\underline{m}})_F/(\xi)_{\underline{m}}$$

and for $f, g\in M_j^{(\lambda)}$ we define

$$(f, g)_{\lambda, j} = \lim_{\xi\to\lambda} (\xi-\lambda)^j (f, g)_\xi$$

then $(\cdot,\cdot)_{\lambda,j}$ is a $U^{(\lambda)}$-invariant Hermitian form on $M_j^{(\lambda)}$ with

$$\{f\in M_j^{(\lambda)};\ (g, f)_{\lambda,j} = 0,\ \forall g\in M_j^{(\lambda)}\} = M_{j-1}^{(\lambda)}.$$

One can compute

$$(f, g)_{\lambda, j} = \sum_{\substack{\underline{m} \\ q(\lambda,\underline{m})=j}} (f_{\underline{m}}, g_{\underline{m}})_F/\langle\lambda\rangle_{\underline{m}, j}$$

where

$$\langle\lambda\rangle_{\underline{m},j} = \lim_{\xi\to\lambda}\frac{(\xi)_{\underline{m}}}{(\xi-\lambda)^j} = \Pi_{\nu=1}^{r}\,\Pi'^{\,m_j-1}_{l=0}\Big(\lambda+l-(j-1)\frac{a}{2}\Big)$$

(where "$\Pi'^{\,m_j-1}_{l=1}$" ranges over all non-zero terms). The Hermitian form $(\cdot,\cdot)_{\lambda,j}$ on $M_j^{(\lambda)}/M_{j-1}^{(\lambda)}$ is *definite* (positive or negative) if and only if either $j=0$, or $j=q(\lambda)$ and $(r-\nu)\frac{a}{2}$ is a non-negative integer. In this case the quotient $M_j^{(\lambda)}/M_{j-1}^{(\lambda)}$ is said to be *unitarizable*, and we denote by $\mathscr{H}_{\lambda,j}$ the completion of $M_j^{(\lambda)}/M_{j-1}^{(\lambda)}$ with respect to $(\cdot,\cdot)_{\lambda,j}$.

Since Z is a JB^*-algebra, we get the following three possibilities.

Corollary 11. (1) *If a is even then $M_{q(\lambda)}^{(\lambda)}/M_{q(\lambda)-1}^{(\lambda)}$ is always unitarizable, i.e. $(\cdot,\cdot)_{\lambda,q(\lambda)}$ is definite.*

(2) *If $D=D(\mathrm{III}_r)$ and $a=1$ then $M_{q(\lambda)}^{(\lambda)}/M_{q(\lambda)-1}^{(\lambda)}$ is unitarizable if and only if* $r\equiv\nu \pmod 2$. *In this case $q(\lambda)=\frac{r-\nu}{2}+1$.*

(3) *If $D=D(\mathrm{IV}_n)$, n odd, then $q(\lambda)=1$, $M_1^{(\lambda)}/M_0^{(\lambda)}$ is unitarizable for $\lambda=\frac{n-2}{2}$ and not unitarizable for $\lambda=0$.*

The following theorem is the main result of this section.

Theorem 12. *Let* $\lambda=(\nu-1)\frac{a}{2}$, $1\leqq\nu\leqq r$, *and assume that the highest quotient* $M^{(\lambda)}_{q(\lambda)}/M^{(\lambda)}_{q(\lambda)-1}$ *is unitarizable. Then*

$$\|f\|^2_{\lambda,q(\lambda)} = \gamma(N^s\partial_{N^s}f, f)_{L^2(S)}$$

where $s=(r-\nu)\frac{a}{2}+1=\frac{p}{2}-\lambda$, *and* $\gamma=\Pi'^{r}_{j=1}\Pi^{(r-\nu)(a/2)}_{l=0}((\nu-j)\frac{a}{2}+l)$, *the product ranges over non-zero terms. Consequently,* $\mathscr{H}_{\lambda,q(\lambda)}$ *is identified with the space of analytic functions* f *on* D *for which* $(N^s\partial_{N^s})^{1/2}f\in H^2(S)$.

Proof. Let $f=\sum_{\underline{m}}f_{\underline{m}}$ be analytic in a neighborhood of $\bar{D}$. Then

$$\|f\|^2_{\lambda,q(\lambda)} = \sum_{q(\lambda,m)=q(\lambda)}\frac{(f_m, f_m)_F}{\langle\lambda\rangle_{\underline{m},q(\lambda)}}$$

$$= \sum_{m_r\geqq s}(f_{\underline{m}}, f_{\underline{m}})_F\Big/\Pi'^{r}_{j=1}\Pi'^{m_j-1}_{l=0}\left((\nu-1)\frac{a}{2}+l-(j-1)\frac{a}{2}\right).$$

Also, by Corollary 10′

$$(N^s\partial_{N^s}f,f)_{L^2(S)} = \sum_{m_r\geqq s}\Pi^{r}_{j=1}\Pi^{s}_{i=1}\left(\frac{d}{r}+m_j-i-(j-1)\frac{a}{2}\right)(f_{\underline{m}}, f_{\underline{m}})_{L^2(S)}$$

$$= \sum_{m_r\geqq s}\frac{\Pi^{r}_{j=1}\Pi^{s}_{i=1}\left(\frac{d}{r}+m-i-(j-1)\frac{a}{2}\right)}{\Pi^{r}_{j=1}\Pi^{m_j-1}_{l=0}\left(\frac{d}{r}+l-(j-1)\frac{a}{2}\right)}(f_{\underline{m}}, f_{\underline{m}})_F$$

by [FK, Corollary 3.5]. Now, if $m_r\geqq s$ then

$$\frac{\Pi^{r}_{j=1}\Pi^{s}_{i=1}\left(\frac{d}{r}+m-i-(j-1)\frac{a}{2}\right)}{\Pi^{r}_{j=1}\Pi^{m_j-1}_{l=0}\left(\frac{d}{r}+l-(j-1)\frac{a}{2}\right)}$$

$$=\left(\Pi^{r}_{j=1}\Pi^{m_j-s-1}_{l=0}\left(\frac{d}{r}+l-(j-1)\frac{a}{2}\right)\right)^{-1}$$

$$=\left(\Pi^{r}_{j=1}\Pi^{m_j-1}_{l=s}\left((\nu-1)\frac{a}{2}+l-(j-1)\frac{a}{2}\right)\right)^{-1}=\frac{1}{\gamma}\cdot\frac{1}{\langle\lambda\rangle_{\underline{m},q(\lambda)}}.$$

Hence

$$\gamma(N^s\partial_{N^s}f,f)_{L^2(S)} = \|f\|^2_{\lambda,q(\lambda)}. \quad \blacksquare$$

Remark. If $r=1$ (and then $a=0$), $Z=\mathbf{C}$ and $D=\{z\in\mathbf{C}, |z|<1\}$, $\lambda=0$ and $\mathcal{H}_{0,1}$ is the Dirichlet space. Theorem 12 yields in this case the known result

$$\|f\|^2_{\text{Dirichlet}} = \int_{|z|<1} |f'(z)|^2 dA(z) = \int_{\Pi} e^{i\sigma} f'(e^{i\sigma}) \overline{f(e^{i\sigma})}^d \frac{\sigma}{2\pi}.$$

Corollary 13. *Let λ, s, γ be as in Theorem 12. Then*

$$(f, g)_{\lambda, q(\lambda)} = \gamma_\nu (N^s \partial_{N^s} f, g)_{L^2(S)}.$$

Thus $(N^S \partial_{N^S} f, g)_{L^2(S)}$ is invariant under the action of G given by $U^{(\lambda)}(\varphi)f=(f\circ\varphi)(J\varphi)^{\lambda/p}$.

§ 4. Characterization of the invariant inner product in terms of integration over D

The Dirichlet semi-norm in the unit disk Δ in $\mathbf{C}$,

$$\|f\|_{\text{Dirichlet}} = \Big(\int_{|z|<1} |f'(z)|^2 dA(z)\Big)^{1/2}$$

can be written as

$$\|f\|_{\text{Dirichlet}} = \Big(\int_{|z|<1} |(f\circ\varphi_a)'(0)|^2 d\mu(a)\Big)^{1/2} = \|(f\circ\varphi)'\|_{L^2(G, d\varphi)}.$$

Here the Möbius transformation $\varphi_a(z)=(a-z)/(1-\bar{a}z)$ is the biholomorphic symmetry of Δ which interchanges 0 and a, $d\mu(a)=(1-|a|^2)^{-2}dA(a)$ is the Möbius-invariant measure on the unit disk and $d\varphi$ is the Haar measure of $G=\text{Aut}(\Delta)=(\text{Aut}(\Delta))_0$.

In this section we study the generalizations of this formula to other tube domains and group actions. Let D be a tube domain in $\mathbf{C}^d$ and fix a point $\lambda=\lambda_\nu=(\nu-1)\frac{a}{2}$, $(1\leqq\nu\leqq r)$ in $W_d(D)$. Recall that $U^{(\lambda)}(\varphi)f=(f\circ\varphi)(J\varphi)^{\lambda/p}$, $\varphi\in G$.

We restrict our attention to the case where the highest quotient $M^{(\lambda)}_{q(\lambda)}/M^{(\lambda)}_{q(\lambda)-1}$ is unitarizable, i.e. $(r-1)\frac{a}{2}-\lambda=(r-\nu)\frac{a}{2}$ is a non-negative integer.

Define

$$s = s(\lambda) = \min\{l\in\mathrm{N};\ N^l \notin M^{(\lambda)}_{q(\lambda)-1}\}.$$

Thus, s is the first positive integer for which

$$\text{span}\{U^{(\lambda)}(\varphi) N^s;\ \varphi\in G\} = M^{(\lambda)}_{q(\lambda)} = P^{(\lambda)}.$$

It is not hard to see that in fact

$$s = \frac{p}{2} - \lambda.$$

Define a differential operator $D_s^{(\lambda)}$ form the analytic functions on D into the real analytic functions on G by

$$D_s^{(\lambda)}(f)(\varphi) = \partial_{N^s}(U^{(\lambda)}(\varphi)f)(0) = (U^{(\lambda)}(\varphi)f, N^s)_F.$$

$D_s^{(\lambda)}$ is *invariant* in the sense that

$$D_s^{(\lambda)}(U^{(\lambda)}(\varphi)f)(\psi) = \overline{c(\varphi, \psi, \lambda)}\, D_s^{(\lambda)}(f)(\varphi\psi), \quad \varphi, \psi \in G$$

where $c(\varphi, \psi, \lambda)$ is the unimodular function introduced in Section 1. Also, for $k \in K$

$$R_R D_s^{(\lambda)} = \chi(k)^s J(k)^{\lambda/p} D_s^{(\lambda)} = \chi(k)^{s+\lambda/2} D_s^{(\lambda)}$$

where $R_k(u)(\psi) = u(\psi k)$ is the operator of right translation by k.

Definition. $\mathcal{H}^{(\lambda)}$ is the space of all analytic functions f on D for which $D_s^{(\lambda)}(f) \in L^2(G) = L^2(G, d\varphi)$, with the seminorm

$$\|f\|_{\mathcal{H}^{(\lambda)}} = \|D_s^{(\lambda)}(f)\|_{L^2(G)}.$$

Here $d\varphi$ is the Haar measure of G.

The main result of this section is the following

Theorem 14. *Let D be a tube domain in $\mathbf{C}^d$, and let λ be a point in $W_d(D)$. Assume that the highest quotient $M_{q(\lambda)}^{(\lambda)}/M_{q(\lambda)-1}^{(\lambda)}$ is unitarizable. Then $\mathcal{H}^{(\lambda)}$ is non-trivial if and only if $\lambda < 1$, and is this case $\mathcal{H}^{(\lambda)} = \mathcal{H}_{\lambda, q(\lambda)}$ with proportional seminorms*

$$\|f\|_{\mathcal{H}^{(\lambda)}} = \alpha \|f\|_{\mathcal{H}_{\lambda, q(\lambda)}}$$

where α is a constant, independent of f.

We prove the theorem in several steps, where the most substantial one deals with the non-triviality of $\mathcal{H}^{(\lambda)}$ for $\lambda < 1$ (i.e. that $\mathcal{H}^{(\lambda)}$ contains some function with non-zero semi-norm). Let us begin with the easier parts.

Step 1. Characterizing $\mathcal{H}^{(\lambda)}$ by integration over D.

Define for an analytic function f on D

$$\tilde{D}_s^{(\lambda)} f(a) = |D_s^{(\lambda)}(f)(\varphi)|$$

where $\varphi \in G$ satisfies $\varphi(0) = a$. The right K-invariance of $D_s^{(\lambda)}$ shows that $\tilde{D}_s^{(\lambda)} f(a)$ is well-defined, i.e. independent of the choice of φ which satisfies $\varphi(0) = a$. Also, $\tilde{D}_s^{(\lambda)} U^{(\lambda)}(\varphi) = L\varphi \tilde{D}_s^{(\lambda)}$, where $(L_\varphi u)(\psi)$ is the operator left translation. Let $\varphi_a \in G$ be the symmetry which interchanges 0 and a. Then $\mathcal{H}^{(\lambda)}$ consists of all analytic functions f on D for which $\tilde{D}_s^{(\lambda)}(\varphi_a) f \in L^2(D, \mu)$, where $d\mu(a) = K(a, a)dV(a)$ in the G-invariant measure on D.

Step 2. $\mathcal{H}^{(\lambda)}$ is invariant with respect to the isometric action $U^{(\lambda)}$ of G.

Indeed, for every $f\in\mathcal{H}^{(\lambda)}$ and $\varphi\in G$,

$$\begin{aligned}\|U^{(\lambda)}(\varphi)f\|_{\mathcal{H}^{(\lambda)}} &= \|D_s^{(\lambda)}U^{(\lambda)}(\varphi)f\|_{L^2(G)}\\ &= \|L_\varphi(D_s^{(\lambda)}f)\|_{L^2(G)}, \quad \text{by the invariance of } D_s^{(\lambda)}\\ &= \|D_s^{(\lambda)}f\|_{L^2(G)}, \quad \text{by the left invariance of the Haar measure}\\ &= \|f\|_{\mathcal{H}^{(\lambda)}}.\end{aligned}$$

Step 3. $\|f\|_{\mathcal{H}^{(\lambda)}}=0$ *if and only if f is supported on* $M_{q(\lambda)-1}^{(\lambda)}$, *i.e.* $f=\sum_{q(\lambda,\underline{m})<q(\lambda)} f_{\underline{m}}, f_{\underline{m}}\in P_{\underline{m}}$.

Indeed, let $f\in\mathcal{H}^{(\lambda)}$. Then $\|f\|_{\mathcal{H}^{(\lambda)}}=0$ if and only if $(U^{(\lambda)}(\varphi)f, N^s)_F=0$ for all $\varphi\in G$. This is equivalent to the orthogonality of N^s in the Fischer inner product to $\overline{\text{span}}\,\{U^{(\lambda)}(\varphi)f;\ \varphi\in G\}$. Since the latter space is $U^{(\lambda)}$-invariant and the decomposition series $M_0^{(\lambda)}\subset M_1^{(\lambda)}\subset\ldots\subset M_{q(\lambda)}^{(\lambda)}$ exhaust all the nontrivial $U^{(\lambda)}$-invariant subspaces of $P^{(\lambda)}$ we see that $\overline{\text{span}}\,\{U^{(\lambda)}(\varphi)f;\ \varphi\in G\}=\overline{M}_j^{(\lambda)}$ for a unique $j\leqq q(\lambda)$. However $N^s\in M_{q(\lambda)-1}^{\perp}$. Hence $\|f\|_{\mathcal{H}^{(\lambda)}}=0$ if and only if $j<q(\lambda)$ (i.e. $f=\sum_{q(\lambda,\underline{m})\leqq j} f_{\underline{m}}$, $f_{\underline{m}}\in P_{\underline{m}}$ and $j<q(\lambda)$).

Step 4. If $\mathcal{H}^{(\lambda)}$ is non trivial, then $\mathcal{H}^{(\lambda)}=\mathcal{H}_{\lambda,q(\lambda)}$ with $\|f\|_{\mathcal{H}^{(\lambda)}}=\alpha(\lambda)\|f\|_{\mathcal{H}_{\lambda,q(\lambda)}}$ for all $f\in\mathcal{H}^{(\lambda)}$.

This is the special case of the uniqueness theorem of [AF]. We sketch the short proof for the convenience of the reader.

Since we assume that $\mathcal{H}^{(\lambda)}$ is non-trivial we get by step 3 that $M_{q(\lambda)}^{(\lambda)}$ is dense in $\mathcal{H}^{(\lambda)}$ and that $\|N^s\|_{\mathcal{H}^{(\lambda)}}>0$. Moreover, by the orthogonality of the Peter—Weyl decomposition $\sum_{\underline{m}}\oplus P_{\underline{m}}$ in both of $\mathcal{H}^{(\lambda)}$ and $\mathcal{H}_{\lambda,q(\lambda)}$, we get

$$(N^s, f)_{\mathcal{H}^{(\lambda)}} = \alpha(\lambda)^2(N^s, f)_{\mathcal{H}_{\lambda,q(\lambda)}}$$

for every function f which is analytic in a neighborhood of $\bar{D}$, where

$$\alpha(\lambda) = \|N^s\|_{\mathcal{H}^{(\lambda)}}/\|N^s\|_{\mathcal{H}_{\lambda,q(\lambda)}}.$$

The semi-inner products of $\mathcal{H}^{(\lambda)}$ and $\mathcal{H}_{\lambda,q(\lambda)}$ are $U^{(\lambda)}$-invariant. Thus for all $\varphi,\psi\in G$:

$$\begin{aligned}(U^{(\lambda)}(\varphi)N^s, U^{(\lambda)}(\psi)N^s)_{\mathcal{H}^{(\lambda)}} &= (N^s, U^{(\lambda)}(\varphi^{-1})U^{(\lambda)}(\psi)N^s)_F\\ &= \alpha(\lambda)^2(N^s, U^{(\lambda)}(\varphi^{-1})U^{(\lambda)}(\psi)N^s)_{\mathcal{H}_{\lambda,q(\lambda)}}\\ &= \alpha(\lambda)^2(U^{(\lambda)}(\varphi)N^s, U^{(\lambda)}(\psi)N^s)_{\mathcal{H}_{\lambda,q(\lambda)}}\end{aligned}$$

since span $\{U^{(\lambda)}(\varphi)N^s;\ \varphi\in G\}$ is dense in both $\mathcal{H}^{(\lambda)}$ and $\mathcal{H}_{\lambda,q(\lambda)}$, we see that $\mathcal{H}_{(\lambda)}=\mathcal{H}_{\lambda,q(\lambda)}$ and that

$$(f, g)_{\mathcal{H}^{(\lambda)}} = \alpha(\lambda)^2(f, g)_{\mathcal{H}_{\lambda,q(\lambda)}}$$

for all $f, g\in\mathcal{H}^{(\lambda)}$.

The last and the most important part of the proof of Theorem 14 characterizes the λ's for which $\mathscr{H}^{(\lambda)}$ is non-trivial.

We begin with the transformation rule of the determinant polynomial N under composition with automorphism of D. This is interesting in its own right.

As in [FK], let $h(z, w)$ be the sesquiholomorphic extension of the unique K-invariant polynomial h on Z whose restriction to $\mathrm{span}_{\mathbf{R}}\{e_j\}_{j=1}^r$ is given by

$$h\left(\sum_{j=1}^r t_j e_j\right) = N\left(\sum_{j=1}^r (1-t_j)e_j\right) = \prod_{j=1}^n (1-|t_j|^2).$$

It is known that

$$K(z, w) = h(z, w)^{-p}, \quad z, w \in D.$$

Recall that S is the Shilov boundary of D.

Lemma 15. *Let $a \in D$. Then*

(i) $N(-\varphi_a(u))\overline{N(u)} = h(a, u)h(u, a)^{-1}, \quad u \in S.$

(ii) $N(\varphi_a(z)) = N(a-z)h(z, a)^{-1}, \quad z \in D.$

We sketch the *proof.* In the matrix tube domains (types $\mathrm{I}_{n,n}$, II_n (n even), and III_n) φ_a can be written as

$$\varphi_a(z) = (I-aa^*)^{-1/2}(a-z)(I-a^*z)(I-a^*a)^{1/2}, \quad z \in D.$$

The determinant polynomial N is very closely related to the ordinary determinant:

$$N(z) = \det(z), \quad \text{in types } \mathrm{I}_{n,n} \text{ and } \mathrm{III}_n,$$

$$N^2(z) = \det(z), \quad \text{in type } \mathrm{II}_n \ (n \text{ even}).$$

The desired formulas in Lemma 15 follow now by the multiplicativity of the ordinary determinant function, the formula

$$\det(I-xy) = \det(I-yx)$$

and the fact that

$$h(z, w) = \begin{cases} \det(I-zw^*), & \text{types } \mathrm{I}_{n,n} \text{ and } \mathrm{II}_n, \\ \det(I-zw^*)^{1/2}, & \text{type } \mathrm{II}_n \ (n \text{ even}). \end{cases}$$

Formula (i) in Lemma 15 is proved in full generality in [Y]. Formulas (i) and (ii) in the Lemma are equivalent. Indeed, since both sides of (ii) are analytic in z, (ii) is equivalent to

(ii′) $$N(-\varphi_a(u)) = N(u-a)h(u, a)^{-1}, \quad a \in D, \ u \in S.$$

We claim that

$$N(u-a) = N(u)h(a, u), \quad a \in D, \ u \in S.$$

Clearly, this establishes the equivalence of (i) and (ii′). For $u=e$, this is well known (see [FK]). For $u=k(e)$, $k\in K$, this follows by the K-invariance of N and h and the previous case. Since $S=\{k(e);\, k\in K\}$ the proof is complete. ∎

Lemma 16. *Let* $z, a\in D$, *then*

$$J\varphi_a(z) = (-1)^d K(a,a)^{-1/2} K(z,a).$$

Proof. By the transformation rule of the Bergman kernel

$$\begin{aligned} K(z,a) &= K(\varphi_a(\varphi_a(z)), \varphi_a(0)) \\ &= \left(J\varphi_a(\varphi_a(z))\,\overline{J\varphi_a(0)}\right)^{-1} = J\varphi_a(z)/\overline{J\varphi_a(0)} \end{aligned}$$

letting $z=0$, we get that $J\varphi_a(0)$ is real. Letting $z=a$, we get $J\varphi_a(0)^2=K(a,a)^{-1}$. Thus $J\varphi_a(0)=\varepsilon(a)K(a,a)^{-1/2}$, with $\varepsilon(a)=\pm 1$. Clearly, $\varepsilon(a)$ is a continuous function of a. Where $a=0$, $\varphi_a=-\mathrm{id}$, so $J\varphi_0(0)=(-1)^d$. Thus $\varepsilon(a)=(-1)^d$ identically. It follows that $J\varphi_a(0)=(-1)^d K(a,a)^{-1/2}$. This completes the proof. ∎

Remarks. (1) The above proof holds in any bounded symmetric domain.

(2) The argument determining $\varepsilon(a)$ is due to A. Koranyi [K].

(3) Lemma 16 is proved in [Y] by a different method.

Lemma 17. *For every* $a\in D$

$$(N^s, U^{(\lambda)}(\varphi_a)N^s)_F = c\beta K(a,a)^{\lambda-p/2p}$$

where c is a unimodular constant and

$$\beta = \|N^s\|_F^2/\|N^s\|_{H^2(S)}^2 = \left(\frac{p}{2}\right)_{(s,s,\ldots,s)}.$$

Proof. By the arguments in step 4 above

$$(N^s, f)_F = b(N^s, f)_{H^2(S)}$$

for every function f which is analytic in the neighborhood of $\bar{D}$. Recall that the reproducing kernel (i.e. the Szegö kernel) is

$$S(u,a) = K(u,a)^{1/2};\quad u\in S,\ a\in D.$$

Using Lemmas 15 and 16 and the fact that $s+\lambda=p/2$, we get

$$\begin{aligned} (N^s, U^{(\lambda)}(\varphi_a)(N^s))_{H^2(S)} &= \int_S N(u)^s\,\overline{N(\varphi_a(u))^s J\varphi_a(u)^{\lambda/p}}\,d\sigma(u) \\ &= (-1)^l \int_S h(u,a)^s h(a,u)^{-s} K(a,u)^{\lambda/p}\,d\sigma(u)\cdot K(a,a)^{-\lambda/2p} \\ &= (-1)^l \int_S K(u,a)^{-s/p} K(a,u)^{1/2}\,d\sigma(u)\,K(a,a)^{-\lambda/2p} \\ &= (-1)^l K(a,a)^{\lambda-p/2p} \end{aligned}$$

where

$$l = rs + \frac{d\lambda}{p}. \quad \blacksquare$$

Step 5. Completing the proof of Theorem 14.

Since span $\{U^{(\lambda)}(\varphi)N^s;\ \varphi\in G\} = M^{(\lambda)}_{q(\lambda)} = P^{(\lambda)}$, it is clear that $\mathcal{H}^{(\lambda)}$ is not trivial if and only if $0 < \|N^s\|_{\mathcal{H}^{(\lambda)}} < \infty$. However, by Lemma 17 and step 1 of the proof

$$\|N^s\|^2_{\mathcal{H}^{(\lambda)}} = \int_D |(U^{(\lambda)}(\varphi_a)N^s, N^s)_F|^2\, d\mu(a) = \beta^2 \int_D K(a,a)^{\lambda/p}\, dV(a).$$

It is well known (see [FK]) that this integral is finite if and only if $\lambda < 1$. Thus $\mathcal{H}^{(\lambda)}$ is not trivial for all tube domains different from $D(\mathrm{III}_n)$ precisely when $\lambda = 0$. For $D(\mathrm{III}_n)$, $\mathcal{H}^{(\lambda)}$ is not trivial for $\lambda = 0$ if n is odd, or for $\lambda = 1/2$ if n is even. This completes the proof. $\blacksquare$

Remark. The proof yields in fact the value of the constant

$$\alpha^2 = \|N^s\|^2_{\mathcal{H}^{(\lambda)}} / \|N^s\|^2_{\mathcal{H}_{\lambda, q(\lambda)}}$$

in Theorem 14. The case where $D = D(\mathrm{III}_n)$, n even and $\lambda = 1/2$ requires the computation of $\int_D h(z,z)^{1/2} dV(z)$, using either [H] or formulas (3.7), (3.8) in [FK].

Corollary 18. *In the context of Theorem* 14, *let* $\lambda < 1$. *Then the inner product in the highest quotient* $\mathcal{H}_{\lambda, q(\lambda)}$ *is*

$$(f, g)_{\mathcal{H}_{\lambda, q(\lambda)}} = \frac{1}{\alpha^2}(D_s^{(\lambda)} f, D_s^{(\lambda)} g)_{L^2(G)}.$$

§ 5. Characterization of the invariant inner product in terms of integration over D×D

A less well-known formula for the Dirichlet semi-norm in the unit disk Δ in **C** is

$$\|f\|^2_{\mathrm{Dirichlet}} = \iint_{\Delta\times\Delta} |f(z) - f(w)|^2\, |K_\nu(z,w)|^2\, d\nu(z)\, d\nu(w)$$

where ν is a very general finite measure on Δ and $K_\nu(z,w)$ is the reproducing kernel of $L^2_a(\nu)$, the space of analytic functions in $L^2(\nu)$. See [AFP] for the special case $d\nu(z) = (\alpha-1)(1-|z|^2)^{\alpha-2} dA(z)$ $(\alpha > 1)$ and [AFJP1] for the general case and extensions to other planar domains. One interpretation of this formula is that the Hilbert—Schmidt norm of the *Hankel operator*

$$(H_f h)(z) = \int_\Delta \overline{(f(z) - f(w))}\, h(w)\, K_\nu(z,w)\, d\nu(w)$$

is given by $\|H_f\|_{S_2} = \|f\|_{\mathrm{Dirichlet}}$, independently of the measure ν.

In this section we extend these results to the context of all tube domains.

Fix $\alpha > p+(r-1)\frac{a}{2}$ and let

$$d\mu_\alpha(z) = c(\alpha)K(z, z)^{1-\alpha/p}\,dV(z)$$

where $c(\alpha)^{-1}=\int_D K(z, z)^{1-\alpha/p}dV(z)$. Let $L^2_a(\mu_\alpha)$ be the subspace of $L^2(\mu_\alpha)$ consisting of analytic functions. Its reproducing kernel is $K(z, w)^{\lambda/p}$. G acts isometrically on $L^2(\mu_\alpha)$ and $L^2_a(\mu_\alpha)$ via $U^{(\alpha)}(\varphi)f=(f\circ\varphi)(J\varphi)^{\alpha/p}$, $\varphi\in G$.

Next, let $\lambda=(\nu-1)a/2$ $(1\leqq\nu\leqq r)$ be a point in the discrete part of the Wallach set. Let $Q^{(\lambda)}$ be the orthogonal projection on the highest quotient $M^{(\lambda)}_{q(\lambda)}/M^{(\lambda)}_{q(\lambda)-1}$, that is

$$Q^{(\lambda)}\left(\sum_{\underline{m}} f_{\underline{m}}\right) = \sum_{q(\lambda,\underline{m})=q(\lambda)} f_{\underline{m}}.$$

We assume in the sequel that $M^{(\lambda)}_{q(\lambda)}/M^{(\lambda)}_{q(\lambda)-1}$ *is unitarizable,* i.e. $(r-1)a/2-\lambda$ is a nonnegative integer.

Definition. $\mathcal{H}^{(\lambda)}(\alpha)$ is the space of all analytic functions f on D for which

$$\|f\|_{\mathcal{H}^{(\lambda)}(\alpha)} := \left(\int_G \|Q^{(\lambda)}(U^{(\lambda)}(\varphi))f\|^2_{L^2(\mu_\alpha)}\,d\varphi\right)^{1/2}$$

is finite.

Again $d\varphi$ is the Haar measure of G. Since $Q^{(\lambda)}$ commutes with the action of the subgroup K it is clear that

$$\|f\|^2_{\mathcal{H}^{(\lambda)}(\alpha)} = \int_D \|Q^{(\lambda)}(U^{(\lambda)}(\varphi_z))f\|^2_{L^2(\mu_\alpha)}\,d\mu(z)$$

where $d\mu(z)=K(z, z)dV(z)$ is the G-invariant measure on D. Also, by Lemma 16 one obtains

$$d\mu_\alpha(\varphi_z(w)) = |(J\varphi_a(w))|^{2\alpha/p}\,d\mu_\alpha(w) = \frac{|K(z, w)|^{2\alpha/p}}{K(z, z)^{\alpha/p}}\,d\mu_\alpha(w).$$

Hence,

$$\|f\|^2_{\mathcal{H}^{(\lambda)}(\alpha)} = c(\alpha)^{-1}\iint_{D\times D}|(Q^{(\lambda)}(U^{(\lambda)}(\varphi_z))f)(\varphi_z(w))|^2\,|K(z, w)|^{2\alpha/p}\,d\mu_\alpha(w)\,d\mu_\alpha(z)$$

Example. Let $D=\Delta$ be the unit disk in $\mathbf{C}$ and let $\lambda=0$. Then $p=2$ and $(Q^{(0)}f)(\xi)=f(\xi)-f(0)$. Hence

$$Q^{(0)}(f\circ\varphi_z)(\varphi_z(w)) = f(w)-f(z)$$

and the last formula becomes

$$\|f\|^2_{\mathcal{H}^{(\lambda)}(\alpha)} = c(\alpha)^{-1}\iint_{\Delta\times\Delta}|f(z)-f(w)|^2\,|K(z, w)^{\alpha/2}|^2\,d\mu_\alpha(z)\,d\mu_\alpha(w)$$

which is (up to the proportionality constant) the formula for the Dirichlet semi-norm.

Remark. If the rank r of D is 2 or more we cannot take in the definition of $\mathcal{H}^{(0)}(\alpha)$ the projection $(Q_0^{(0)}f)\xi=f(\xi)-f(0)$ on $M_{q(0)}^{(0)}/M_0^{(0)}$. Unless f is constant, the integral

$$\iint_{D\times D} |f(z)-f(w)|^2 |K(z,w)^{\alpha/p}|^2 \, d\mu_\alpha(z)\, d\mu_\alpha(w)$$

is infinite, because it is the Hilbert—Schmidt norm of the ordinary, *Hankel operator*

$$H_f h = (I-P_\alpha)\bar{f}h, \quad h\in L_a^2(\mu_a)$$

where $P_\alpha\colon L^2(\mu_\alpha)\to L_a^2(\mu_\alpha)$ is the orthogonal projection. It is well-known that if $r\geqq 2$ no non-trivial Hankel operator H_f with f analytic is compact, see [BCZ].

The projection $Q^{(0)}$ onto $M_{q(0)}^{(0)}/M_{q(0)-1}^{(0)}$ used in the definition of $\mathcal{H}^{(0)}(\alpha)$ is much smaller than $Q_0^{(0)}$, yet preserves the "essential contents" of f, i.e. the component of f in the highest quotient $M_{q(0)}^{(0)}/M_{q(0)-1}^{(0)}$. This observation is the key to our definition of the generalized Hankel operator, see section 6.

The main result of this section is the following.

Theorem 19. *Let D be a tube domain and let $\lambda\in W_d(D)$. Assume that the highest quotient $M_{q(\lambda)}^{(\lambda)}/M_{q(\lambda)-1}^{(\lambda)}$ is unitarizable. Then $\mathcal{H}^{(\lambda)}(\alpha)$ is non-trivial if and only if $\lambda<1$. In this case $\mathcal{H}^{(\lambda)}(\alpha)=\mathcal{H}_{\lambda,q(\lambda)}$ with proportional seminorms*

$$\|f\|_{\mathcal{H}^{(\lambda)}(\alpha)} = c\,\|f\|_{\mathcal{H}_{\lambda,q(\lambda)}}$$

where

$$c = \|N^s\|_{\mathcal{H}^{(\lambda)}(\alpha)}/\|N^s\|_{\mathcal{H}_{\lambda,q(\lambda)}}.$$

As in the proof of Theorem 14, it is easy to verify that $\mathcal{H}^{(\lambda)}(\alpha)$ is invariant under the isometric action $U^{(\lambda)}$ of G, that $\|f\|_{\mathcal{H}^{(\lambda)}(\alpha)}=0$ if and only if $f=\sum_{q(\lambda,\underline{m})<q(\lambda)} f_{\underline{m}}$, and that if $\mathcal{H}^{(\lambda)}(\alpha)$ is not trivial then it must coincide with $\mathcal{H}_{\lambda,q(\lambda)}$ with proportional semi-norms. It is also clear that $\mathcal{H}^{(\lambda)}(\alpha)$ is non-trivial if and only if $N^s\in\mathcal{H}^{(\lambda)}(\alpha)$, where as in § 4

$$s = s(\lambda) = \min\{l\in\mathbf{N};\ N^l\notin M_{q(\lambda)-1}^{(\lambda)}\} = \frac{p}{2}-\lambda.$$

It remains to check when is $N^s\in\mathcal{H}^{(\lambda)}(\alpha)$.

Recall that for every signature $\underline{m}$, $K^{\underline{m}}(z,w)$ is the reproducing kernel of $P_{\underline{m}}$ in the Fischer inner product.

Lemma 20. *For any signature $\underline{m}$, all $z,w\in D$ and $l\in\mathbf{N}$,*

$$\left(\frac{d}{r}\right)_{\underline{m}} K^{\underline{m}}(z,w)N(z)^l\,\overline{N(w)}^l = \left(\frac{d}{r}\right)_{\underline{m}+l} K^{\underline{m}+l}(z,w)$$

where $\underline{m}+l=(m_1+l, m_2+l, \ldots, m_r+l)$.

Proof. By the L-invariance of N and the definition of $\varphi_{\underline{m}}$ (see the introduction),

$$\varphi_{\underline{m}} N^l = \varphi_{\underline{m}+l}.$$

Let $t=\sum_{j=1}^r t_j e_j$, $t_j \geq 0$ (where $\{e_j\}_{j=1}^r$ is the fixed frame of orthogonal minimal tripotents). Denote $t^2=\sum_{j=1}^r t_j^2 e_j$. If $k \in K$, $z=k(t)$ then $N(z)=\chi(k)\, N(t)=\chi(k) \prod_{j=1}^r t_j$ and by [FK], Lemmas 3.1, 3.2 and Theorem 3.6:

$$K^{\underline{m}}(z, z) = K^{\underline{m}}(t, t) = \frac{\varphi_{\underline{m}}(t^2) \dim (P_{\underline{m}})}{\left(\frac{d}{r}\right)_{\underline{m}}}.$$

Thus, using Proposition 2

$$\left(\frac{d}{r}\right)_{\underline{m}} K^{\underline{m}}(z, z)\, N(z)^l\, \overline{N(z)}^l = \dim (P_{\underline{m}}) \cdot \varphi_m(t^2)\, N(t^2)^l$$

$$= \dim (P_{\underline{m}+l}) \varphi_{\underline{m}+l}(t^2) = \left(\frac{d}{r}\right)_{\underline{m}+l} K^{\underline{m}+l}(z, z).$$

The functions $\left(\frac{d}{r}\right)_{\underline{m}} K^{\underline{m}}(z, w) N(z)^l \overline{N(w)}^l$ and $\left(\frac{d}{r}\right)_{\underline{m}+l} K^{\underline{m}+l}(z, w)$ are analytic in z, conjugate analytic in w and coincide for $z=w$. Hence they coincide for all $z, w \in D$. ∎

Let $S(z, u)=K(z, u)^{1/2}$, $z \in D$, $u \in S$, be the Szegö kernel. It admits an expansion

$$S(z, u) = \sum_{\underline{m}} \left(\frac{d}{r}\right)_{\underline{m}} K^{\underline{m}}(z, u).$$

Lemma 21. *Let D be a tube domain and let $f=\sum_{\underline{m}} f_{\underline{m}} \in H^2(S)$. Then for $l \in \mathbf{N}$*

$$\int_S f(u)\, S(z, u)\, N(z)^l\, \overline{N(u)}^l\, d\sigma(u) = \sum_{m_r \geq l} f_{\underline{m}}(z)$$

Proof. $\left(\frac{d}{r}\right)_{\underline{m}} K^{\underline{m}}(z, u)$ is the reproducing kernel of $P_{\underline{m}}$ in the norm of $H^2(S)$. Hence

$$\int_S f(u)\, S(z, u)\, N(z)^l\, \overline{N(u)}^l\, d\sigma(u) = \sum_{\underline{m}} \int_S f(u) \left(\frac{d}{r}\right)_{\underline{m}} K^{\underline{m}}(z, u)\, N(z)^l\, \overline{N(u)}^l\, d\sigma(u)$$

$$= \sum_{\underline{m}} \int_S f(u) \left(\frac{d}{r}\right)_{\underline{m}+l} K^{\underline{m}+l}(z, u)\, d\sigma(u)$$

$$= \sum_{\underline{m}} f_{\underline{m}+l}(z) = \sum_{m_r \geq l} f_{\underline{m}}(z).$$

Corollary 22. *Let f be an analytic function in a neighborhood of $\bar{D}$, $s=\frac{p}{2}-\lambda=\frac{d}{r}-\lambda$. Then*

$$(Q^{(\lambda)} f)(z) = \int_S f(u)\, S(z, u)\, N(z)^s\, \overline{N(z)}^s\, d\sigma(u).$$

Proof. It is obvious that

$$M^{(\lambda)}_{q(\lambda)-1} = \sum_{m_r<s} \oplus P_m.$$

Hence, using Lemma 21 with $l=s$, we get for $f=\sum_{\underline{m}} f_{\underline{m}}$

$$(Q^{(\lambda)} f)(z) = \sum_{m_r \geq l} f_{\underline{m}}(z) = \int_S f(u)\, S(z, u)\, N(z)^s \overline{N(u)}^s\, d\sigma(u) = \theta N(\varphi_a(z) - z)^s. \quad \blacksquare$$

Lemma 23. *For all $a, z \in D$,*

$$Q^{(\lambda)}\big(U^{(\lambda)}(\varphi_a) N^s\big)(z) = \theta \cdot N(z)^s K(a, a)^{-(p-\lambda/2p)} K(z, a)^{1/2},$$

where θ is a unimodular constant independent of a and z.

Proof. By Lemmas 15 and 16 and Corollary 22, we get

$$Q^{(\lambda)}\big(U^{(\lambda)}(\varphi_a) N^s\big)(z)$$
$$= \int_S N(\varphi_a(u))^s J\varphi_a(u)^{\lambda/p} K(z, u)^{1/2} N(z)^s \overline{N(u)}^s\, d\sigma(u)$$
$$= \theta \int_S h(a, u)^s h(u, a)^{-s} K(z, u)^{1/2} K(u, a)^{\lambda/p}\, d\sigma(u)\, N(z)^s K(a, a)^{-\lambda/2p}$$
$$= \theta \int_S K(a, u)^{-s/p} S(u, a)\, K(z, u)^{1/2}\, d\sigma(u)\, N(z)^s K(a, a)^{-\lambda/2p}$$
$$= \theta \cdot K(a, a)^{-(s+(\lambda/2)/p)} K(z, a)^{1/2} N(z)^s$$
$$= \theta \cdot K(a, a)^{-(p-\lambda/2p)} K(z, a)^{1/2} N(z)^s$$

where $\theta = (-1)^{rs+d\lambda/p} = (-1)^{r((p-\lambda)/2)}$. $\blacksquare$

Lemma 24. *Let $\alpha > p-1$. Then there exists a positive constant C such that for every $f \in L^2_a(\mu_\alpha)$*

$$C\, \|f\|_{L^2(\mu_\alpha)} \leq \|N^s f\|_{L^2(\mu_\alpha)} \leq \|f\|_{L^2(\mu_\alpha)}.$$

Proof. Let $0 < \varepsilon < 1$, then

$$\Big(\int_{D \setminus \varepsilon D} |f(z)|^2\, d\mu_\alpha(z)\Big)^{1/2}$$

is an equivalent norm on $L^2_a(\mu_\alpha)$. Since $|N(z)| \geq \varepsilon^{-r}$ on $D \setminus \varepsilon D$ and $|N(z)| \leq 1$ for all $z \in D$ we get the desired inequality. $\blacksquare$

Conclusion of the proof of Theorem 19. By Lemmas 23 and 24,

$$\|N^s\|^2_{\mathscr{H}^{(\lambda)}(a)} \approx \iint_{D\times D} K(a, a)^{-(p-\lambda/p)} |K(z, a)|\, d\mu_\alpha(z)\, d\mu(a)$$
$$= \iint_{D\times D} |K(z, a)|\, d\mu_\alpha(z)\, K(a, a)^{\lambda/p}\, dV(a).$$

We claim that there exists $1 \leq C < \infty$ so that

$$1 \leq \int_D |K(z, a)|\, d\mu_\alpha(z) \leq C$$

for all $a\in D$. The lower estimate is trivially,

$$\int_D |K(z,a)|\,d\mu_\alpha(z) \geqq \left|\int_D K(z,a)\,d\mu_\alpha(a)\right| = |K(0,a)| = 1.$$

The upper estimate follows from [FK], Theorem 4.1 since $\alpha > p+(r-1)\frac{a}{2}$. Thus

$$\|N^s\|^2_{\mathscr{H}^{(\lambda)}(\alpha)} \approx \int_D K(a,a)^{\lambda/p}\,dV(a).$$

The integral on the right-hand side is finite if and only if $\lambda<1$. This completes the proof. ∎

Remark. It is possible to compute $\|N^s\|_{\mathscr{H}^{(\lambda)}(\alpha)}$ explicitly in terms of the Gindikin's Gamma function and the generalized hypergeometric functions, see [FK1] for general information and Proposition 2.2 there for the actual computation. This gives the value of the constant c in Theorem 19.

§ 6. Concluding remarks and open problems

The most interesting problem left for future study is to extend our results to the non-tube cases and to obtain "canonical" formulas (involving derivatives, integrals, etc.) for the invariant inner-products on the highest quotients $M^{(\lambda)}_{q(\lambda)}/M^{(\lambda)}_{q(\lambda)-1}$. This seems to require some new ideas if $r>1$. In the case of the unit ball B of $\mathbf{C}^d$ (which is the only Cartan domain of rank 1), $W_d=\{0\}$ and the invariant Hilbert space $\mathscr{H}_{0,q(0)}=\overline{\mathscr{P}^{(0)}/\mathbf{C}1}$ consists of all analytic functions $f(z)=\sum_\alpha c_\alpha z^\alpha$ on B so that $\|f\|^2_{\mathscr{H}_{0,q(0)}}:=\sum_\alpha |\alpha|\,\frac{\alpha!}{|\alpha|!}\,|c_\alpha|^2$ is finite, see [Z]. J. Peetre [P] obtained integral formulas for the invariant inner product $(\cdot,\cdot)_{0,q(0)}$ by analytic continuation of the inner products of $L^2_a(B,\mu_\lambda)$, $d<\lambda$. See [A] for the details. A similar formula was obtained independently by M. Peloso [Pe] by different methods.

Both Theorems 14 and 19 provide integral formulas for the highest quotient $M^{(\lambda)}_{q(\lambda)}/M^{(\lambda)}_{q(\lambda)-1}$ only for $\lambda=0$ and the special case of $D(\mathrm{III}_2)$ and $\lambda=1/2$. It is interesting to find the modifications of our formulas which will hold for more (or, all) $\lambda\in W_d(D)$. In Theorem 19, it seems that one can modify $Q^{(\lambda)}$ by subtracting terms of low degree, to improve the chance of convergence of the integrals.

What is behind the seemingly different descriptions of $\mathscr{H}_{\lambda,q(\lambda)}$ (Theorems 12, 14, and 19) is its uniqueness with respect to the isometric action $U^{(\lambda)}$ of G (see [AF] and step 4 of the proof of Theorem 14). One can obtain many other equivalent descriptions. For instance, let H be an auxiliary K-invariant Hilbert space of analytic functions on D with some natural properties, and consider the space $\mathscr{H}^{(\lambda)}(H)$ of all analytic functions f on D for which $Q^{(\lambda)}(U^{(\lambda)}(\varphi)f)\in H$ for all $\varphi\in G$ and

$$\|f\|_{\mathscr{H}^{(\lambda)}(H)}:=\left(\int_G \|Q^{(\lambda)}(U^{(\lambda)}(\varphi)f)\|^2_H\,d\varphi\right)^{1/2}$$

is finite. By Lemma 23, $\mathcal{H}^{(\lambda)}(H)$ is non-trivial if and only if

$$\int_D \|N^s K_a^{1/2}\|_H^2 K(a, a)^{\lambda/p} dV(a) < \infty$$

and in this case $\mathcal{H}^{(\lambda)}(H)=\mathcal{H}_{\lambda, q(\lambda)}$ with proportional semi-norms. In Theorem 19 we study the case where $H=L_a^2(\mu_\alpha)$ and $\alpha>p+(r-1)\frac{a}{2}$ (thus $\mathcal{H}^{(\lambda)}(\alpha)=\mathcal{H}^{(\lambda)}(L_a^2(\mu_\alpha))$. Also, it is easy to verify that $\mathcal{H}^{(\lambda)}(H^2(S))$ is always trivial.

Theorems 12, 14, and 19 justify the notation

$$B_2^{(\lambda)} = \mathcal{H}_{\lambda, q(\lambda)},$$

the *Besov-2 space* associated with the isometric action $U^{(\lambda)}$ of G. One can define the other *Besov-p spaces* associated with $U^{(\lambda)}$ (here $0<p\leqq\infty$, and the genus of D is denoted by g), by either

$$B_p^{(\lambda)} = \{f \text{ analytic in } D;\ \|f\|_{B_p^{(\lambda)}} := \|D_s^{(\lambda)}(f)\|_{L^p(G)} < \infty\}$$

or

$$B_p^{(\lambda)}(X) = \left\{f \text{ analytic in } D;\ \|f\|_{B_p^{(\lambda)}(X)} := \left(\int_G \left\|Q^{(\lambda)}(U^{(\lambda)}(\varphi)f)\right\|_X^p d\varphi\right)^{1/p} < \infty\right\}$$

where X is an auxiliary Banach space of analytic functions on D. It is elementary to use the proofs of Theorems 14 and 19 to characterize the non-triviality of these spaces. Thus $B_p^{(\lambda)}$ and $B_p^{(\lambda)}(L_a^p(\mu_\alpha))$ for $\alpha>\frac{gp}{2}+(r-1)\frac{a}{2}$ are non-trivial if and only if $\lambda<g-2(g-1)/p$. It is interesting to study the spaces $B_p^{(\lambda)}$ and $B_p^{(\lambda)}(X)$ from the usual point of views in the theory of Besov spaces, and in particular to establish our conjecture that $B_p^{(\lambda)}=B_p^{(\lambda)}(X)$ for interesting spaces X for which $B_p^{(\lambda)}(X)$ is non-trivial.

Motivated by our Theorem 19, and by [AFP], [AFJP1], and [AFJP2] we define the *(generalized) Hankel operator* H_f with an analytic symbol f as the operator H_f: $L^2(\mu_\alpha)\to L^2(\mu_\alpha)$ $(\alpha>g+(r-1)\frac{a}{2})$ given by

$$(H_f h)(z) = \int_D h(w) A_f(z, w) K(z, w)^{\alpha/g} d\mu_\alpha(w)$$

where

$$A_f(z, w) = \overline{\left(Q^{(0)}(f\circ\varphi_z)\right)(\varphi_z(w))}$$

and $Q^{(0)}$ is the orthogonal projection on the highest quotient $M_{q(0)}^{(0)}/M_{q(0)-1}^{(0)}$ (thus $\lambda=0$). It is easy to establish some of the usual properties of (ordinary) Hankel operators, for instance $H_f|_{L_a^2(\mu)^\perp}=0$ and

$$U^{(\alpha)}(\varphi) H_f U^{(\alpha)}(\varphi^{-1}) = H_{f\circ g}, \quad \varphi\in G.$$

It is interesting to investigate the question of boundness, compactness, and the membership in Schatten ideals S_p of the generalized Hankel operators. Theorem 19 says that $H_f \in S_2$ (=Hilbert—Schmidt class) if and only if $f \in B_2^{(0)}$ and that

$$\|H_f\|_{S_2} = c \cdot \|f\|_{B_2^{(0)}}.$$

We conjecture that (at least for $p > 2(g-1)/g$)

$$\|H_f\|_{S_p} \approx \|f\|_{B_p^{(0)}} \approx \|f\|_{B_p^{(0)}(X)} \approx \left(\int_D \|H_f k_z\|^p \, d\mu(z)\right)^{1/p}$$

where $k_z = K_z^{\alpha/g} / \|K_z^{\alpha/g}\|_{L^2(\mu_\alpha)}$ is the normalized kernel and $d\mu(z) = K(z,z)dV(z)$ is the G-invariant measure on D.

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Received Jan. 23, 1990
in revised form Aug. 14, 1990

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